

CM3110  
 Transport I  
 Part I: Fluid Mechanics

**MichiganTech**

**Microscopic Momentum  
 Balance Equation  
 (Navier-Stokes)**

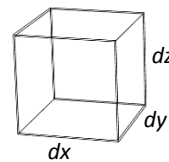


**Professor Faith Morrison**  
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 Michigan Technological University

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**Microscopic Balances**

We have been doing a microscopic *control volume* balance; these are specific to whatever problem we are solving.

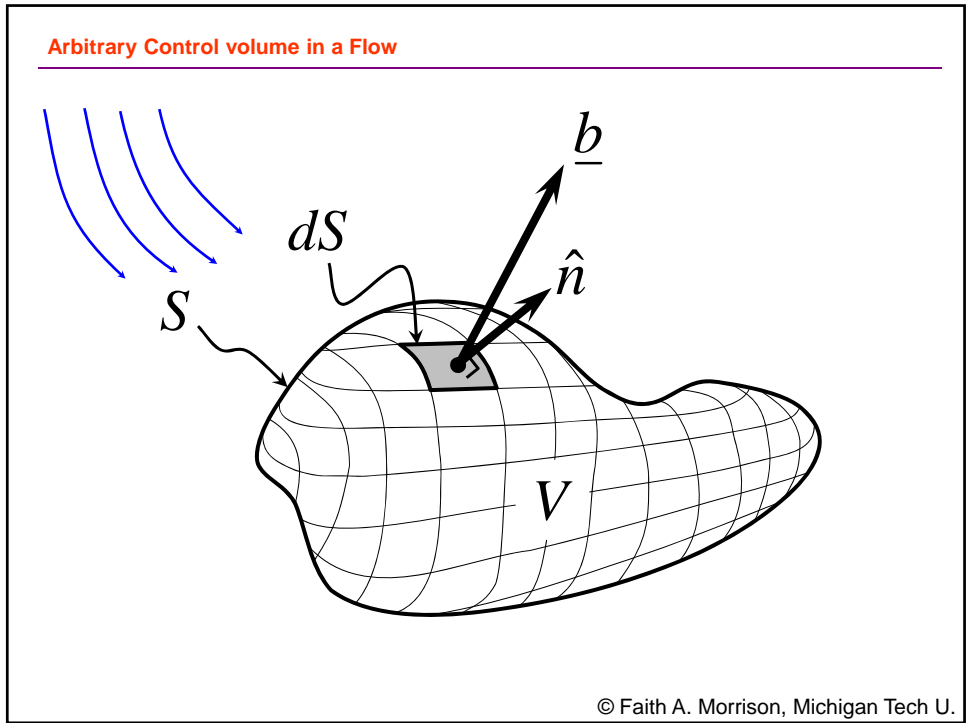


We seek equations for microscopic mass, momentum (and energy) balances that are *general*.



- equations must not depend on the choice of the control volume,
- equations must capture the appropriate balance

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Mass Balance

On an arbitrary control volume:

$$\left( \begin{array}{c} \text{rate of} \\ \text{increase of mass} \\ \text{in control volume} \end{array} \right) = \left( \begin{array}{c} \text{net flux of} \\ \text{mass into C.V.} \end{array} \right)$$

(details in the book)

$$\iiint_V \frac{\partial}{\partial t} (\rho) dV = - \iiint_V \nabla \cdot (\rho \underline{v}) dV$$

Rate of increase of mass
Net convection in

⇓

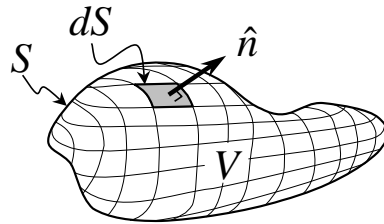
(just as we did with the individual control volume balance)

Microscopic mass balance for any flow

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### Continuity Equation

Microscopic **mass** balance written on an arbitrarily shaped control volume,  $V$ , enclosed by a surface,  $S$



$$\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)$$

Gibbs notation:  $\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$

Microscopic mass balance is a scalar equation.

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### Momentum Balance

On an arbitrary control volume:

(details in the book)  $\left( \begin{matrix} \text{rate of increase} \\ \text{of momentum} \\ \text{in C.V.} \end{matrix} \right) = \left( \begin{matrix} \text{net flux of} \\ \text{momentum} \\ \text{into C.V.} \end{matrix} \right) + \left( \begin{matrix} \text{sum of forces} \\ \text{on C.V.} \end{matrix} \right)$

$$\iiint_V \frac{\partial}{\partial t} (\rho \underline{v}) dV = - \iiint_V \nabla \cdot (\rho \underline{v} \underline{v}) dV + \iiint_V \rho \underline{g} dV - \iiint_V \nabla \cdot \underline{\underline{\Pi}} dV$$

Rate of increase of momentum

Net convection in

Force due to gravity

Viscous forces and pressure forces

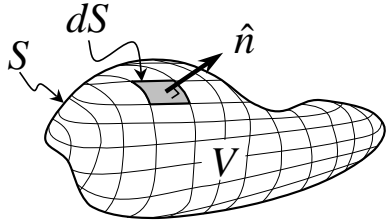


(just as we did with the individual control volume balance)

**Microscopic momentum balance for any flow**

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### Equation of Motion



Microscopic **momentum** balance written on an arbitrarily shaped control volume,  $V$ , enclosed by a surface,  $S$

Gibbs notation:  $\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$  general fluid

Gibbs notation:  $\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$  Newtonian fluid

Navier-Stokes Equation

Microscopic momentum balance is a vector equation. © Faith A. Morrison, Michigan Tech U.

### Continuity Equation (And Non-Newtonian Equation) on the FRONT

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

$\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = -\rho(\nabla \cdot \underline{v})$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

The one with "τ" is for non-Newtonian fluids

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**Navier-Stokes (Newtonian Fluids Only) is on the back:**

There are no " $\tilde{\tau}$ "'s on this side

$$\rho \left( \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial(rv_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r \\ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial(rv_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta \\ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

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**Problem-Solving Procedure – solving for velocity and stress fields**

1. sketch system
2. choose coordinate system
3. simplify the continuity equation (mass balance)
4. simplify the 3 components of the equation of motion (momentum balance) (note that for a Newtonian fluid, the equation of motion is the Navier-Stokes equation)
5. solve the differential equations for velocity and pressure (if applicable)
6. apply boundary conditions
7. calculate any engineering values of interest (flow rate, average velocity, force on wall)

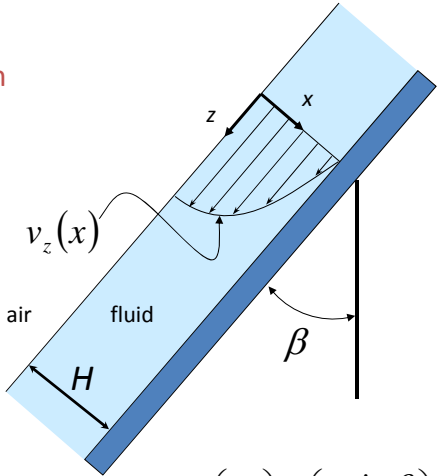
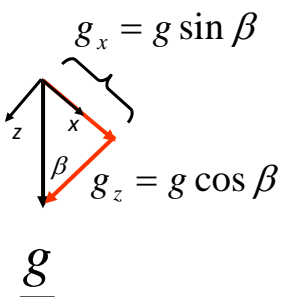
amended: when using the microscopic balances

$$\underline{\underline{\tilde{\tau}}} = \mu (\nabla \underline{v} + (\nabla \underline{v})^T)$$

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**EXAMPLE I:** Flow of a Newtonian fluid down an inclined plane

**Revisited**

$$\underline{g} = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix} = \begin{pmatrix} g \sin \beta \\ 0 \\ g \cos \beta \end{pmatrix}$$

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**EXAMPLE I:** Flow of a Newtonian fluid down an inclined plane

**Revisited**

(see hand notes)

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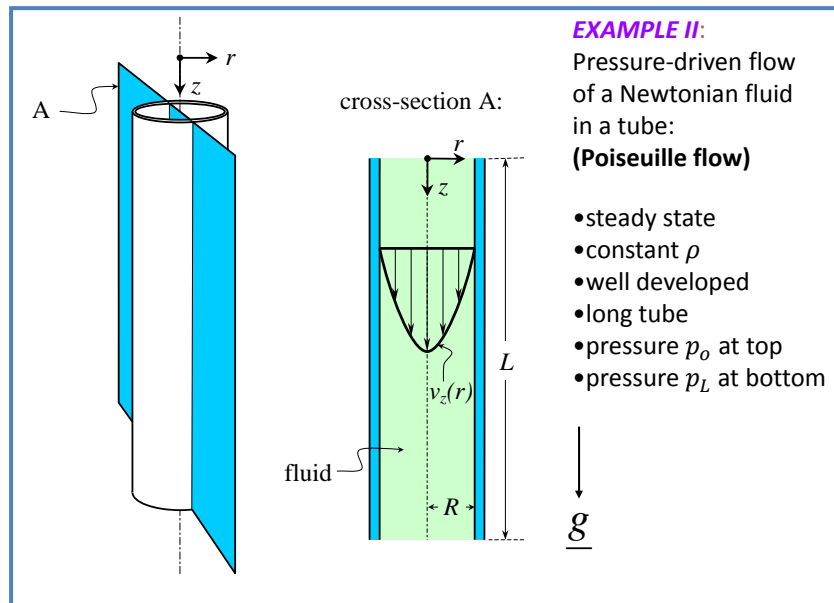
As with balance we performed with a control volume we selected, we made modelling assumptions along the way that we can collect and associate with the final result:

**Model Assumptions:** (laminar flow down an incline, Newtonian)

1. no velocity in the  $x$ - or  $y$ -directions (laminar flow)
2. well developed flow
3. no edge effects in  $y$ -direction (width)  $v_z \neq f(y)$
4. constant density
5. steady state
6. Newtonian fluid
7. no shear stress at interface
8. no slip at wall

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz}$$

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The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

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Continuity Equation, Cartesian coordinates

$$\frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r} \frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

Continuity Equation, spherical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(\rho v_\phi)}{\partial \phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x \\ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho g_y \\ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \end{aligned}$$

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**Navier-Stokes:**

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\ \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\ \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial P}{\partial r} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial(r v_r)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r \\ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial P}{\partial \theta} + \mu \left( \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial(r v_\theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta \\ \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \end{aligned}$$

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Poiseuille flow of a Newtonian fluid:

See hand notes

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<p><b>List of Common Integrals</b></p> <p>Prof. Faith Morrison, Michigan Technological University CM3110 Transport Processes I CM4650 Polymer Rheology</p> <hr/> $\int du = u + C$ $\int u \, du = \frac{u^2}{2} + C$ $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$ $\int \frac{1}{u} \, du = \ln u  + C$ $\int \frac{1}{u^2} \, du = -\frac{1}{u} + C$ <hr/> $\int e^u \, du = e^u + C$ $\int \ln u \, du = u \ln u - u + C$ $\int a^u \, du = \left(\frac{1}{\ln a}\right) a^u + C$ <hr/> $\int \sin u \, du = -\cos u + C$ $\int \cos u \, du = \sin u + C$ <hr/> $\int u e^u \, du = e^u(u-1) + C$ $\int u^2 e^u \, du = e^u(x^2 - 2x + 2) + C$ $\int e^{au} \sin u \, du = \frac{e^{au}(\sin u - \cos u)}{2} + C$ $\int e^{au} \cos u \, du = \frac{e^{au}(\sin u + \cos u)}{2} + C$	$\int e^{ax} \sin bx = \frac{e^{ax}}{(a^2+b^2)}(a \sin bx - b \cos bx) + C$ $\int e^{ax} \cos bx = \frac{e^{ax}}{(a^2+b^2)}(b \sin bx + a \cos bx) + C$ <hr/> <p>Integration by parts:</p> $\int u \, dv = uv - \int v \, du$ <hr/> <p>Note: Many more integrals can be solved with these formulas by using substitution. For example, evaluate <math>I</math>:</p> $I = \int (1-2x)^2 dx$ <p>Let <math>u = (1-2x)</math>, then</p> $\frac{du}{dx} = -2, \quad du = -2 \, dx$ <p>Substituting into the integral, we obtain</p> $I = -\frac{1}{2} \int u^2 du = \left(-\frac{1}{2}\right) \frac{u^3}{3} + C$ <p>Substituting back (eliminating <math>u</math>) gives the answer:</p> $I = -\frac{1}{6} (1-2x)^3 + C$ <hr/> <p>29 April 2014</p>	<p><b>List of Common Integrals</b></p> <p><a href="http://www.chem.mtu.edu/~fmorriso/cm310/2014CommonIntegrals.pdf">www.chem.mtu.edu/~fmorriso/cm310/2014CommonIntegrals.pdf</a></p> <p>© Faith A. Morrison, Michigan Tech U. <sup>18</sup></p>
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### Poiseuille flow of a Newtonian fluid:

What is the force on the walls in this flow?

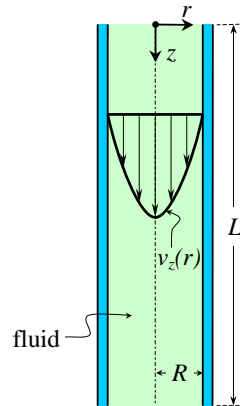
$$\int \left( \frac{\text{force}}{\text{area}} \right) (\text{area})$$

Total wetted area

$$= \int (\tilde{\tau}_{\text{?}}) (\text{area})$$

Inside surface of tube

cross-section A:

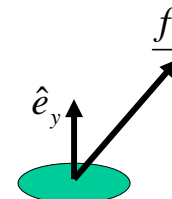
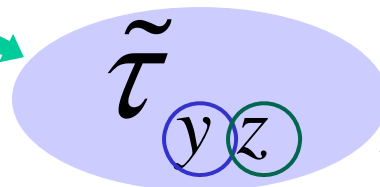


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$$\tilde{\tau}_{yz} = \frac{\text{force}}{\text{area}} = \frac{\text{kg m} / \text{s}^2}{\text{area}} = \frac{(\text{kg})(\text{m} / \text{s})}{(\text{s})(\text{area})}$$

**Momentum Flux**

9 stresses at a point in space



$$\underline{f} = A(\tilde{\tau}_{yz}\hat{e}_x + \tilde{\tau}_{yy}\hat{e}_y + \tilde{\tau}_{yz}\hat{e}_z)$$

stress on a y-surface → in the z-direction

A surface whose unit normal is in the y-direction

← flux of z-momentum

$$-\tilde{\tau}_{yz} =$$

(See discussion of sign convention of stress; this is the tension-positive convention,  $\tilde{\tau}_{ij}$ )

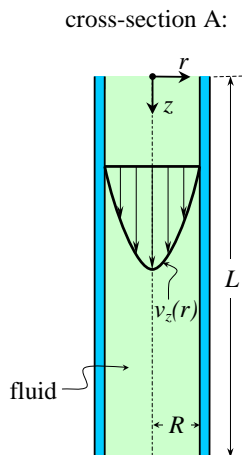
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Poiseuille flow of a Newtonian fluid:

What is the **shear stress** in this flow?

$$\tilde{\tau}_{rz} = \mu \frac{\partial v_z}{\partial r}$$

Stress on an r surface  
in the z direction



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Poiseuille flow of a Newtonian fluid:

Force on the walls:

See hand notes

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Poiseuille flow of a Newtonian fluid:

$$p(z) = \frac{-(p_0 - p_L)}{L}z + p_0$$

$$\tilde{\tau}_{rz}(r) = \frac{-(\rho gL + p_0 - p_L)}{2L}r$$

$$v(r) = \frac{R^2(\rho gL + p_0 - p_L)}{4\mu L} \left(1 - \left(\frac{r}{R}\right)^2\right)$$

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**Engineering  
Quantities of  
Interest**

(tube flow)

average  
velocity

$$\langle v_z \rangle = \frac{\int_0^{2\pi} \int_0^R v_z r dr d\theta}{\int_0^{2\pi} \int_0^R r dr d\theta}$$

volumetric  
flow rate

$$Q = \int_0^{2\pi} \int_0^R v_z r dr d\theta = \pi R^2 \langle v_z \rangle$$

z-component  
of force on  
the wall

$$F_z = \int_0^L \int_0^{2\pi} \tilde{\tau}_{rz} \Big|_{r=R} R d\theta dz$$

*Must work these out for each problem in the coordinate system in use; see inside back cover of book.*

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**Engineering  
Quantities of  
Interest**

(any flow)

volumetric  
flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average  
velocity

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component  
of force on  
the wall

$$F_z = \hat{e}_z \cdot \iint_S [\hat{n} \cdot (-p\underline{I} + \underline{\underline{\tau}})]_{surface} dS$$

For more complex flows, we use the **Gibbs notation** versions (will discuss soon).

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Poiseuille flow of a Newtonian fluid:

Volumetric Flow Rate:

volumetric  
flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

Let's try

See hand notes

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Poiseuille flow of a Newtonian fluid:

$$Q = \int_0^{2\pi} \int_0^R v_z(r) r dr d\theta$$

$$= \frac{R^2(\rho g L + p_o - p_L)}{4\mu L} \int_0^{2\pi} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr d\theta$$

$$Q = \frac{\pi R^4(\rho g L + p_o - p_L)}{8\mu L}$$

Hagen-Poiseuille  
Equation\*\*

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Poiseuille flow of a Newtonian fluid:

$$v_{av} = \int_0^{2\pi} \int_0^R v_z(r) r dr d\theta$$

$$= \frac{R^2(L\rho g + P_o - P_L)}{4\mu L} \int_0^{2\pi} \int_0^R \left(1 - \left(\frac{r}{R}\right)^2\right) r dr d\theta$$

$$= \frac{R^2(L\rho g + P_o - P_L)}{8\mu L}$$

$$= \frac{v_{z,max}}{2}$$

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