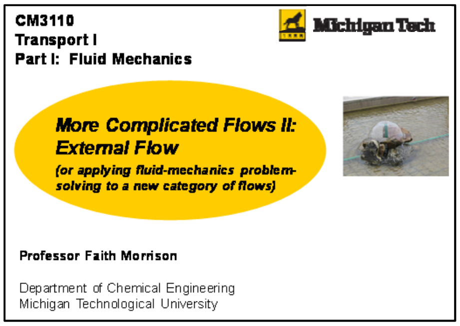
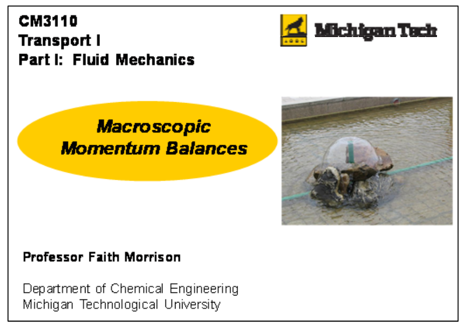
Before continuing on to External flows



Let's take the time to explore a different type of momentum balance



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Michigan Tech

CM3110
Transport I
Part I: Fluid Mechanics

Macroscopic Momentum Balances

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Michigan Technological University

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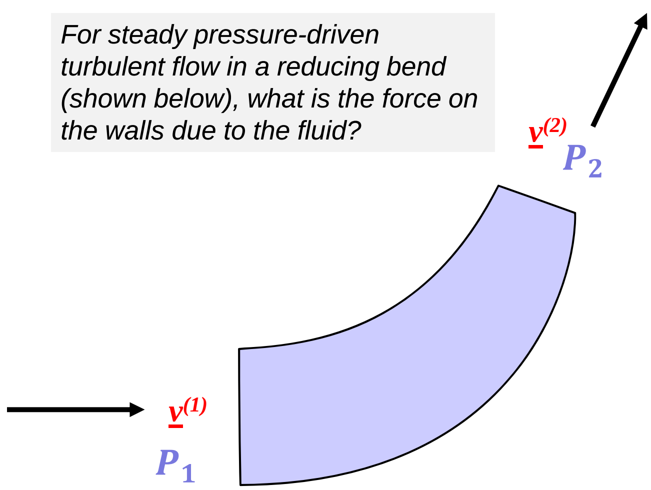
Macroscopic Momentum Balance Example:
Drag on the walls of a pipe

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

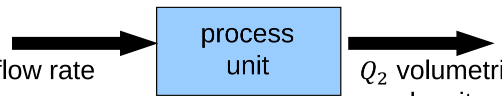
For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?



Macroscopic Balances

- Use when we do not need the details of the velocity profile
- 3 types:
  - -mass (CM2110, Felder and Rousseau)
  - -momentum ←Now
  - -energy (CM2110, Felder and Rousseau)

Macroscopic Mass Balance:



Q_1 volumetric flow rate
ρ_1, density

Q_2 volumetric flow rate
ρ_2, density

Macroscopic Momentum Balance

$\beta_{laminar} = 0.75$
$\beta_{turbulent} \sim 1$

$\underline{\mathbf{P}}$ = fluid momentum vector

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{\underline{v}}\right]_{A_i}$$

$$= \sum_{i=1}^{\#streams} \left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$\underline{R}$ = net force on fluid due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal

See inside front Cover
of Morrison, 2013

Compare with the (more familiar) Navier-Stokes

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

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Rate of change of momentum with time

Macroscopic Momentum Balance

Ma ntum Balance

Convective terms

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma ntum

Convective terms

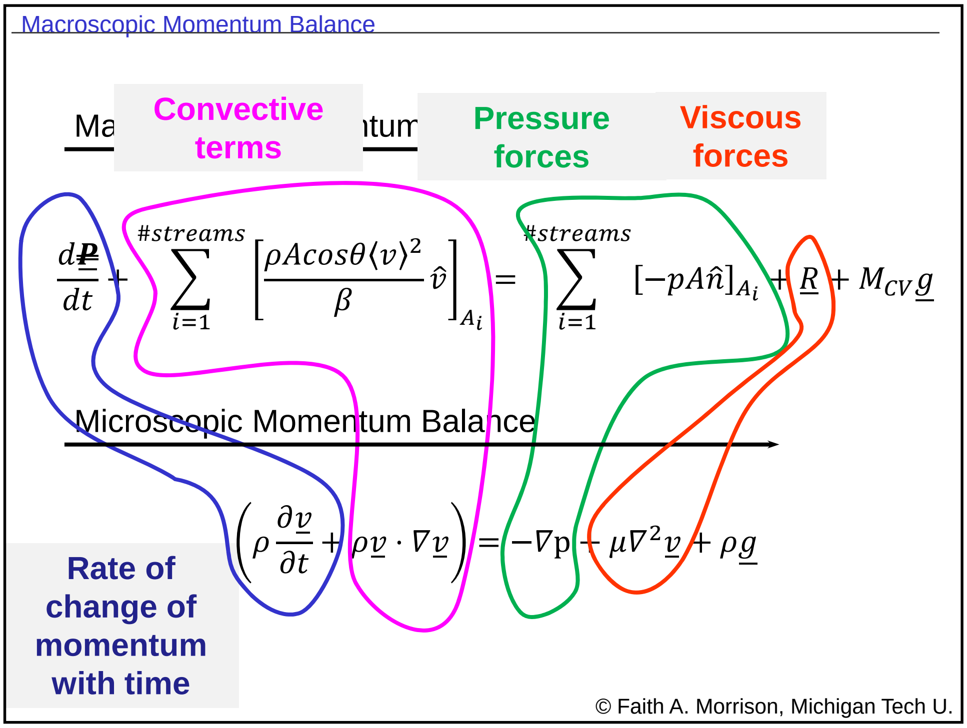
Pressure forces

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time


Macroscopic Momentum Balance
Convective terms
Pressure forces
Viscous forces
$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{\underline{v}}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{\underline{n}}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$
Microscopic Momentum Balance
$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$
Rate of change of momentum with time
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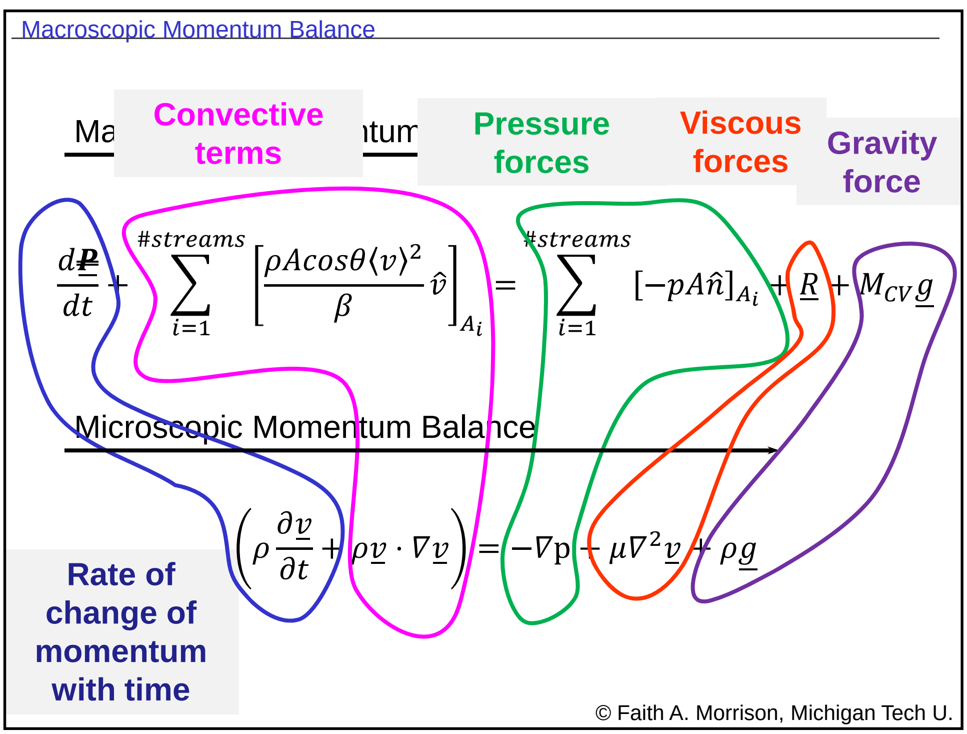

Macroscopic Momentum Balance
Convective terms
Pressure forces
Viscous forces
Gravity force
$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{\underline{v}}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{\underline{n}}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$
Microscopic Momentum Balance
$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$
Rate of change of momentum with time
© Faith A. Morrison, Michigan Tech U.

Macroscopic Momentum Balance

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \mathrm{p} + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

We know how to apply this

Now we need to learn when and how to apply this

Macroscopic Momentum Balance

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams} \left[\frac{\rho A cos\theta \langle v \rangle^2}{\beta} \hat{v} \right]_{A_i} = \sum_{i=1}^{\#streams} [-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} \right) = -\nabla \mathrm{p} + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

We know how to apply this

We begin with the

Macroscopic Mass Balance

Mass accumulation = Mass in - Mass out

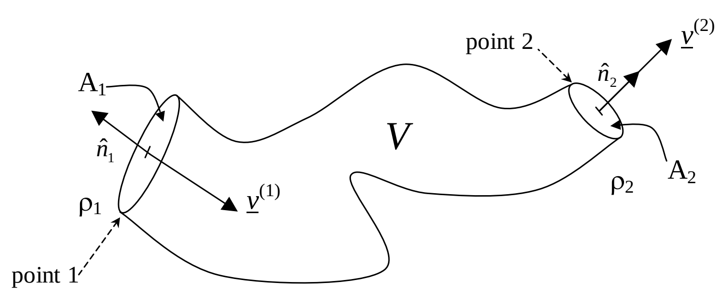
To obtain a general equation (first for mass, then for momentum) we first consider the following case:

- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV

Macroscopic Mass Balance:

Mass in = Mass out

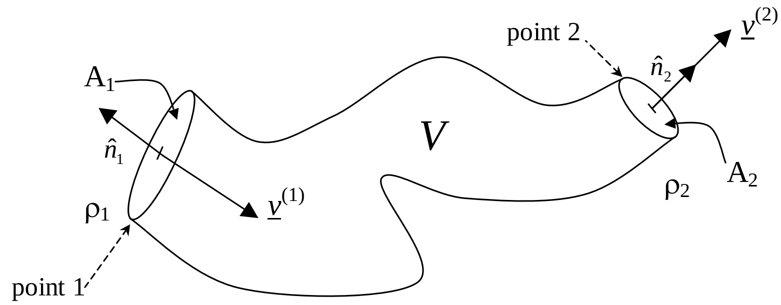
- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV



Macroscopic Mass Balance:

See Chapter 9 for detailed derivation

Arbitrary, single-input, single-output system: *special case* of velocity perpendicular to control surfaces (CS) A_i



Special case:

Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **perpendicular to** A_i
- ρ constant across surface

Macroscopic Mass Balance:

Mass in = Mass out

$$\rho_1 \langle v^{(1)} \rangle A_1 = \rho_2 \langle v^{(2)} \rangle A_2$$

average velocity through surface A_1

cross-sectional area, in

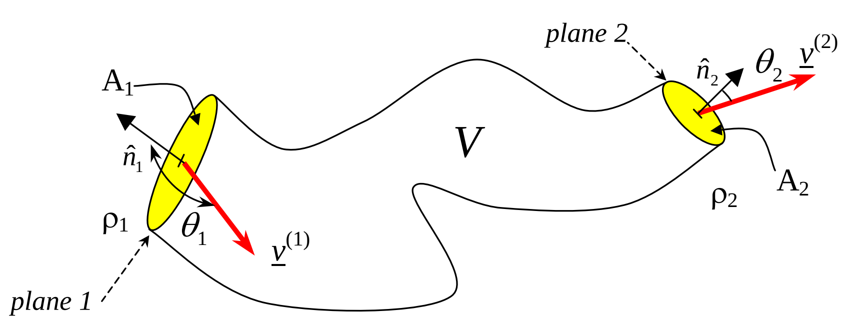
average velocity through surface A_2

cross-sectional area, out

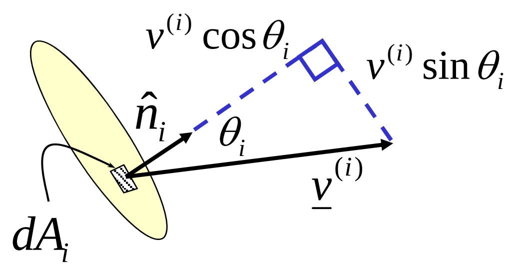
Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **perpendicular to** A_i
- ρ constant across surface

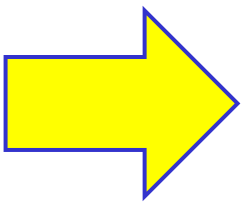
Arbitrary, single-input, single-output system:

velocity is *NOT perpendicular* to control surfaces A_i



Macroscopic Mass Balance





$\hat{\boldsymbol{n}}_{\boldsymbol{i}} =$ ***outwardly*** **pointing (with respect to the control volume CV) unit normal**

Macroscopic Mass Balance

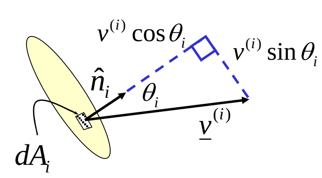
Macroscopic Mass Balance:

This takes care of 'out' or 'in'

0= net mass out

$$0 = \rho_1 \langle v^{(1)} \rangle \cos\theta_1\, A_1 + \rho_2 \langle v^{(2)} \rangle \overbrace{\cos\theta_2}\, A_2$$

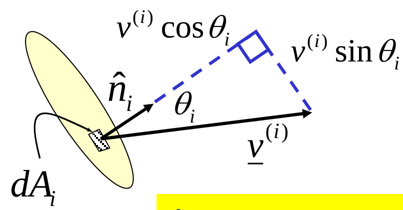
Assumptions:
- steady state
- single-input, single output
- **$\underline{v}^{(i)}$NOT perpendicular to A_i**
- ρ_i constant across surface



Macroscopic Mass Balance

Reminder:

θ relates to the orientation of inlet and outlet surfaces in the chosen coordinate system



$\hat{n}_i$ = *outwardly* pointing unit normal at in/outlet of CV

Macroscopic Momentum Balance

Macroscopic Momentum Balance:

$$\boxed{\sum \underline{F}_{on\,CV} = \cancel{\frac{d\underline{P}}{dt}} + \underbrace{\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS}_{\left(\text{net momentum convected out}\right)}}$$

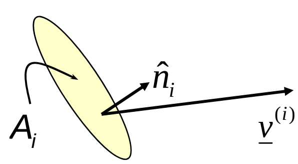
steady state

Momentum balance on fluid in a control volume

We can specialized the ***convective term*** for macroscopic control volumes

$$\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{i=1}^{N}\left[\iint_{CS} (\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i$$

N bounding control surfaces



steady state

See Chapter 9 for detailed derivation

N bounding control surfaces

Macroscopic Momentum Balance
(continued)

$$\begin{pmatrix} \text{net momentum} \\ \text{out of CV} \end{pmatrix} = \iint_{CS} (\hat{n} \cdot \underline{v}) \rho \underline{v} \, dS = \sum_{i=1}^{N} \left[\iint_{CS} (\hat{n} \cdot \underline{v}) \rho \underline{v} \, dS \right]_i$$

Input, output surfaces A_i

$$\sum_{i=1}^{N} \left[\iint_{CS} (\hat{n} \cdot \underline{v}) \rho \underline{v} \, dS \right]_i = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)} \right) \left(\hat{n}_i \cdot \underline{v}^{(i)} \, dA \right)$$

We can now specify for each A_i:

$$\underline{v}^{(i)} = v^{(i)} \hat{v}^{(i)}$$

We now separate velocity *magnitude* from the direction

Macroscopic Momentum Balance
(continued)

We now separate velocity *magnitude* from the direction

For each inlet or outlet surface A_i:

$$\boxed{\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}}$$

- $v^{(i)}$ = magnitude of velocity through A_i
- $\hat{v}^{(i)}$ = unit vector in direction of velocity through A_i
- $\hat{n}\cdot\underline{v}^{(i)} = v^{(i)}\cos(\theta_i)$ = component of velocity "through" A_i

Macroscopic Momentum Balance
(continued)

$$\begin{pmatrix}\text{net momentum}\\ \text{out of CV}\end{pmatrix} = \iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS = \sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i$$

Input, output surfaces A_i

$$\sum_{i=1}^{N}\left[\iint_{CS}(\hat{n}\cdot\underline{v})\rho\underline{v}\,dS\right]_i = \sum_i\iint_{A_i}\left(\rho\underline{v}^{(i)}\right)\left(\hat{n}_i\cdot\underline{v}^{(i)}dA\right)$$

$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$
$$\hat{n}\cdot\underline{v}^{(i)} = v^{(i)}\cos\theta_i$$

We now separate velocity *magnitude* from the direction

Macroscopic Momentum Balance

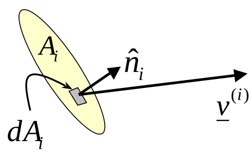
group the velocity magnitudes together

$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

$$\begin{pmatrix} \textit{net momentum} \\ \textit{convected out} \end{pmatrix} = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)}\right)\left(\underbrace{\hat{n}_i \cdot \underline{v}^{(i)}}_{v^{(i)}\cos\theta_i} dA\right)$$

$$= \sum_i \iint_{A_i} \left(\rho_i v^{(i)}\hat{v}^{(i)}\right)\left(v^{(i)}\cos\theta_i \, dA_i\right)$$

$$= \sum_i \rho_i \hat{v}^{(i)}\cos\theta_i \left(\iint_{A_i} \left(v^{(i)}\right)^2 dA_i\right)$$



only the velocity magnitudes vary across dA_i; they appear as ${v^{(i)}}^2$

Macroscopic Momentum Balance

group the velocity magnitudes together

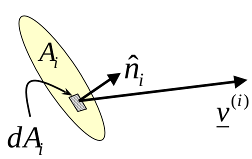
$$\underline{v}^{(i)} = v^{(i)}\hat{v}^{(i)}$$

$$\begin{pmatrix} \textit{net momentum} \\ \textit{convected out} \end{pmatrix} = \sum_i \iint_{A_i} \left(\rho_i \underline{v}^{(i)}\right)\left(\underbrace{\hat{n}_i \cdot \underline{v}^{(i)}}_{v^{(i)}\cos\theta_i} dA\right)$$

$$= \sum_i \iint_{A_i} \left(\rho_i v^{(i)}\hat{v}^{(i)}\right)\left(v^{(i)}\cos\theta_i \, dA_i\right)$$

We have assumed that the direction of $\underline{v}^{(i)}$ does not vary across A_i.

$$= \sum_i \rho_i \hat{v}^{(i)}\cos\theta_i \left(\iint_{A_i} \left(v^{(i)}\right)^2 dA_i\right)$$



only the velocity <u>magnitudes</u> vary across dA_i; they appear as ${v^{(i)}}^2$

Macroscopic Momentum Balance

Assumptions:
- steady state
- single-input, single output
- $\underline{v}^{(i)}$ **NOT perpendicular to A_i**
- ρ_i constant across surfaces
- $\hat{v}^i$ constant across surfaces

Sum of the forces on the fluid in the CV

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \left[\iint_{A_1} \left(v^{(1)}\right)^2 dA \right] - \rho_2 \cos\theta_2 \hat{v}^{(2)} \left[\iint_{A_2} \left(v^{(2)}\right)^2 dA \right] + \sum_i \underline{F}_{i,on}$$

Macroscopic Momentum Balance

We can write these terms compactly as $\frac{\langle v \rangle^2}{\beta}$, as we now show

Sum of the forces on the fluid in the CV

$$0 = -\rho_1 \cos\theta_1 \hat{v}^{(1)} \left[\iint_{A_1} \left(v^{(1)}\right)^2 dA \right] - \rho_2 \cos\theta_2 \hat{v}^{(2)} \left[\iint_{A_2} \left(v^{(2)}\right)^2 dA \right] + \sum_i \underline{F}_{i,on}$$

Recall that the average of a function *f* is calculated from:

$$\langle f(x,y) \rangle = \frac{\iint_A f \, dA}{\iint_A dA} = \frac{1}{A} \iint_A f \, dA$$

Macroscopic Momentum Balance

Sum of the forces on the fluid in the CV

$$0=-\rho_1\cos\theta_1\hat{v}^{(1)}\underbrace{\left[\iint_{A_1}\left(v^{(1)}\right)^2 dA\right]}_{=\left\langle\left(v^{(1)}\right)^2\right\rangle A_1}-\rho_2\cos\theta_2\hat{v}^{(2)}\underbrace{\left[\iint_{A_2}\left(v^{(2)}\right)^2 dA\right]}_{=\left\langle\left(v^{(2)}\right)^2\right\rangle A_2}+\sum_i \underline{F}_{i,on}$$

$$\boxed{0=-\rho_1 A_1\left\langle\left(v^{(1)}\right)^2\right\rangle\cos\theta_1\hat{v}^{(1)}-\rho_2 A_2\left\langle\left(v^{(2)}\right)^2\right\rangle\cos\theta_2\hat{v}^{(2)}+\sum_i \underline{F}_{i,on}}$$

But what is this?

We can make this look more like other convective terms we have seen by introducing a factor relating $\langle v^2\rangle$ to average velocity squared.

Macroscopic Momentum Balance

β quantifies the variation of the true velocity profile from **plug flow** (flat profile). $\beta \equiv$ ***Velocity Correction Factor***

$$\sum_i \underline{F}_{i,on}=\rho_1 A_1\left\langle\left(v^{(1)}\right)^2\right\rangle\cos\theta_1\hat{v}^{(1)}+\rho_2 A_2\left\langle\left(v^{(2)}\right)^2\right\rangle\cos\theta_2\hat{v}^{(2)}$$

define: $$\boxed{\beta\equiv\frac{\langle v\rangle^2}{\langle v^2\rangle}}$$

experimental result

$\beta_{turbulent} = 0.95\text{-}0.99$

$\beta_{laminar} = 0.75$

Result: Steady State Macroscopic Momentum Balance (convective terms)

$$\boxed{\sum_i \underline{F}_{i,on}=\frac{\rho_1 A_1\left\langle v^{(1)}\right\rangle^2\cos\theta_1}{\beta_1}\hat{v}^{(1)}+\frac{\rho_2 A_2\left\langle v^{(2)}\right\rangle^2\cos\theta_2}{\beta_2}\hat{v}^{(2)}}$$

vector equation

Macroscopic Momentum Balance

Force Terms

$$\sum_i \underline{F}_{i,on} = \frac{\rho_1 A_1 \langle v^{(1)} \rangle^2 \cos\theta_1}{\beta_1}\hat{v}^{(1)} + \frac{\rho_2 A_2 \langle v^{(2)} \rangle^2 \cos\theta_2}{\beta_2}\hat{v}^{(2)}$$

Sum of the forces on the fluid in the CV

$$\sum_i \underline{F}_{i,on} = contact + noncontact$$

$$\underline{F}_{contact} = \iint_S \left[\hat{n}\cdot\underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

Molecular forces

(viscosity and pressure)

$$= M_{CV}\,\underline{g}$$

gravity

Macroscopic Momentum Balance

Contact Forces = pressure + viscous

Viscous: $\underline{R}$

This is the force on the fluid (force on walls is $-\underline{R}$)

Pressure:

$$\underline{F}_{contact} = \iint_S \left[\hat{n}\cdot\underline{\underline{\tilde{\Pi}}}\right]_{surface} dS$$

$$= \sum_i \left[\iint_S \left[\hat{n}\cdot(-p\underline{\underline{I}})\right] dS\right]_i$$

$$= \sum_i \left[(-p)\hat{n}\right]\left[\iint_A dS\right]_i$$

$$= \sum_i \left[(-p)\hat{n}A\right]$$

Macroscopic Momentum Balance

$\beta_{laminar} = 0.75$
$\beta_{turbulent} \sim 1$

$\underline{\underline{P}}$ = fluid momentum

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i}$$
$$= \sum_{i=1}^{\#streams}[-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$\underline{R}$ = net force on fluid due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal of the macroscopic control volume, CV

See inside front Cover of Morrison, 2013
And the exam formula sheet

Compare with the (more familiar) Navier-Stokes

Macroscopic Momentum Balance

$$\frac{d\underline{\underline{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}[-pA\hat{n}]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial\underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\boldsymbol{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma **Convective terms** ntum Balance

$$\frac{d\underline{\boldsymbol{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta \langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma ntum

Convective terms

Pressure forces

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma ntum

Convective terms

Pressure forces

Viscous forces

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Ma **Convective terms** ntum **Pressure forces** **Viscous forces** **Gravity force**

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

Rate of change of momentum with time

Macroscopic Momentum Balance

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A\cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla \mathrm{p} + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

Macroscopic Momentum Balance

Now we need to learn when and how to apply this

Macroscopic Momentum Balance

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

Microscopic Momentum Balance

$$\left(\rho\frac{\partial \underline{v}}{\partial t} + \rho\underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$$

We know how to apply this

www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Macroscopic Momentum Balance

$\underline{\mathbf{P}}$ = *fluid* momentum

$\beta_{laminar} = 0.75$
$\beta_{turbulent} \sim 1$

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

$$\begin{pmatrix} \frac{d\mathbf{P}_x}{dt} \\ \frac{d\mathbf{P}_y}{dt} \\ \frac{d\mathbf{P}_z}{dt} \end{pmatrix}_{xyz} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\begin{pmatrix}\hat{v}_x \\ \hat{v}_y \\ \hat{v}_z\end{pmatrix}_{xyz}\right]_{A_i}$$

$$= \sum_{i=1}^{\#streams}\left[-pA\begin{pmatrix}\hat{n}_x \\ \hat{n}_y \\ \hat{n}_z\end{pmatrix}_{xyz}\right]_{A_i} + \begin{pmatrix}R_x \\ R_y \\ R_z\end{pmatrix}_{xyz} + M_{CV}\begin{pmatrix}g_x \\ g_y \\ g_z\end{pmatrix}_{xyz}$$

$\underline{R}$ = net force *on fluid* due to walls
M_{CV} = mass of control volume
$\hat{n}$ = outwardly pointing unit normal of CV

See inside front cover of Morrison, 2013

Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Drag on the walls of a pipe

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

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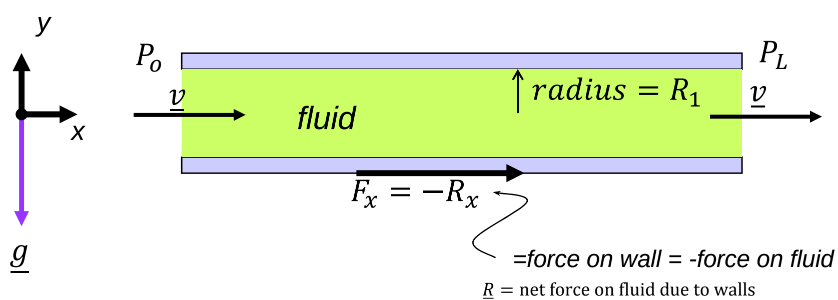
Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the drag on the walls of a pipe

Assume:
- steady state
- turbulent
- neglect gravity



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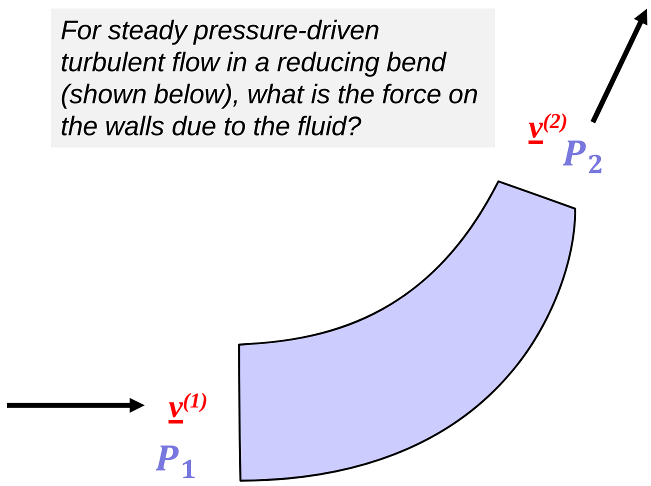
Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?



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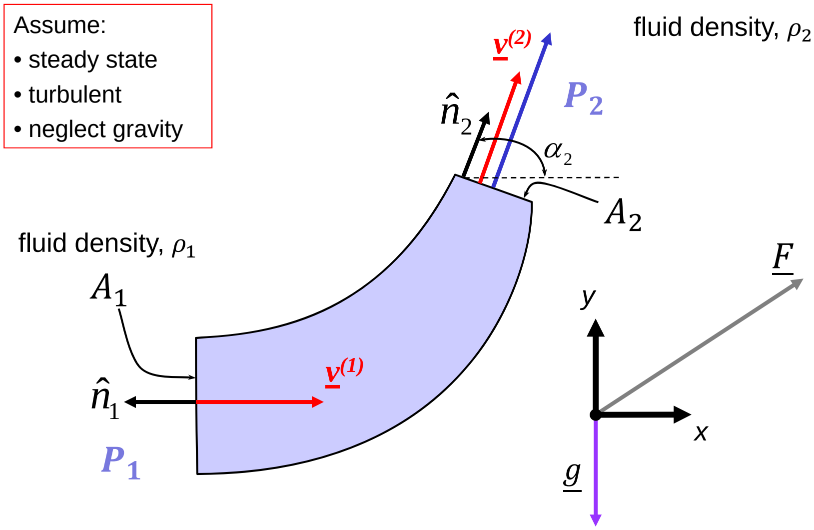
Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

Assume:
- steady state
- turbulent
- neglect gravity



Types of Momentum Transfer

Microscopic		**Macroscopic**
convection (momentum flows in)	→	convection (momentum flows in)
pressure forces	→	pressure forces
viscous forces (or viscous flux)	→	wall forces (due to viscosity)
body forces (gravity)	→	body forces (gravity)

After calculating the flow field with microscopic balances you can calculate wall forces

With macroscopic balances you can often **calculate wall forces directly**

Problem-Solving Procedure -

Steady State Macroscopic Momentum Problems

$$\frac{d\underline{\mathbf{P}}}{dt} + \sum_{i=1}^{\#streams}\left[\frac{\rho A cos\theta\langle v\rangle^2}{\beta}\hat{v}\right]_{A_i} = \sum_{i=1}^{\#streams}\left[-pA\hat{n}\right]_{A_i} + \underline{R} + M_{CV}\underline{g}$$

1. sketch system; choose CV on which you will perform balance
2. choose coordinate system
3. perform macroscopic mass balance

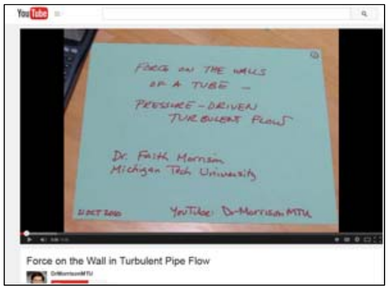
Consider angles carefully

4. perform macroscopic momentum balance (vector equation; forces are pressure, gravity, force on the wall; all forces ON the fluid in CV)
5. solve (usually for force on the wall)

Macroscopic Momentum Balance www.chem.mtu.edu/~fmorriso/cm310/MacroMomentumBalance2015.pdf

Solution to force on a reducing bend:

www.chem.mtu.edu/~fmorriso/cm310/reducing_bend.pdf



Dr. Morrison doing a Macro-Momentum Balance on YouTube:

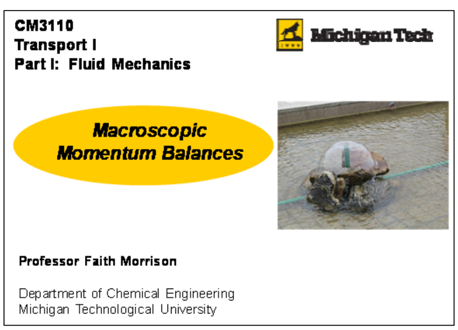
(DrMorrisonMTU)

www.youtube.com/watch?v=jXNkN7NMINM

(note that there is a sign error in the gravity term; sorry about that; gravity is negligible)

Many useful handouts:

www.chem.mtu.edu/~fmorriso/cm310/handouts.html



Done

(just need to practice; see HW4)