More Complicated Flows
(Dimensional Analysis,
rough pipes, hydraulic
diameter, porous media)

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What's the plan?

Outline

1. Solving complicated flows with Navier-Stokes
2. Dimensional Analysis for
   a. Design
   b. Scale up
   c. Data correlations
3. How to Do Dimensional Analysis
4. What Dimensional Analysis looks like when it works
EXAMPLE: Pressure-driven flow of a Newtonian fluid in a rectangular duct: Poiseuille flow

- steady state
- well developed
- long tube
- $P(0) = P_0$, $P(L) = P_L$

Velocity varies in two directions

Let’s try.
The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

CM3110 Fall 2011 Faith A. Morrison

Continuity Equation, Cartesian coordinates
\[ \frac{\partial \rho}{\partial t} + \left( v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} \right) + \rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right) = 0 \]

Continuity Equation, cylindrical coordinates
\[ \frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0 \]

Continuity Equation, spherical coordinates
\[ \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 \rho v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_\phi)}{\partial \phi} = 0 \]

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates
\[
\begin{align*}
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z 
\end{align*}
\]

Navier-Stokes:

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation) 3 components in Cartesian coordinates
\[
\begin{align*}
\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial P}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \\
\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial P}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \\
\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial P}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z 
\end{align*}
\]

Equation of Motion for incompressible, Newtonian fluid (Navier-Stokes equation), 3 components in cylindrical coordinates
\[
\begin{align*}
\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_\theta \frac{\partial v_r}{\partial \theta} - \frac{v_r^2}{r} + \frac{v_\theta^2}{r} \right) &= \frac{\partial P}{\partial r} + \mu \left( \frac{\partial}{\partial r} \left( \frac{\partial v_r}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial v_r}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial P}{\partial r} + \frac{2}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\partial v_r}{\partial \theta} \right) + \frac{\partial^2 v_r}{\partial \theta^2} + \rho g_r \\
\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + v_\theta \frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta v_r}{r} + \frac{v_\theta}{r} \right) &= -\frac{\partial P}{r \partial \theta} + \mu \left( \frac{\partial}{\partial r} \left( \frac{\partial v_\theta}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial v_\theta}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial P}{\partial r} + \frac{2}{r \sin \theta} \frac{1}{\partial \theta} \left( \frac{\partial v_\theta}{\partial \theta} \right) + \frac{\partial^2 v_\theta}{\partial \theta^2} + \rho g_\theta \\
\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_\theta \frac{\partial v_z}{\partial \theta} + \frac{v_z}{r} + v_z \right) &= \frac{\partial P}{\partial z} + \mu \left( \frac{\partial}{\partial r} \left( \frac{\partial v_z}{\partial r} \right) + \frac{\partial}{\partial \theta} \left( \frac{\partial v_z}{\partial \theta} \right) \right) + \frac{1}{r^2} \frac{\partial P}{\partial r} + \frac{\partial^2 v_z}{\partial \theta^2} + \rho g_z 
\end{align*}
\]
Boundary Conditions:?

What about more complicated Newtonian problems?

Example: Pressure-driven flow of a Newtonian fluid in a rectangular duct: Poiseuille flow

- steady state
- well developed
- long tube
- $P(0) = P_0$, $P(L) = P_L$

Cross-section A:

Velocity varies in two directions

Pressure-driven flow in a rectangular duct

\[ \langle v \rangle_{\text{slit}} = \frac{48}{\pi^3} \sum_{n=1,3,5,...}^\infty (-1)^{n-1} \left( \frac{n}{\pi W} \xi_3 / 2H \right)^2 \frac{\cosh(n \pi W / 2H)}{\cosh(n \pi / 2)} \cos \left( \frac{mn \xi_2}{2} \right) \]

\[ \xi_2 \equiv \frac{x_2}{H} \quad \xi_3 \equiv \frac{x_3}{W} \quad \langle v \rangle_{\text{slit}} = \frac{H^2 \Delta p}{3\mu L} \]
What about more complicated Newtonian problems?

What does this show?

**Tricky step:** Solving for \( \mathbf{v} \) and \( \mathbf{\tau} \) can be difficult

- partial differential equation in up to three variables
- boundaries may be complex
- multiple materials, multiple phases present
- non-Newtonian fluids

**Solution strategies:**

- advanced analytical techniques for solving partial differential equations (PDEs)
- numerical techniques (Ansys, Comsol Multiphysics)
  - www.ansys.com
  - www.comsol.com/

What about more complicated Newtonian problems?

**Ansys Fluent**: Solutions: Example
X19 Fluid Handling and Flow
Distribution

Streamlines depict the flow of regenerated catalyst through a slide valve, revealing the source of erosion problems.

"Transport and storage of gases, liquids, or slurries represents a large capital and operating expense in process plants. Fluent's CFD software helps you to design for flow uniformity, balance flows in manifolds, minimize pressure drop, design storage tanks, and accurately size blowers, fans, and pumps. High-speed nozzles and spray systems can be analyzed in order to optimize performance."

www.ansys.com
What about more complicated Newtonian problems?

So far:

• We’ve learned to set-up and sometimes solve flow problems (conservation of mass, momentum)

Question:

• Do we always need to solve the modeling problems that real systems present?

• Can we solve them?
Most industrial flows are not simple:

- piping
- pumps
- mixers
- flow in an engine
- fluidized beds
- flow in a packed bed (catalytic reactor)
- two-phase flows (extractors)
- jets (jet engines, ink-jet printing)
- coating flows
- evaporators
- heat exchangers

Most of these flows are impossible to solve in detail

Exception: plastics, high viscosity flows

Questions:

- Do we always need to solve the modeling problems that real systems present?  
  - No, not always.
- Can we solve them?  
  - No, not always.

What do we do instead?  
- Experiments, scale-models, and data-correlations

What experiments do we do?  
- Random experiments and hope for the best
What about more complicated Newtonian problems?

Questions:

- Do we always need to solve the modeling problems that real systems present? 
  - No, not always.
- Can we solve them? 
  - No, not always.

What do we do instead? 
- Experiments, scale-models, and data-correlations

What experiments do we do? 
- Random experiments and hope for the best
- Small-scale pilot experiments that can scale to the real system

Better choice, but how?

Designing a Device

• Dimensional similarity 
  = similar proportions

• Dynamic similarity 
  = similar behavior

How systems behave depends on the laws of physics.
GOALS of Dimensional Analysis:

To use our knowledge of physical laws (mass, momentum, energy conservation) to guide our studies, modeling, and experimentation on complex (real engineering) flows (i.e. to save us trial-and-error work)

Specifically:

• To be able to design devices in which the flow is expected to be complex
• To scale-up (relate) any experiments to similar flows that are not (yet) available for experimentation
• To guide the use and production of data correlations (i.e. the plotting and reporting of experimental data)

San Francisco Bay Model, Sausalito, CA.

• distorted scale: the dimension in the vertical direction is 1/10th the scale in the horizontal direction.
• US Army Corps of Engineers
• used to evaluate proposed changes to the bay such as dams and other types of development.
Dimensional Analysis

**Principle:** even in complex systems, the same equations still apply:

- **continuity equation** (mass conservation)
- **equation of motion** (momentum conservation)

**Strategy:** render the governing equations dimensionless to identify the important parameters that apply in every situation.

→ rely on experiments and data correlations

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What about more complicated Newtonian problems?

**Continuity Equation**

Microscopic mass balance written on an arbitrarily shaped control volume, $V$, enclosed by a surface, $S$:

\[
\frac{\partial \rho}{\partial t} + v_x \frac{\partial \rho}{\partial x} + v_y \frac{\partial \rho}{\partial y} + v_z \frac{\partial \rho}{\partial z} = -\rho \left( \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right)
\]

Gibbs notation:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho = -\rho \cdot \nabla ( \cdot \mathbf{v} )
\]

*Microscopic mass balance is a scalar equation.*
**Equation of Motion**

Microscopic **momentum** balance written on an arbitrarily shaped control volume, \( V \), enclosed by a surface, \( S \)

Gibbs notation:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

**general fluid**

Gibbs notation:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

**Newtonian fluid**

**Dimensional Analysis**

**Principle**: even in complex systems, the same equations still apply:

Mass is conserved:

**Continuity Equation**

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho (\nabla \cdot \mathbf{v})
\]

Momentum is conserved:

**Navier-Stokes Equation**

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

For a complex problem,

**Which terms dominate?**

**How can we simplify?**
### Dimensional Analysis

Principle: even in complex systems, the same equations still apply:

**Mass is conserved:**

Continuity Equation

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{v}
\]

Depends on how big \( \mathbf{v} \) is

**Momentum is conserved:**

Navier-Stokes Equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla P + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Depends on how big \( \mu \) is

For a complex problem,

Which terms dominate?  
How can we simplify?

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**Dimensional Analysis**

**Principle:** even in complex systems, the same equations still apply:

**Mass is conserved:**
Continuity Equation
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = -\rho \nabla \cdot \mathbf{v}
\]

Depends on how big \( \rho \) is

**Momentum is conserved:**
Navier-Stokes Equation
\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Depends on how big \( \rho \) is
Depends on how big \( \mu \) is
Depends on how fast \( \mathbf{v} \) is changing
Depends on how big \( \nabla^2 \mathbf{v} \) is

For a complex problem,

Which terms dominate? How can we simplify?

---

**Variable, constants:**
\( \mathbf{v}, t, p, x, y, z, \nabla, \nabla^2 \)
\( \mu, \rho, g \)

- Choose “typical” values (scale factors)
- Use them to scale the equations
- Deduce which terms dominate

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Note that once the variables are non-dimensionalized, the scale factors and constants will form **dimensionless groups**

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Dimensional Analysis

**Principle:** even in complex systems, the same equations still apply:

**Procedure:**
1. select appropriate differential equations and boundary conditions
2. select characteristic quantities with which to scale the variables, e.g., \( v, x, P \)
   - characteristic quantities must be constant
   - must be representative of the system
3. scale all variables in the governing equations; yields dimensionless equation as a function of dimensionless groups.
   - The values of the dimensionless groups determine the properties of the differential equations.
4. design scaled-down experiments to develop data correlations for the system of interest
5. use data correlations to design and evaluate systems

**OR**

4. perform experiments on an existing system and correlate results using dimensionless groups

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**EXAMPLE I:** Pressure-driven flow of an incompressible Newtonian fluid in a tube:
- NOT Laminar (not unidirectional)
- steady state
- well developed
- long tube
- incompressible
locally the flow is 3D:

\[
\mathbf{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}
\]

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z-component of the Navier-Stokes Equation:

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right)$$

$$= -\frac{\partial P}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

**Need a “Plan B” (Dimensional Analysis)**

Choose:
- \(D\) = characteristic length
- \(V\) = characteristic velocity
- \(D/V\) = characteristic time
- \(\rho V^2\) = characteristic pressure

- Choose “typical” values (scale factors)
- Use them to scale the equations
- Deduce which terms dominate

**Dimensional Analysis**

non-dimensional variables:

- time:
  $$t^* \equiv \frac{tV}{D}$$

- position:
  $$r^* \equiv \frac{r}{D}$$
  $$z^* \equiv \frac{z}{D}$$

- velocity:
  $$v_z^* \equiv \frac{v_z}{V}$$
  $$v_r^* \equiv \frac{v_r}{V}$$
  $$v_\theta^* \equiv \frac{v_\theta}{V}$$

- driving force:
  $$P^* \equiv \frac{P}{\rho V^2}$$
  $$g_z^* \equiv \frac{g_z}{g}$$

- Choose “typical” values (scale factors)
- Use them to scale the equations
- Deduce which terms dominate

“slash & burn”-like
Dimensional Analysis

Choose "typical" values (scale factors)
Use them to scale the equations
Deduce which terms dominate

z-component of the **nondimensional**
Navier-Stokes Equation:

\[
\frac{Dv_z^*}{Dt} = \frac{-\partial P^*}{\partial z^*} + \frac{\mu}{\rho VD} (\nabla^2 v_z^*)^* + \frac{gD}{V^2} g^*
\]

\[
(\nabla^2 v_z^*)^* = \frac{1}{r} \frac{\partial}{\partial r} \left( r^* \frac{\partial v_z^*}{\partial r^*} \right) + \frac{1}{r^2} \frac{\partial^2 v_z^*}{\partial \theta^2} + \frac{\partial^2 v_z^*}{\partial z^2}
\]

\[
\frac{Dv_z^*}{Dt} = \left( \frac{\partial v_z^*}{\partial t^*} + v_r^* \frac{\partial v_z^*}{\partial r^*} + v_\theta^* \frac{\partial v_z^*}{\partial \theta^*} + v_z^* \frac{\partial v_z^*}{\partial z^*} \right)
\]

Two dimensionless groups appear:

\[
Re = \frac{\rho VD}{\mu}
\]

Reynolds number = ratio of inertial to viscous forces

\[
Fr = \frac{V^2}{gD}
\]

Froude number = ratio of inertial to gravity forces ("frood")

If for two systems \(Re\) and \(Fr\) are the same, the two systems are governed by the same momentum, same mass balance.

If the dimensionless boundary conditions are also the same, the two systems are mathematically identical

= Dynamic Similarity
Scale-up

• Dimensional similarity
  - similar proportions

• Dynamic similarity
  - similar behavior

Dimensionless Navier-Stokes:

\[
\frac{D v^*_z}{Dt} = - \frac{\partial p^*}{\partial z^*} + \frac{1}{Re} (\nabla^* v^*_z) + \frac{1}{Fr} g^*
\]

We match dimensionless numbers Re and Fr to achieve this.

Dimensional Analysis

Dimensionless Navier-Stokes:

\[
\frac{D v^*_z}{Dt} = - \frac{\partial p^*}{\partial z^*} + \frac{1}{Re} (\nabla^* v^*_z) + \frac{1}{Fr} g^*
\]

We can also use non-dimensionalization to help us to correlate experimental results.

What does it mean to correlate?
If we need to know about the operation of an apparatus and we have the apparatus, then we can learn whatever we need to know about the apparatus by conducting experiments.

What if we don’t have the apparatus? (we’re designing one or comparing the possible performance of one with another)

**Answer:** we can build a scale model and scale up the experimental results on that;

**Or**

**Answer:** we can use others’ results on scale models and scale up their experimental results to our needs (no point in re-inventing the wheel)

Either way, this is called creating a data correlation.

**EXAMPLE:**
Pressure-driven flow of an incompressible Newtonian fluid in a tube:

NOT Laminar (not unidirectional)

- steady state
- well developed
- long tube
- incompressible

Locally the flow is 3D:

\[
\mathbf{v} = \begin{pmatrix}
    v_r \\
    v_\theta \\
    v_z
\end{pmatrix}
\]

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Laminar flow in a pipe

For laminar flow of a Newtonian fluid we can calculate the relationship between pressure drop and flow rate. We can also calculate the frictional force on the wall, which is related to these.

\[ F_z = \int_0^L \int_0^{2\pi} \tau_{rz} \bigg|_{r=R} R \, d\theta \, dz = \pi R^2 \Delta P \]

\[ \tau_{rz} = \frac{r \Delta P}{2L} \]

To build a correlation, we start with a simple, model flow

\[ Q = \int_0^L \int_0^{2\pi} v_z \, rdr \, d\theta = \pi R^2 \langle v_z \rangle = \frac{\pi \Delta P R^4}{8 \mu L} \]

Hagen-Poiseuille equation

\[ F_z = \Delta P \pi R^2 = \left( \frac{8 \mu L}{R^2} \right) Q \]

(Tells us what pump we need for a given flow rate, for example)

What about turbulent flow?
Creating a Data Correlation

**Turbulent flow in a pipe**

The frictional force on the wall is again related to this expression:

\[ F_z = \int_{0}^{L} \int_{0}^{2\pi} \tau_{rz} \left|_{r=R} \right. R \, d\theta \, dz \]

z-component of force on the wall

but without the solution for \( v_z(r) \), where will we get this?

If we cannot solve for \( F_z \), how will we get \( \Delta p \) as a function of \( Q \) when the flow is turbulent?

---

Creating a Data Correlation

We do not know \( \tau_{rr} \), but we do know that it comes from the solution to the Navier-Stokes equation and the continuity equation.

(we just cannot solve it because turbulent flow is way too complicated.)

**What then?**

- We could do experiments.
  
  (but what if we do not have the system?
  what if it is a design problem?)

- We (or someone else) could do experiments on a similar system and then we could scale the results.

  ahhh ... DIMENSIONAL ANALYSIS
Creating a Data Correlation

Dimensionless Force on the Wall

\[ F_z = \int_0^L \int_0^{2\pi} r \left. R \, d\theta \, dz \right|_{r=R} \]

\[ = \int_0^L \int_0^{2\pi} \mu \left( \frac{\partial v_z}{\partial r} \right)_{r=R} \, R \, d\theta \, dz \]

How shall we nondimensionalize \( F_z \)?

Nondimensionalize:

- **position**: \( r^* \equiv \frac{r}{D} \)
- **velocity**: \( v_z^* \equiv \frac{v_z}{V} \)

Creating a Data Correlation

Nondimensional Wall Friction

\[ f = \frac{F_z}{\left( \text{area} \right) \left( \text{kinetic energy} \right)} \]

\[ = \frac{F_z}{2\pi RL \left( \frac{1}{2} \rho v^2 \right)} \]

Fanning friction factor

dimensionless wall friction in a tube
Creating a Data Correlation

Non-dimensional force on the wall:

\[ f = \frac{1}{\pi} \frac{1}{L} \frac{1}{\text{Re}} \int_0^L \left( \frac{\partial v_z^*}{\partial r^*} \right) r^* \frac{1}{2} d\theta dz^* \]

\[ \Rightarrow f = f\left( \text{Re}, \frac{L}{D} \right) \]

for well developed flow

expts show there is no L/D dependence

\[ \Rightarrow f = f(\text{Re}) \]

Conclusion: wall friction, \( f \), should only correlate (vary) with \( \text{Re} \)

Creating a Data Correlation

One final question:

How do we measure \( f \) ?

Answer:

We can see how to measure \( f \) by performing a macroscopic momentum balance on a straight pipe (incompressible fluid).

\[ -P_0 \hat{n}_\text{inlet} - \rho v^2 \pi R^2 = \text{force on wall} = -\text{force on fluid} \]

\[ -P_0 \hat{n}_\text{outlet} - \rho v^2 \pi R^2 \]
Result of macroscopic momentum balance on straight pipe:

\[ F_z = (P_o - P_L) \pi R^2 \]

Fanning Friction Factor

\[ f = \frac{F_z}{(\text{area})(\text{kinetic energy})} = \frac{F_z}{(2\pi R L) \left( \frac{1}{2} \rho v^2 \right)} \]

(this is one of the equations in the front cover of the book)

Data correlation for friction factor (\(\Delta P\)) versus Re (flow rate) in a pipe

Moody Chart

(Geankoplis, 2003)
Flow Regimes in a Pipe

- **Re < 2100 Laminar**
  - smooth
  - one direction only
  - predictable

- **2100 < Re < 4000 Transitional**

- **4000 < Re Turbulent**
  - chaotic - fluctuations within fluid
  - transverse motions
  - unpredictable - deal with average motion
  - most common

What is the Fanning Friction Factor for Laminar Flow?

\[
F_z = \Delta P \pi R^2 = \left(\frac{8 \mu L}{R^2}\right)Q
\]

\[
f = \frac{\left(P_0 - P_L\right) \frac{1}{4}}{\left(\frac{L}{D}\right) \left(\frac{1}{2} \rho v^2\right)} = \frac{16 \mu}{\rho v D} = \frac{16}{Re}
\]

TRUE!
Q: What have we done so far?
A: learn to non-dimensionalize

WHY?

• When flow problems are too complex for analytical or numerical solution, use experimental data correlations. Non-dimensionalization guides the production and use of these data correlations.
How can we apply this approach to a new problem?

Understanding a new system
- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved
Real Flows (continued)

Other internal flows:
- **rough pipes** - need an additional dimensionless group

- characteristic size of the surface roughness

\[ \varepsilon / D \] - relative roughness (dimensionless roughness)

\[
\frac{1}{\sqrt{f}} = -4.0 \log_{10} \left( \frac{\varepsilon}{D} \right) + \frac{4.67}{Re^{\sqrt{f}}} + 2.28
\]

Colebrook correlation (Re>4000)

### Surface Roughness for Various Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$\varepsilon$ (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drawn tubing (brass, lead, glass, etc.)</td>
<td>$1.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>Commercial steel or wrought iron</td>
<td>0.05</td>
</tr>
<tr>
<td>Asphalted cast iron</td>
<td>0.12</td>
</tr>
<tr>
<td>Galvanized iron</td>
<td>0.15</td>
</tr>
<tr>
<td>Cast iron</td>
<td>0.46</td>
</tr>
<tr>
<td>Wood stave</td>
<td>0.2–9</td>
</tr>
<tr>
<td>Concrete</td>
<td>0.3–3</td>
</tr>
<tr>
<td>Riveted steel</td>
<td>0.9–9</td>
</tr>
</tbody>
</table>

**Notes:**
- Drawn tubing (brass, lead, glass, etc.) $1.5 \times 10^{-3}$
- Commercial steel or wrought iron 0.05
- Asphalted cast iron 0.12
- Galvanized iron 0.15
- Cast iron 0.46
- Wood stave 0.2–9
- Concrete 0.3–3
- Riveted steel 0.9–9

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Real Flows (continued)

Other internal flows:

- rough pipes -

What else?

Real Flows (continued)

Other internal flows:

- rough pipes -

- flow through noncircular conduits
  - Steady
  - Unidirectional
  - Long (no end effects)
  - Incompressible

Let's go back through the analysis and see where we assumed the pipe was circular
Result of macroscopic momentum balance on straight pipe:

\[ F_z = (P_o - P_L)\pi R^2 \]

Fanning Friction Factor

\[ f \equiv \frac{F_z}{(\text{area})(\text{kinetic energy})} = \frac{F_z}{(2\pi RL)\left(\frac{1}{2} \rho V^2\right)} \]

Straight Pipe: where did we assume circular?

Result of macroscopic momentum balance on straight pipe:

\[ F_z = (P_o - P_L)\pi R^2 \]

If we change these to general relations (good for any shape, . . .)
Real Flows (continued)

Other internal flows:
• flow through noncircular conduits

We can show:

Drag in conduit:
\[ F_{drag} = F_z = \Delta P A_{crosssection} \]

Wetted surface:
\( (perimeter)L \)

Carrying out the dimensional analysis, we see that a good characteristic length is given by:
\[ D \equiv D_H = \frac{4A_{xs}}{perimeter} \]
Real Flows (continued)

Other internal flows:
• flow through noncircular conduits

We can show:

Drag in conduit:

\[ F_{\text{drag}} = F_z = \Delta P A_{\text{crosssection}} \]

Wetted surface:

\[ (\text{perimeter})L \]

Carrying out the dimensional analysis, we see that a good characteristic length is given by:

\[ D \equiv D_H = \frac{4A_{\text{xs}}}{\text{perimeter}} \]

It works! (for both laminar and turbulent)

Empirically, it is found that \( f \) vs. \( \text{Re} \) correlations for circular conduits matches the data for noncircular conduits if \( D \) is replaced with equivalent hydraulic diameter \( D_H \).

\[ D_H \equiv \left( \frac{4(\text{cross-sectional area})}{\text{wetted perimeter}} \right) = 4R_H \]

Equivalent hydraulic diameter

Hydraulic radius

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Real Flows (continued)

Laminar flow

\[ f_D R_D = \text{constant} = Po \]

\[ Po \equiv \text{Poiseuille number} \]

- Circle = 16
- Slit = 24
- Ellipse = function of a, b
- Triangle = 13.33

Flow Through Noncircular Conduits - Turbulent

- Flow through equilateral triangular conduit
- \( f \) and \( Re \) calculated with \( D_H \)
- Solid lines are for circular pipes

Can be corrected to be more exact:
See section 7.2.2 (Morrison, p570)

\[
\frac{1}{\sqrt{f_{D_H}}} = 4.0 \log \left( \frac{Re_{D_H} \sqrt{f_{D_H}}}{Po_{duct}} \right) - 0.40
\]
Non-Circular Cross-sections have application in the new field of microfluidics
Real Flows (continued)

Other internal flows:
- rough pipes -
- flow through noncircular conduits

What else?
Real Flows (continued)

Other internal flows:
- Rough pipes
- Flow through noncircular conduits
- Flow through a packed bed

Commonly used as:
- Reactors
- Separators
  (distillation columns, absorbers)

Understanding a new system
- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved

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Flow Through Packed Beds

Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit

Use Hydraulic Diameter

\[ D_H \equiv \frac{4A_{ss}}{p} \]

(see Example 7.16, page 564)
Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit

\[ D_H \equiv \frac{4A_{ss}}{\bar{p}} = \frac{4\varepsilon}{(1-\varepsilon)A_v} \]

(see text for details)

\[ \varepsilon = \text{void fraction} \]
\[ A_v = \text{specific surface area of the packing} \]
\[ v_0 = \text{superficial velocity} = \frac{Q}{V_{ol/L}} \]

\[ f_{D_H} = \left( \frac{\Delta p}{L} \right) \frac{D_H \varepsilon^2}{2\rho v_0^2} \]
\[ \text{Re}_{D_H} = \frac{\rho V D_H}{\mu} \]

Does it work?

(see Example 7.16, page 564) © Faith A. Morrison, Michigan Tech U.
Flow Through Packed Beds

Hypothesis:
Flow through a packed bed is like flow through any other non-circular conduit

Ergun Equation
Friction factor/Re number relationship for flow through packed beds

\[
\frac{100/3}{Re_{DH}} + \frac{1.75}{3} = f_{DH}
\]

(see text for details)

\[
f_{DH} = \left(\frac{\Delta p}{L}\right) \frac{D_H \varepsilon^2}{2 \rho v_0^2}
\]

\[
Re_{DH} = \frac{\rho V D_H}{\mu}
\]

(see Example 7.16, page 564) © Faith A. Morrison, Michigan Tech U.

Flow Through Noncircular Conduits – All Flow Regimes

For flow through packed beds, in laminar flow the flow is like flow through an irregular cross section.

While in turbulent flow, the passage is like through extremely rough pipes.

Morrison, An Introduction to Fluid Mechanics, Cambridge, 2013
Summary

Other internal flows:

- rough pipes
- flow through noncircular conduits
- flow through a packed bed

Commonly used as

- Reactors
- Separators
  (distillation columns, absorbers)

Summary

- Complex flows are governed by the same physics as simple flows, but the math (programming) is harder

- We can leverage our knowledge about the physics through **Dimensional Analysis (DA)**

  For a complex problem:
  
  Which terms dominate?
  How can we simplify?

- DA reveals the dimensionless numbers that govern dynamic similarity, e.g. Re, Fr

- Experiments on geometrically and dynamically similar systems can be used to correlate results, e.g. Moody plot, and to perform scale-up, e.g. pilot plant studies.
Summary (continued)

• In addition to using the data correlations of others, we can use **Dimensional Analysis** to analyze new, never-before-studied systems (like in a plant, or novel device design)

**Understanding a new system**
- Propose a simplified system (ignoring end effects, minor complications, imposing symmetry, etc.)
- Solve (analytically, numerically)
- Nondimensionalize (must choose characteristic values)
- Test if identified dimensionless numbers do capture essential physics
- Refine model until success is achieved

We discussed how this procedure worked for: (1) rough pipes, (2) noncircular cross sections, (3) flow through packed beds (all internal flows, Chapter 7).

We will also use this procedure for external flows: skydiving, automotive drag, etc. (Chapter 8)