CM3110
Transport I
Part II: Heat Transfer

One-Dimensional Heat Transfer

(part 2: cylindrical shell)

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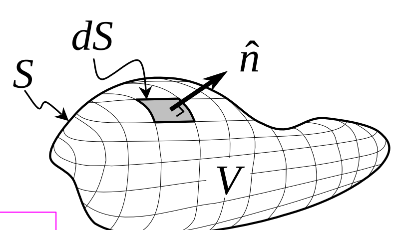
General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, V, enclosed by a surface, S.



Gibbs notation:

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T \right) = k \nabla^2 T + S_e$$

see handout for component notation

General Energy Transport Equation

(microscopic energy balance)

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S_e$$

- $\frac{\partial T}{\partial t}$: rate of change
- $\underline{v}\cdot\nabla T$: convection
- $k\nabla^2 T$: conduction (all directions)
- S_e: source (energy generated per unit volume per time)

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

The Equation of Energy for systems with constant k

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

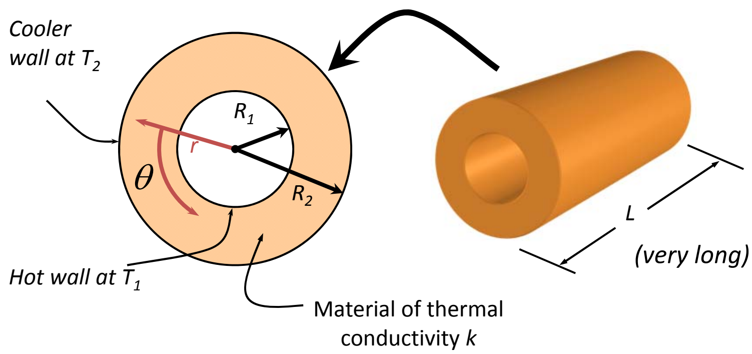
$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right)$$
$$= k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}\right) + S$$

Note: this handout is also on the web

Example 3: Heat flux in a cylindrical shell – Temp BC

Assumptions:
- long pipe
- steady state
- k = thermal conductivity of wall

*What is the steady state temperature profile in a cylindrical shell (pipe) if the **inner wall** is at T_1 and the **outer wall** is at T_2? ($T_1 > T_2$)*



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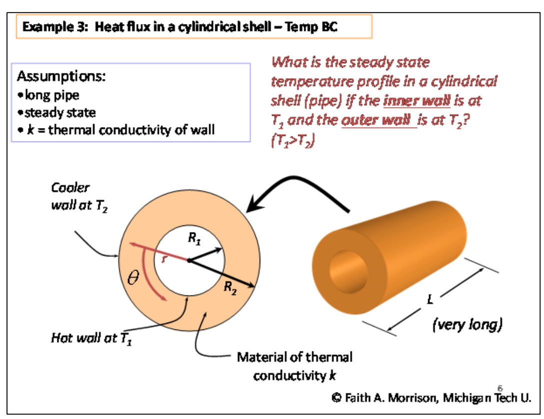
Example 3: Heat flux in a cylindrical shell – Temp BC

Let's try.



Example 3: Heat flux in a cylindrical shell – Temp BC

Solution:

$$\frac{q_r}{A} = \frac{c_1}{r}$$

Not constant

$$T = -\frac{c_1}{k}\ln r + c_2$$

Boundary conditions?

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Example 3: Heat flux in a cylindrical shell – Temp BC

Solution for Cylindrical Shell:

NOT constant

$$\frac{q_r}{A} = \frac{T_1 - T_2}{\ln\frac{R_2}{R_1}}\left(\frac{k}{r}\right)$$

The heat flux $\frac{q_r}{A}$ **DOES** depend on, *k;* also $\frac{q_r}{A}$ decreases as *1/r*

NOT linear

$$\frac{T_2 - T}{T_2 - T_1} = \frac{\ln\frac{R_2}{r}}{\ln\frac{R_2}{R_1}}$$

Note that *T(r)* does not depend on the thermal conductivity, *k* (steady state)

Pipe with temperature BCs

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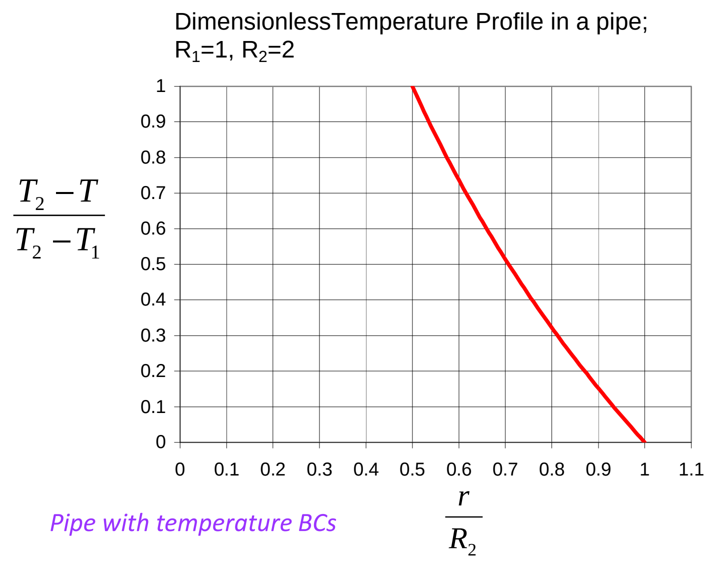
Example 3: Heat flux in a cylindrical shell – Temp BC



Pipe with temperature BCs

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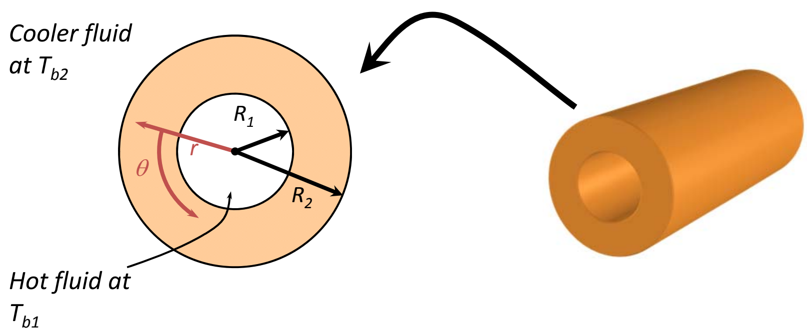
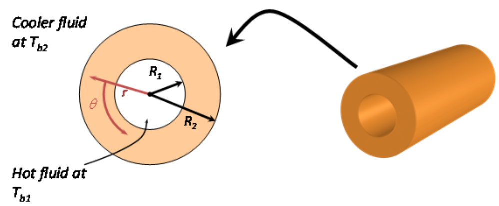
Example 4: Heat flux in a cylindrical shell – Newton's law of cooling

Assumptions:
- long pipe
- steady state
- k = thermal conductivity of wall
- h_1, h_2 = heat transfer coefficients

What is the steady state temperature profile in a cylindrical shell (pipe) if the ***fluid on the inside*** *is at* T_{b1} *and the* ***fluid on the outside*** *is at* T_{b2}*?* ($T_{b1} > T_{b2}$)



Example 4: Heat flux in a cylindrical shell



You try.

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Example 4: Heat flux in a cylindrical shell

Solution:

$$\frac{q_r}{A} = \frac{c_1}{r}$$

Not constant

$$T = -\frac{c_1}{k} \ln r + c_2$$

Boundary conditions?

Example 4: Heat flux in a cylindrical shell

$$\frac{c_1}{R_1} = h_1\left(T_{b1} - T_{w1}\right)$$

$$\frac{c_1}{R_2} = h_2\left(T_{w2} - T_{b2}\right)$$

$$T_{w1} = -\frac{c_1}{k}\ln R_1 + c_2$$

$$T_{w2} = -\frac{c_1}{k}\ln R_2 + c_2$$

4 equations

4 unknowns; c_1, T_{w1}, c_2, T_{w2}

SOLVE

Example 4: Heat flux in a cylindrical shell

Newton's law of cooling boundary conditions

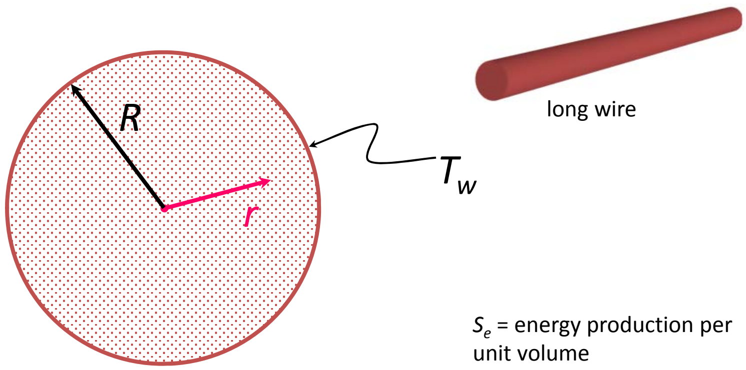
Solution: Radial Heat Flux in an Annulus

$$T - T_{b2} = \frac{(T_{b1} - T_{b2})\left(\ln\left(\frac{R_2}{r}\right) + \frac{k}{h_2 R_2}\right)}{\frac{k}{h_2 R_2} + \ln\left(\frac{R_2}{R_1}\right) + \frac{k}{h_1 R_1}}$$

$$\frac{q_r}{A} = \frac{(T_{b1} - T_{b2})}{\frac{1}{h_2 R_2} + \frac{1}{k}\ln\left(\frac{R_2}{R_1}\right) + \frac{1}{h_1 R_1}}\left(\frac{1}{r}\right)$$

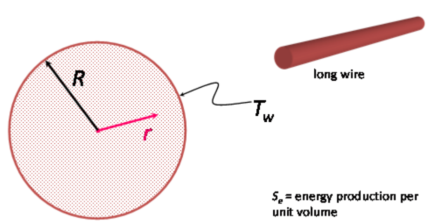
Example 5: Heat Conduction with Generation

What is the steady state temperature profile in a wire if heat is generated uniformly throughout the wire at a rate of S_e W/m³ and the outer radius is held at T_w?



S_e = energy production per unit volume

Example 5: Heat conduction with generation



You try.

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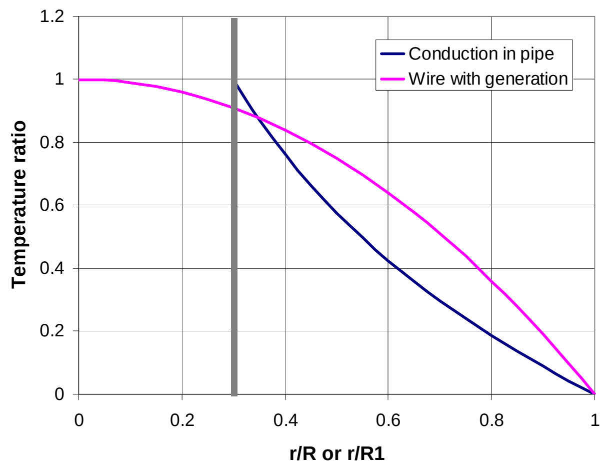
Compare solutions

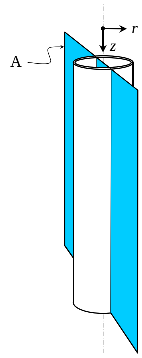


$$\frac{T-T_1}{T_1-T_2}\frac{1}{\ln\frac{R_1}{R_2}}=\ln\frac{r}{R_1}$$

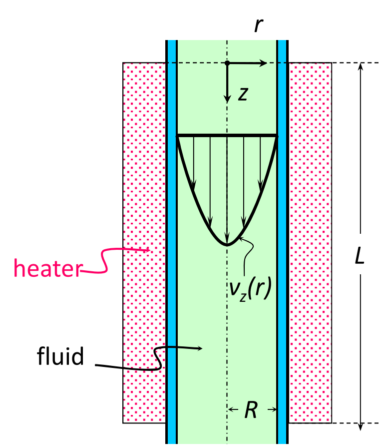
$$\frac{T-T_w}{S_e R^2/4k}=1-\left(\frac{r}{R}\right)^2$$

Example 6: **Wall heating of laminar flow**. *What is the steady state temperature profile in a flowing fluid in a tube if the walls are heated (constant flux, q_1/A) and if the fluid is a Newtonian fluid in laminar flow?*

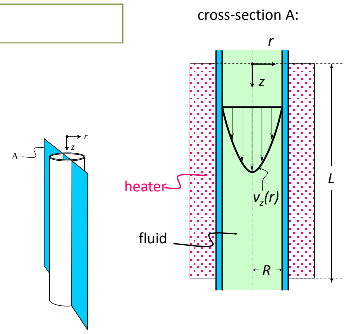
assume:
constant viscosity



cross-section A:



Example 5: Wall heating of laminar flow



You try.

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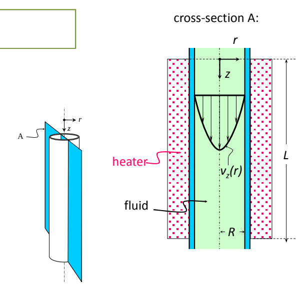
Example 5: Wall heating of laminar flow



We need to solve this partial differential equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial T}{\partial r}r\right)+\frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)-\frac{\rho\hat{C}_p}{k}v_z(r)\frac{\partial T}{\partial t}=0$$

with the appropriate boundary conditions. To see the solution see:

- R. Siegel, E. M. Sparrow, T. M. Hallman, *Appl. Science Research* A7, 386-392 (1958)
- R. B. Bird, W. Stewart, and E. Lightfoot, *Transport Phenomena*, Wiley, 1960, p295.

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SUMMARY

Steady State Heat Transfer

Example 1: Heat flux in a rectangular solid – Temperature BC

Example 2: Heat flux in a rectangular solid – Newton's law of cooling

Example 3: Heat flux in a cylindrical shell – Temperature BC

Example 4: Heat flux in a cylindrical shell – Newton's law of cooling

Example 5: Heat conduction with generation

Example 6: Wall heating of laminar flow

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Example 6: Wall heating of laminar flow

Conclusion: When we can simplify geometry, assume steady state, assume symmetry, the solutions are easily obtained

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Unsteady State Heat Transfer

???

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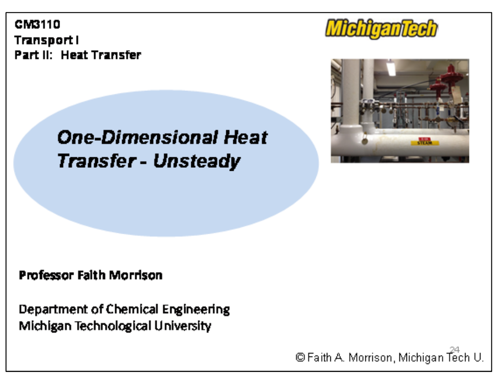

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