**One-Dimensional Heat Transfer - Unsteady**

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**SUMMARY**

**Steady State Heat Transfer**

Example 1: Heat flux in a rectangular solid – Temperature BC
Example 2: Heat flux in a rectangular solid – Newton’s law of cooling
Example 3: Heat flux in a cylindrical shell – Temperature BC
Example 4: Heat flux in a cylindrical shell – Newton’s law of cooling
Example 5: Heat conduction with generation
Example 6: Wall heating of laminar flow

**Conclusion:** When we can simplify geometry, assume steady state, assume symmetry, the solutions are easily obtained
SUMMARY

Steady State Heat Transfer

Example 1: Heat flux in a rectangular solid – Temperature BC
Example 2: Heat flux in a rectangular solid – Newton’s law of cooling
Example 3: Heat flux in a cylindrical shell – Temperature BC
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Example 6: Wall heating of laminar flow

Conclusion: When we can simplify geometry, assume steady state, assume symmetry, the solutions are easily obtained.

What about non-steady-state problems (quite common in heat transfer)?

1D Heat Transfer: Unsteady State

Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature $T_0$. At time $t = 0$, the left face of the slab is exposed to an environment at temperature $T_1$. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, $k$, density, $\rho$, and heat capacity, $C_p$. 

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This is code for:

"Newton’s law of cooling boundary conditions:"

$$|\text{flux}| = h(T_{\text{bulk}} - T_{\text{wall}})$$
1D Heat Transfer: Unsteady State

Initial Condition:

\[ t < 0 \]
\[ T = T_0 \]

\[ t \geq 0 \]
\[ T = T_1 \]

\[ t > 0 \]
\[ T = T(x, t) \]

General Energy Transport Equation

(microscopic energy balance)

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, \( V \), enclosed by a surface, \( S \).

Gibbs notation:

\[ \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S \]

see handout for component notation
General Energy Transport Equation
(microscopic energy balance)

\[ \rho \mathcal{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = k \nabla^2 T + S \]

- **Convection**: \( \rho \mathcal{C}_p \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) \)
- **Conduction (all directions)**: \( k \nabla^2 T \)
- **Source** (energy generated per unit volume per time): \( S \)
- **Rate of change**
- **Velocity must satisfy equation of motion, equation of continuity**
- **see handout for component notation**

Equation of energy for Newtonian fluids of constant density, \( \rho \), and thermal conductivity, \( k \), with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).


Gibbs notation (vector notation)

\[ \left( \frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T \right) = \frac{k}{\mathcal{C}_p} \nabla^2 T + \frac{S}{\mathcal{C}_p} \]

Cartesian (xyz) coordinates:

\[ \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\mathcal{C}_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\mathcal{C}_p} \]

Cylindrical (r\( \theta \)z) coordinates:

\[ \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\mathcal{C}_p} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\mathcal{C}_p} \]

Spherical (r\( \theta \)\( \phi \)) coordinates:

\[ \frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} = \frac{k}{\mathcal{C}_p} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right) + \frac{S}{\mathcal{C}_p} \]
Example 1: Unsteady Heat Conduction in a Semi-infinite solid

A very long, very wide, very tall slab is initially at a temperature $T_0$. At time $t = 0$, the left face of the slab is exposed to an environment at temperature $T_1$. What is the time-dependent temperature profile in the slab? The slab is a homogeneous material of thermal conductivity, $k$, density, $\rho$, and heat capacity, $C_p$.

Newton's law of cooling BC's:

$$|q_x| = hA[T_{bulk} - T_{surface}]$$

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho C_p}$$

$$\alpha \equiv \frac{k}{\rho C_p} = \text{thermal diffusivity}$$

what are the boundary conditions? initial conditions?
Example 7: Unsteady Heat Conduction in a Semi-Infinite Solid

You try.

Unsteady State Heat Conduction in a Semi-Infinite Slab

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right)
\]

Initial condition: 
\[ t = 0, \quad T = T_0 \quad \forall \ x \]

Boundary conditions:
\[ x = 0, \quad q_x = hA(T - T_1) \quad \forall \ t > 0 \]
\[ x = \infty, \quad T = T_o \quad \forall \ t \]
1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab

\[ \frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right) \]

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\[ x = 0, \quad q_x = hA(T - T_1) \quad \forall \ t > 0 \]
\[ x = \infty, \quad T = T_o \quad \forall \ t \]

"for all t"

The solution is obtained by combination of variables.

\[ \beta = \frac{h\sqrt{\alpha t}}{k} \]
\[ \zeta = \frac{x}{2\sqrt{\alpha t}} \]

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### 1D Heat Transfer: Unsteady State

#### Unsteady State Heat Conduction in a Semi-Infinite Slab

**Solution:**

\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \, \zeta - e^{\beta (2\zeta + \beta)} \text{erfc}(\zeta + \beta)
\]

where

\[
\beta \equiv \frac{h \sqrt{\alpha}}{k} \quad \text{and} \quad \zeta \equiv \frac{x}{2 \sqrt{\alpha \, t}}
\]

- **erfc(y)** represents the complementary error function of y.
- **erf(y)** represents the error function of y.

\[
\text{erfc}(y) \equiv 1 - \text{erf}(y)
\]

\[
\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-y^2} \, dy'
\]

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1D Heat Transfer: Unsteady State Heat Conduction in a Semi-infinite Slab

This:
\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \, \zeta - e^{\beta(2\zeta + \beta)} \text{erfc}(\zeta + \beta)
\]

Versus this: \[ \zeta \equiv \frac{x}{2\sqrt{\alpha \, t}} \]

To make this solution easier to use, we can plot it.

At various values of this:
\[
\beta \equiv \frac{h\sqrt{\alpha \, t}}{k}
\]

Geankoplis 4th ed., Figure 5.3-3, page 364

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1D Heat Transfer: Unsteady State Heat Conduction in a Semi-Infinite Slab

With modern tools, we can plot the solution directly (evaluated on Mathcad)

**Unsteady State Heat Conduction in a Semi-Infinite Slab**

\[ h = k = \alpha = 1 \]

\[ t = 0.1, 0.2, 0.5, 1, 5, 10, 50, 100 \]

\[
\frac{T - T_o}{T_1 - T_o}
\]

increasing time, \( t \)

---

**How could we use this solution?**

**Example:** Will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 ft under ground? Use the following physical properties:

\[
h = 2.0 \ \frac{BTU}{h \ ft^2 \ ^\circ F}
\]

\[
\alpha_{soil} = 0.018 \ \frac{ft^2}{h}
\]

\[
k_{soil} = 0.5 \ \frac{BTU}{h \ ft \ ^\circ F}
\]
1D Heat Transfer: Unsteady State Conduction in a Semi-infinite Slab

\[
\frac{T - T_0}{T_1 - T_0} = \text{erfc} \zeta - e^{\beta(2\zeta + \beta)} \text{erfc} (\zeta + \beta)
\]

\[
\zeta = \frac{x}{2\sqrt{\alpha t}} \\
\beta = \frac{h\sqrt{\alpha t}}{k}
\]

Both \(\zeta\) and \(\beta\) depend on time

- \(T_0 = ?\)
- \(T_1 = ?\)
- \(T = ?\)
- \(\frac{T - T_0}{T_1 - T_0} = ?\)

(Interative solution)

Geankoplis 4th ed., Figure 5.3-3, page 364

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1D Heat Transfer: Unsteady State Heat Conduction in a Semi-infinite Slab

\[ \frac{T - T_0}{T_1 - T_0} = \operatorname{erfc} \zeta - e^{\beta (2\zeta + \beta)} \operatorname{erfc}(\zeta + \beta) \]

Answer:
\[ t = 480 \text{ hours} \approx 20 \text{ days} \]
1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Unsteady State Heat Transfer
Use same microscopic energy balance eqn as before.

\[ \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v \cdot \nabla T \right) = k \nabla^2 T + S \]

- **Convection**: \( \rho \hat{C}_p \left( \frac{\partial T}{\partial t} + v \cdot \nabla T \right) \)
- **Source**: \( S \)
- **Rate of Change**: \( \frac{\partial T}{\partial t} \)
- **Conduction (all directions)**: \( k \nabla^2 T \)
- See handout for component notation

1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Microscopic Energy Equation in Cartesian Coordinates

\[ \frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p} \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \frac{S}{\rho \hat{C}_p} \]

\[ \alpha \equiv \frac{k}{\rho \hat{C}_p} = \text{thermal diffusivity} \]

**what are the boundary conditions? initial conditions?**
1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Example 8: Unsteady Heat Conduction in a Finite-sized solid

You try.

Unsteady State Heat Conduction in a Finite Slab

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho \bar{C}_p} \left( \frac{\partial^2 T}{\partial x^2} \right) = \alpha \left( \frac{\partial^2 T}{\partial x^2} \right)
\]

Initial condition:
\[ t = 0, \ T = T_0 \ \forall \ x \]

Boundary conditions:
\[ \begin{align*}
  x &= 0, & T &= T_1 \\
  x &= 2H, & T &= T_1
\end{align*} \quad \forall \ t > 0 \]
1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Q: How can two completely different situations give the same governing equation?
A: The boundary conditions make all the difference

\[
\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}
\]


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Unsteady State Heat Conduction in a Finite Slab

\[
\frac{\partial T}{\partial t} = \frac{k}{\rho C_p} \frac{\partial^2 T}{\partial x^2}
\]

The solution is obtained by separation of variables.

Initial condition: \( t = 0, T = T_0 \ \forall \ x \)

Boundary conditions:
\[
\begin{align*}
  x = 0, & \quad T = T_1 \\
  x = 2H, & \quad T = T_f
\end{align*}
\]
\( \forall t > 0 \)

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1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Unsteady State Heat Conduction in a Finite Slab: solution by separation of variables

\[
Y = \left( \frac{T_1 - T}{T_1 - T_0} \right) \quad \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right)
\]

Guess: \( Y = X(x)\Theta(t) \)

Initial condition:
\[
t = 0, \quad T = T_o \quad \forall x \Rightarrow Y = 1
\]

Boundary conditions:
\[
x = 0, \quad T = T_1 \Rightarrow Y = 0 \quad \forall t > 0
\]
\[
x = 2H, \quad T = T_1 \Rightarrow Y = 0 \quad \forall t > 0
\]

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1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

Unsteady State Heat Conduction in a Finite Slab: soln by separation of variables

\[
Y = X(x)\Theta(t) \quad \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right)
\]

\[
\frac{\partial Y}{\partial t} = \frac{\partial}{\partial t} \left( X(x)\Theta(t) \right) = X(x) \frac{d\Theta(t)}{dt}
\]

\[
\frac{\partial Y}{\partial x} = \frac{\partial}{\partial x} \left( X(x)\Theta(t) \right) = \frac{dX(x)}{dx} \Theta(t)
\]

\[
\frac{\partial^2 Y}{\partial x^2} = \frac{d^2 X(x)}{dx^2} \Theta(t)
\]

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1D Heat Transfer: Unsteady Heat Conduction in a Finite Solid

\[ \frac{\partial Y}{\partial t} = \alpha \left( \frac{\partial^2 Y}{\partial x^2} \right) \]

Substituting:

\[ X(x) \frac{d\Theta(t)}{dt} = \alpha \frac{d^2 X(x)}{dx^2} \Theta(t) \]

\[ \frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} \]

\[ \Rightarrow \text{constant} = \lambda \]

The function of two variables is separable into two functions of one variable.

Separates into two ordinary differential equations:

\[ \frac{1}{\Theta(t)} \frac{d\Theta(t)}{dt} = \lambda \]

\[ \alpha \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} = \lambda \]

Solve.

Apply BCs.

Apply ICs.

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We use solutions like this in CM3215 to back calculate $h$ from non-steady data:
CM3110 Heat Transfer Lecture 3

1D Heat Transfer: Unsteady State

Microscopic Energy Balance – is the correct physics for many problems!

Tricky step:
solving for $T$ field; this can be mathematically difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- may not be separable from mass and momentum balances

Strategy:
- Look up solution in literature
- solve using numerical methods (e.g. Comsol)
- Or
- Develop correlations on complex systems by using Dimensional Analysis

Fluid Mechanics: What did we do?

- Turbulent tube flow
- Noncircular conduits
- Drag on obstacles

1. Find a simple problem that allows us to identify the physics
2. Nondimensionalize
3. Explore that problem
4. Take data and correlate
5. Solve real problems

Solve. Real. Problems.
Powerful.

Works on heat transfer too.
More Complex Heat Transfer – Dimensional Analysis

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