CM3110
Transport I
Part I: Fluid Mechanics: *Microscopic Balances*

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What we know about Fluid Mechanics

1. *MEB* (single input, single output, steady, incompressible, no rxn, no phase change, little heat; good for pipes, pumps; Moody chart; Fanning friction factor versus Re)
2. *Fluid Statics* ($P_{bot} = P_{top} + \rho g h$); same elevation, same pressure; good for manometers, water in tanks)
3. *Math* is in our future
4. *Newton's Law of Viscosity* (fluids transmit forces through momentum flux)
5. *Momentum flux* (=stress) has 9 components
6. *Drag* is a consequence of viscosity
7. *Boundary layers* form (viscous effects are confined near surfaces at high speeds)
8. Sometimes *viscous effects* dominate; sometimes *inertial effects* dominate
9.
10.
11.

What we know about Fluid Mechanics

Newton's second law applies: $\underline{f} = m\underline{a}$

1. *MEE*
   rxn,
   Moo
2. *Fluic* ;
   good for manometers, water in tanks)
3. *Math* is in our future
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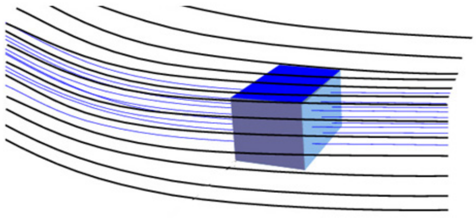
9. **Momentum balance determines velocity distributions**
10.

What we know about Fluid Mechanics

Newton's second law applies: $\underline{f} = m\underline{a}$



1. *MEE*
   rxn,
   Moo
2. *Fluic* ;
   good for manometers, water in tanks)
3. *Mat*
4. *New*
   mon
5. *Mon*
6. *Drag*
7. *Bou*
   surf
8. Som
   effects dominate
9. **Momentum balance determines velocity distributions**
10. **Control volumes are valuable for balances in fluids**

Following a Solid Object

(PH2100)

$$\boxed{\underline{f} = m\underline{a}}$$

- Forces on ***ball***
- Mass of ***ball***
- Acceleration of ***ball***

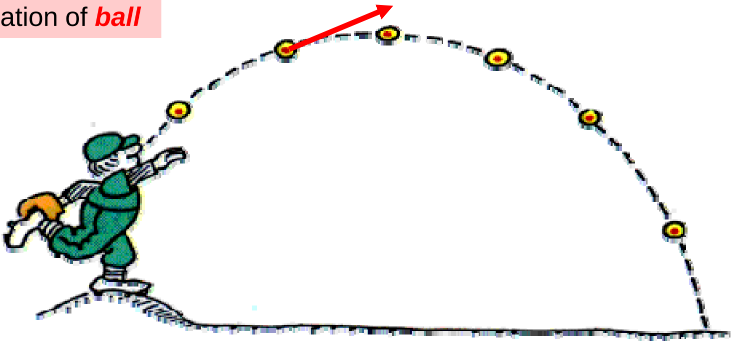
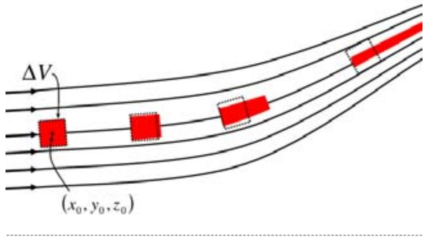


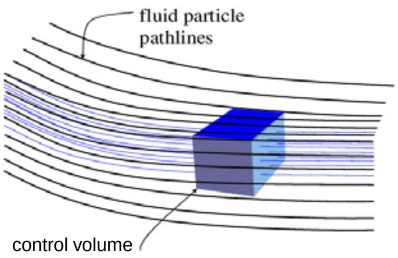
Image from
www.ux1.eiu.edu/~cfadd/1350/09Mom/CoM.html

Following fluid particles is complex:



An Essential Tool: **Control Volume**

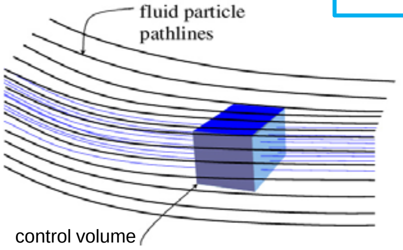
(Ch3)



It is simpler to observe the flow pass through a fixed volume, than to follow fluid particles

Control Volume

A chosen volume in a flow on which we perform balances (mass, momentum, energy)



- Shape, size are arbitrary; choose to be convenient
- Because we are now balancing on *control volumes* instead of on *bodies*, the laws of physics are written differently

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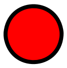
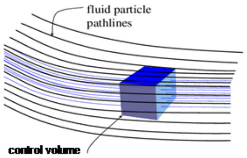
Mass balance, body:



$$M_{body} = \text{constant}$$

$$\frac{dM_{body}}{dt} = 0$$

Mass balance, flowing system (open system; control volume):



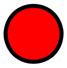
Convective term

$$\left\{\begin{matrix} net\ mass \\ flowing\ in \end{matrix}\right\} = \cancel{\left\{\begin{matrix} rate\ of \\ accumulation \\ of\ \ mass \end{matrix}\right\}}$$

$$\sum_{in} - \sum_{out}$$

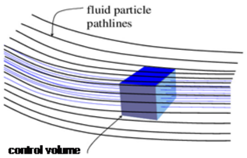
steady state

Momentum balance, **body**:



$$\sum_{on\ body} \underline{f} = M_{body}\underline{a}$$

Momentum balance, flowing system (open system; **control volume**):



Convective term

$$\left\{\begin{matrix}\text{sum of forces}\\ \text{acting } \underline{\text{on}} \text{ control vol}\end{matrix}\right\} + \underbrace{\left\{\begin{matrix}\text{net momentum}\\ \text{flowing in}\end{matrix}\right\}}_{\sum_{in} - \sum_{out}} = \left\{\begin{matrix}\text{rate of}\\ \text{accumulation}\\ \text{of momentum}\end{matrix}\right\}$$

Momentum balance, flowing system (open system; **control volume**):

$$\left\{\begin{matrix}\text{sum of forces}\\ \text{acting } \underline{\text{on}} \text{ control vol}\end{matrix}\right\} + \underbrace{\left\{\begin{matrix}\text{net momentum}\\ \text{flowing in}\end{matrix}\right\}}_{\sum_{in} - \sum_{out}} = \underset{\text{steady state}}{\cancel{\left\{\begin{matrix}\text{rate of}\\ \text{accumulation}\\ \text{of momentum}\end{matrix}\right\}}}$$

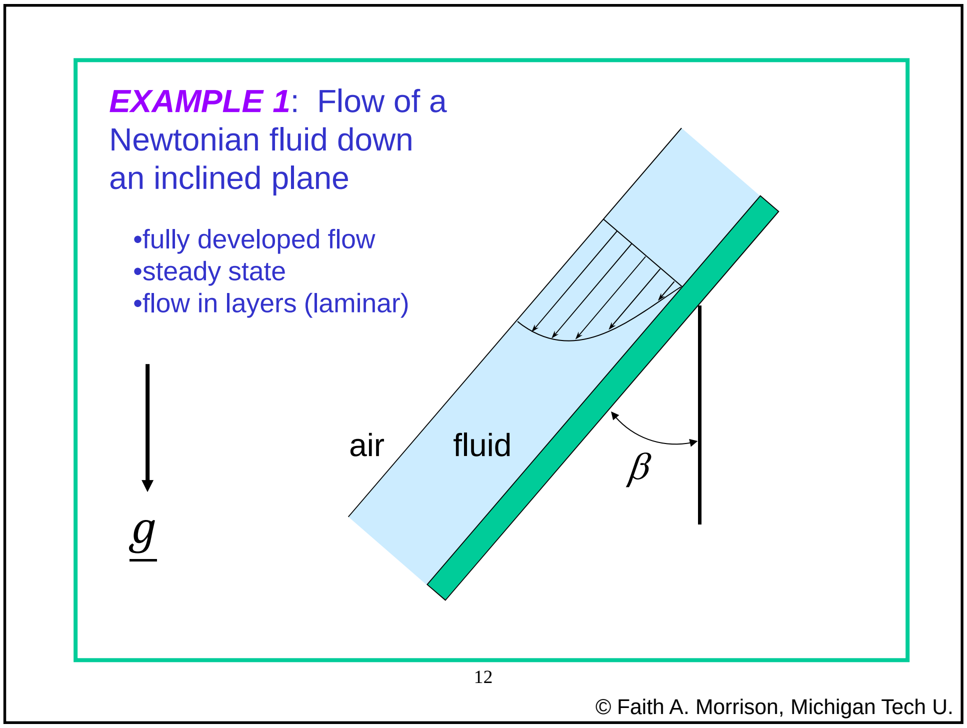
$$\sum_j \underline{F}_{on_i} + \sum_i \left\{\begin{matrix}\text{momentum}\\ \text{flowing } \underline{\text{in}}\\ \text{in the streams}\end{matrix}\right\} - \sum_i \left\{\begin{matrix}\text{momentum}\\ \text{flowing } \underline{\text{out}}\\ \text{in the streams}\end{matrix}\right\} = 0$$

note that **momentum** is a vector quantity

We are ready to try a momentum balance

Tools:

- Mass balance (mass conserved)
- Newton's 2nd law (momentum conserved)
- Control volume (convective term)
- Newton's law of viscosity
- Calc 2, Calc 3, Differential Eqns

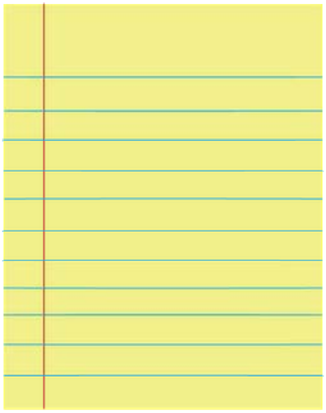


EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

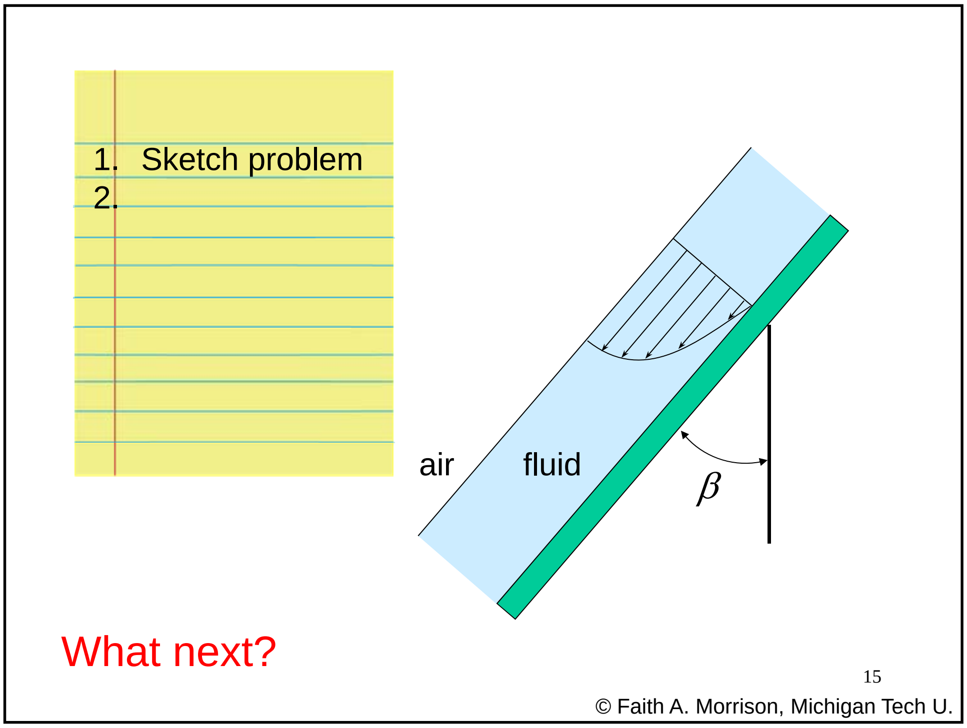
- •fully developed flow
- •steady state
- •flow in layers (laminar)

EXAMPLE 1: Flow of a Newtonian fluid down an inclined plane

What is the velocity field in the steady flow of water down a slope that is wide and long. The fluid properties are constant, and the flow is driven by gravity. The flow is slow so that no waves are formed. What is the force on the surface due to the water flow? What is the flow rate?

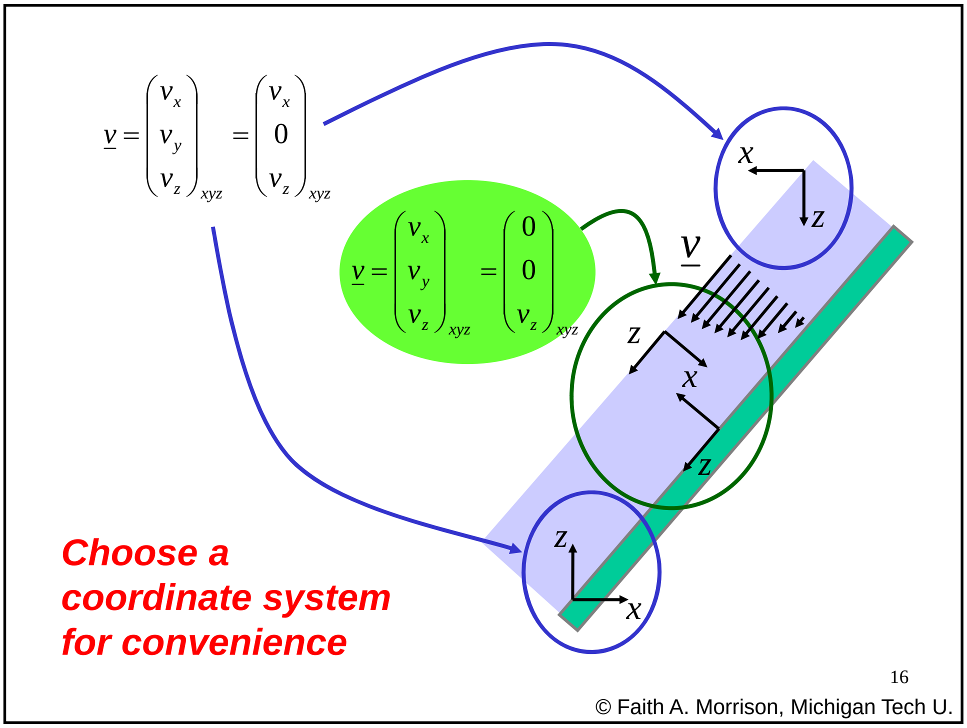


Where do we start?



1. Sketch problem
2.

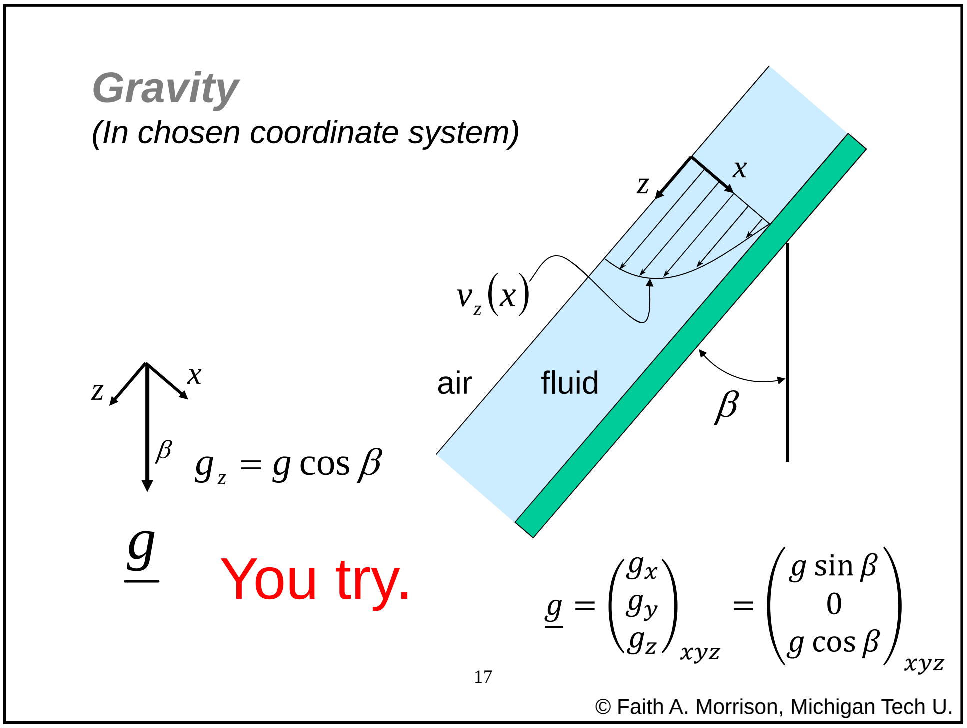
What next?



$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} v_x \\ 0 \\ v_z \end{pmatrix}_{xyz}$$

$$\underline{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_{xyz} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz}$$

Choose a coordinate system for convenience



Gravity

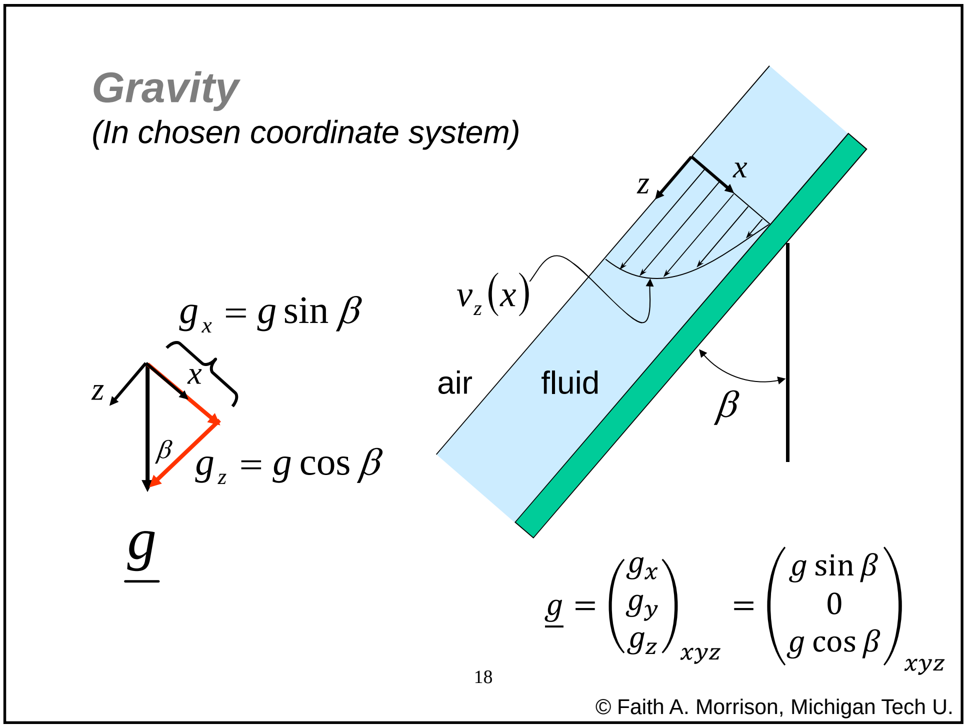
(In chosen coordinate system)

$g_z = g\cos\beta$

$\underline{g}$

You try.

$$\underline{g} = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}_{xyz} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz}$$



Gravity

(In chosen coordinate system)

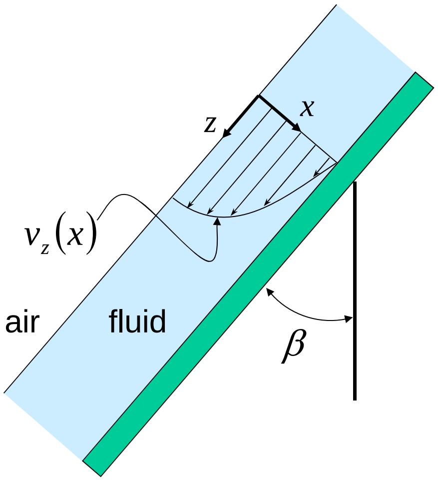
$g_x = g\sin\beta$

$g_z = g\cos\beta$

$\underline{g}$

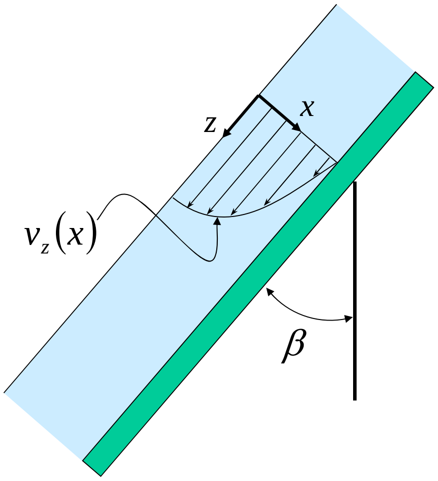
$$\underline{g} = \begin{pmatrix} g_x \\ g_y \\ g_z \end{pmatrix}_{xyz} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz}$$

1. Sketch problem
2. Coordinate sys
3.



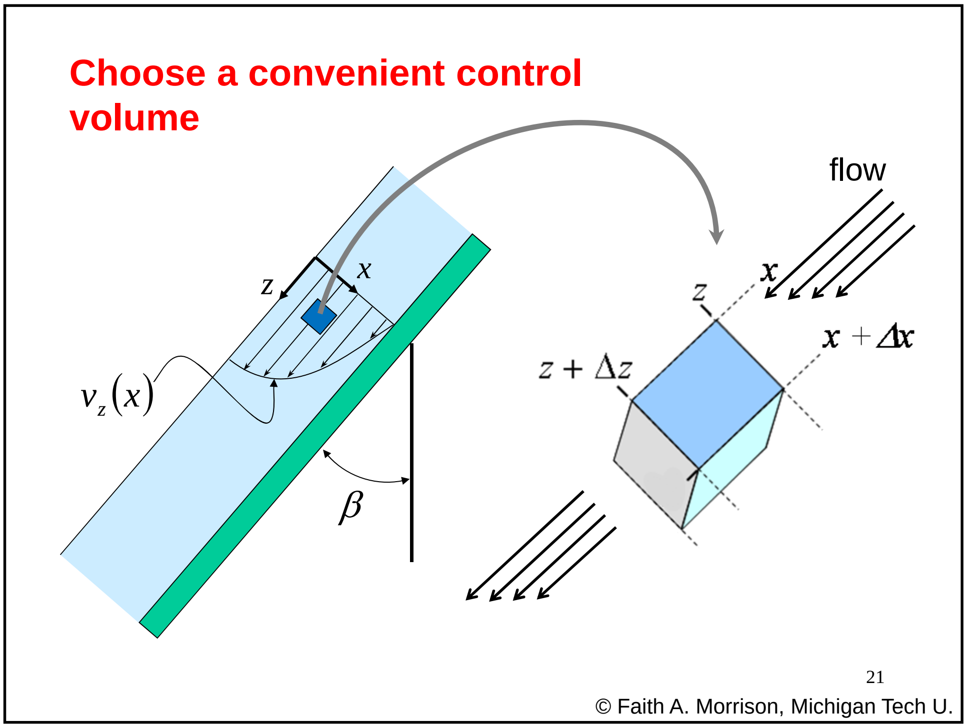
What next?

Choose a convenient control volume

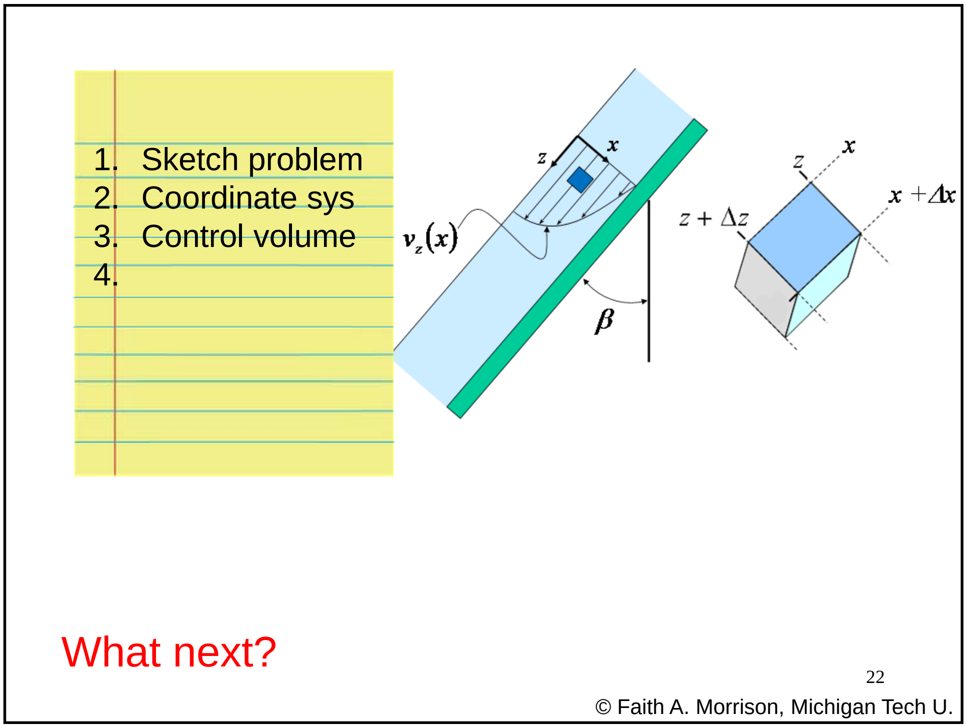


Want it to:

- Lead to what we're looking for
- Be easy to work with



Choose a convenient control volume



1. Sketch problem
2. Coordinate sys
3. Control volume
4.

What next?

Recall:

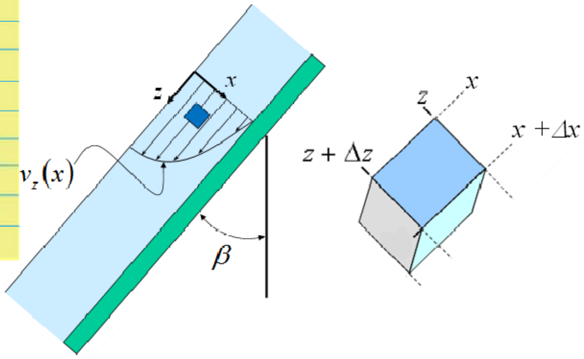
Tools:

- Mass balance (mass conserved)
- Newton's 2nd law (momentum conserved)
- Control volume (convective term)
- Newton's law of viscosity
- Calc 2, Calc 3, Differential Eqns

Solution steps

1. Sketch problem
2. Coordinate sys
3. Control volume
4. Mass balance
5. Momentum bal
6. Solve
7. Plot

Let's do it.



Mass balance, flowing system (open system; control volume):

$$\left\{\begin{matrix}\text{net mass}\\ \text{flowing in}\end{matrix}\right\} = \left\{\cancel{\begin{matrix}\text{rate of}\\ \text{accumulation}\\ \text{of mass}\end{matrix}}\right\}$$

$$\sum_{in} - \sum_{out}$$

steady state

Momentum balance, flowing system (open system; control volume):

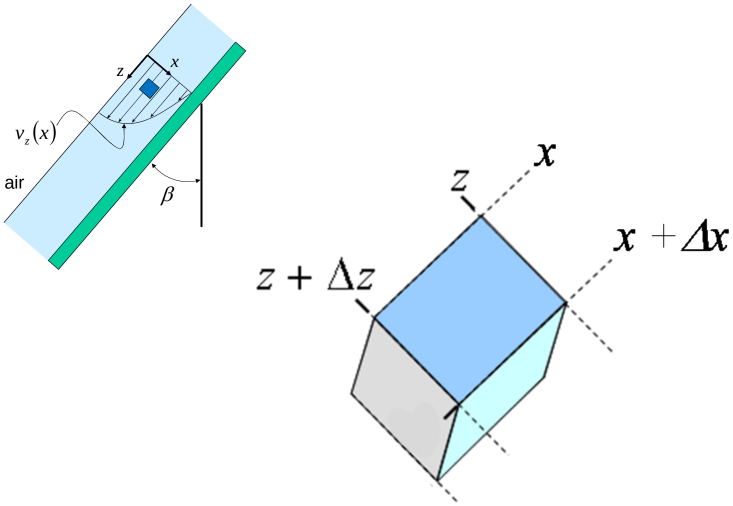
$$\sum_{all\ forces} \underline{f} = m\underline{a} \Rightarrow$$

$$\left\{\begin{matrix}\text{sum of forces}\\ \text{acting }\underline{\text{on}}\text{ control vol}\end{matrix}\right\} + \left\{\begin{matrix}\text{net momentum}\\ \text{flowing in}\end{matrix}\right\} = \left\{\cancel{\begin{matrix}\text{rate of}\\ \text{accumulation}\\ \text{of momentum}\end{matrix}}\right\}$$

$$\sum_{in} - \sum_{out}$$

steady state

$$\sum_i \underline{F}_{on_i} + \sum_i \left\{\begin{matrix}\text{momentum}\\ \text{flowing }\underline{\text{in}}\\ \text{in the streams}\end{matrix}\right\} - \sum_i \left\{\begin{matrix}\text{momentum}\\ \text{flowing }\underline{\text{out}}\\ \text{in the streams}\end{matrix}\right\} = 0$$

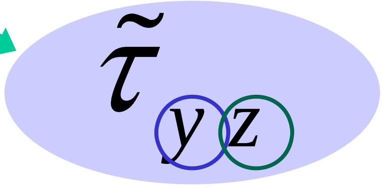


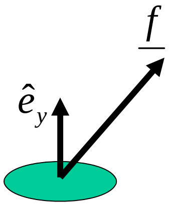
See handwritten notes.

$$\tilde{\tau}_{yz} = \frac{force}{area} = \frac{kg\ m/s^2}{area} = \frac{(kg)(m/s)}{(s)(area)}$$

Momentum Flux

9 stresses at a point in space





$$\underline{f} = A(\tau_{yx}\hat{e}_x + \tau_{yy}\hat{e}_y + \tau_{yz}\hat{e}_z)$$

stress on a *y*-surface → in the *z*-direction

A surface whose unit normal is in the y-direction

$-\tilde{\tau}_{yz} =$ flux of *z*-momentum ← in the *y*-direction

(See discussion of sign convention of stress; this is the tension positive convention)

$$\tau_{yz} = \mu\left(\frac{dv_z}{dy}\right)$$

Newton's Law of Viscosity

(Scalar relationship; specific to one coordinate system)

$$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}_{xyz}$$

We will discuss the general case later

$$\underline{\underline{\tau}} = \mu\underline{\underline{\dot{\gamma}}} = \mu(\nabla\underline{v} + (\nabla\underline{v})^T)$$

Newtonian Constitutive Equation

(Tensor relationship; <u>all</u> coordinate systems)

$$\tau_{xz} = \mu \left(\frac{dv_z}{dx} \right)$$

Newton’s Law of Viscosity

(adapted to our coordinate system)

(Scalar relationship; specific to one particular coordinate system)

Newton’s law gives the link between:

- *Deformation* (change of shape), and
- *Stress*

Flow down an Incline Plane

Boundary conditions:

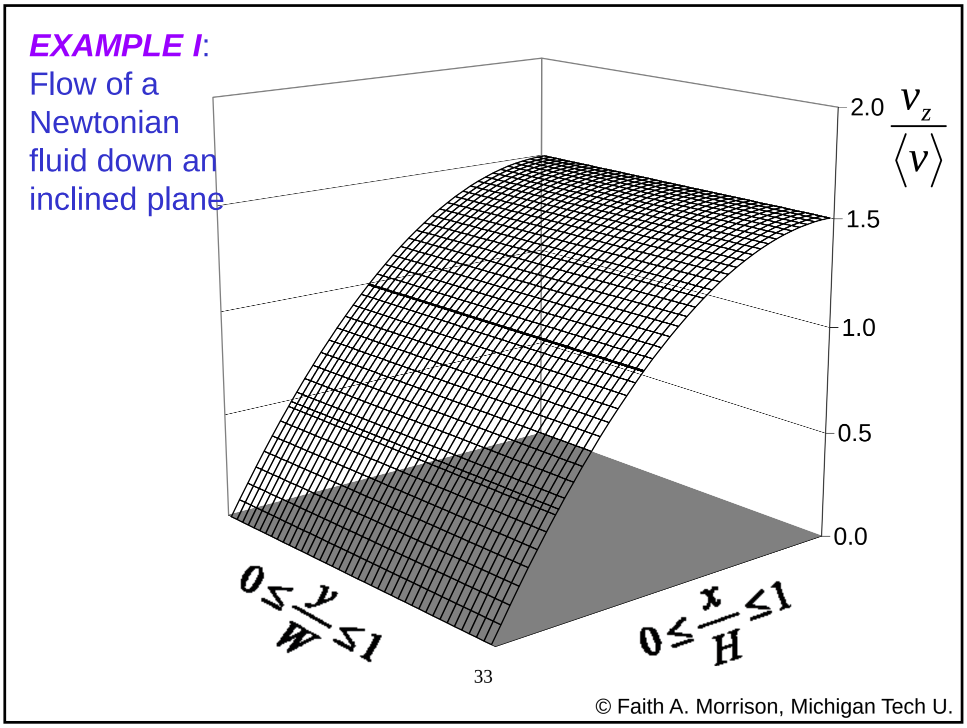
$$\begin{array}{ll} x = 0 & \tilde{\tau}_{xz} = 0 \\ x = H & v_z = 0 \end{array}$$

-stress matches at boundary

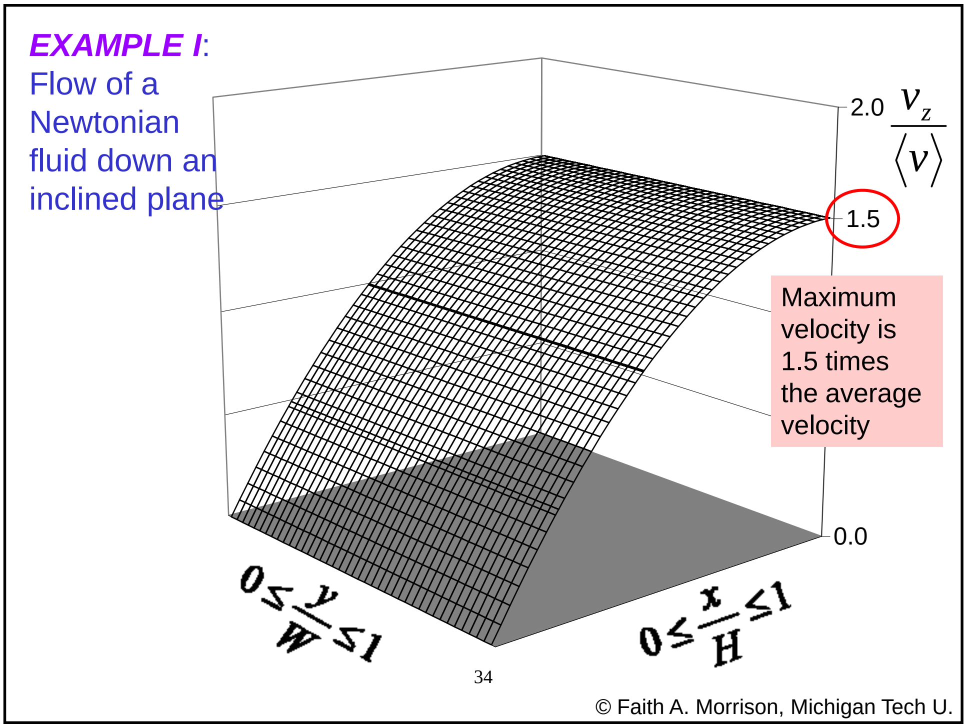
-no slip at the wall

Solution:

$$v_z(x) = \frac{\rho g \cos\beta}{2\mu}(H^2 - x^2)$$



EXAMPLE I: Flow of a Newtonian fluid down an inclined plane



EXAMPLE I: Flow of a Newtonian fluid down an inclined plane

Maximum velocity is 1.5 times the average velocity

From the start of the problem, we developed our **model** step by step. We can collect our modeling assumptions, which are limitations on the result.

Model Assumptions: (laminar flow down an incline, Newtonian)

1. no velocity in the *x*- or *y*-directions (laminar flow)
2. well developed flow
3. no edge effects in *y*-direction (width)
4. constant density
5. steady state
6. Newtonian fluid
7. no shear stress at interface
8. no slip at wall

Calculate: What is the shear stress as a function of position for this flow?

Newton's Law of Viscosity

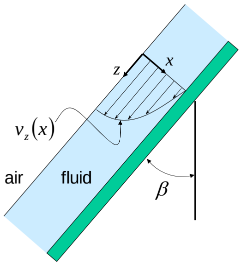
$$\tilde{\tau}_{xz} = \mu \frac{\partial v_z}{\partial x}$$

You try.

We can now calculate:

Engineering Quantities of Interest

- Average velocity
- Volumetric flow rate
- Force on the wall



$$v_z(x) = \frac{\rho g \cos\beta}{2\mu}(H^2 - x^2)$$

Engineering Quantities of Interest

average velocity

$$\langle v_z \rangle \equiv \frac{\int_0^W \int_0^H v_z \, dx \, dy}{\int_0^W \int_0^H dx \, dy}$$

H is the height of the film; W is the width

volumetric flow rate

$$Q = \int_0^W \int_0^H v_x \, dx \, dy = WH \langle v_z \rangle$$

z-component of force on the wall

$$F_z = \int_0^L \int_0^W \tilde{\tau}_{xz}\Big|_{x=H} \, dy \, dz$$

(The expressions are different in different coordinate systems)

Engineering Quantities of Interest

(any flow)

volumetric flow rate

$$Q = \iint_S (\hat{n} \cdot \underline{v}) dS$$

average velocity

$$\langle v_z \rangle \equiv \frac{\iint_S (\hat{n} \cdot \underline{v}) dS}{\iint_S dA} = \frac{Q}{S}$$

z-component of force on the wall

$$F_z = \hat{e}_z \cdot \iint_S \left[\hat{n} \cdot \left(-p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}\right)\right]_{surface} dS$$

For more complex flows, we use the ***Gibbs notation*** *versions (will discuss soon).*

What did we do?

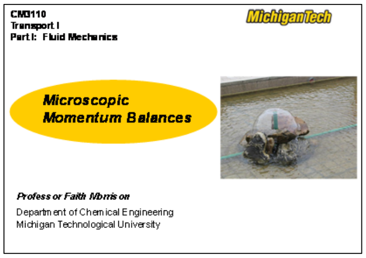
Problem-Solving Procedure – *solving for velocity and stress fields*

1. sketch system
2. choose coordinate system
3. choose a control volume
4. perform a mass balance
5. perform a momentum balance
   (will contain stress)
6. substitute in *Newton's law of viscosity*, e.g. $\boxed{\tilde{\tau}_{xz} = \mu\left(\frac{dv_z}{dx}\right)}$
7. solve the <u>differential</u> equation
8. apply boundary conditions

How can we generalize this soln process?

(to make the process easier)

Answer: Develop a balance that works for all control volumes



CM3110
Transport I
Part I: Fluid Mechanics

Microscopic Momentum Balances

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