Before continuing on to External flows

Let's take the time to explore a different type of momentum balance

CM3110
Transport I
Part I: Fluid Mechanics

More Complicated Flows II: External Flow
(or applying fluid mechanics problem-solving to a new category of flows)

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Macroscopic Momentum Balances

CM3110
Transport I
Part I: Fluid Mechanics

Macroscopic Momentum Balances

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Balances on Control Volumes

What kinds of problems are best approached with the macroscopic momentum balance?
**Macroscopic Momentum Balance Example:**

**Drag on the walls of a pipe**

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

**Questions about devices, especially forces.**

---

**Macroscopic Momentum Balance Example:**

**Calculate the force on a reducing bend**

For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?

**Questions about devices, especially forces.**
Macroscopic Balances

- Use when we do not need the details of the velocity profile
- 3 types:
  - mass (CM2110, Felder and Rousseau)
  - momentum
  - energy (CM2110, Felder and Rousseau)

Macroscopic Mass Balance:

\[
\frac{d}{dt} \sum_{\text{streams}} \left[ \rho A \cos(\theta)(v)^2 \right]_A = \sum_{\text{streams}} \left[ -pA\hat{n} \right]_A + \mathbb{R} + M_{CV}g
\]

\( \mathbb{R} \) = net force on fluid due to walls
\( M_{CV} \) = mass of control volume
\( \hat{n} \) = outwardly pointing unit normal

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Macroscopic Momentum Balance

\[
\frac{d\mathbf{p}}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho \cos \theta (v)^2}{\beta} \right] A_i = \sum_{i=1}^{\text{streams}} [ -pA\hat{n} ] A_i + \mathbf{R} + MCVg
\]

\[
\frac{d\mathbf{n}_k}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho \cos \theta (v)^2}{\beta} \right] \left( \mathbf{n}_k \right)_{A_i} = \sum_{i=1}^{\text{streams}} [ -pA\hat{n} ] A_i \left( \mathbf{n}_k \right)_{A_i} + MCV \left( \mathbf{g}_v \right)_{\mathbf{n}_k}
\]

\( \hat{n} = \text{force on fluid due to walls} \)
\( \beta = \text{laminar} \approx 0.75 \)
\( \beta = \text{turbulence} \approx 1 \)
\( \mathbf{R} = \text{outwardly pointing unit normal} \)
\( MCV = \text{mass of control volume} \)

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Microscopic Momentum Balance

\[
\left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \mathbf{v}^2 + \rho \mathbf{g}
\]

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Macroscopic Momentum Balance

\[
\frac{d\mathbf{P}}{dt} = \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right]_{A_i} - \sum_{i=1}^{\text{#streams}} [-pA\hat{n}]_{A_i} + R + M_{CV}g
\]

Microscopic Momentum Balance

\[
\left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g
\]

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Macroscopic Momentum Balance

\[
\frac{d\mathbf{P}}{dt} = \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right]_{A_i} - \sum_{i=1}^{\text{#streams}} [-pA\hat{n}]_{A_i} + R + M_{CV}g
\]

Microscopic Momentum Balance

\[
\left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g
\]

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Macroscopic Momentum Balance

\[
\frac{d\mathbf{p}}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right]_{A_i} = \sum_{i=1}^{\text{#streams}} \left[ -p \mathbf{A} \mathbf{n} \right]_{A_i} + R + M_{CV} g
\]

Microscopic Momentum Balance

\[
\left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g
\]

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Rate of change of momentum with time

Macroscopic Momentum Balance

\[ \frac{dP}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right] = \sum_{i=1}^{\text{streams}} \left[ -p A \hat{n} \right]_{A_i} + R + M_{CV} g \]

Microscopic Momentum Balance

\[ \left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho g \]

We know how to apply this
Macroscopic Momentum Balance

\[ \frac{d\vec{P}}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right]_{A_i} = \sum_{i=1}^{\text{#streams}} \left[ -pA\vec{h} \right]_{A_i} + R + M_{CVg} \]

Microscopic Momentum Balance

\[ \left( \rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \]

GOAL:
Learn to use the macroscopic momentum balance

Three questions:
1. Why does it work?
2. When do we use it?
3. How do we use it?
We begin with the Macroscopic Mass Balance

Mass accumulation = Mass in - Mass out

To obtain a general equation (first for mass, then for momentum) we first consider the following case:

- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV

Macroscopic Mass Balance: Mass in = Mass out

- Steady
- Arbitrary control volume, CV (macroscopic)
- Direction of flows are perpendicular to inlet/outlet surfaces of CV
**Macroscopic Mass Balance**

See Chapter 9 for detailed derivation

**Arbitrary, single-input, single-output system:** special case of velocity perpendicular to control surfaces (CS) $A_i$

Assumptions:
- steady state
- single-input, single output
- $v^{(i)}$ perpendicular to $A_i$
- $\rho$ constant across surface

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Macroscopic Mass Balance:

Mass in = Mass out
\[ \rho_1 \langle v^{(1)} \rangle A_1 = \rho_2 \langle v^{(2)} \rangle A_2 \]

Assumptions:
- steady state
- single-input, single output
- \( v^{(i)} \) perpendicular to \( A_i \)
- \( \rho \) constant across surface

Arbitrary, single-input, single-output system:
velocity is **NOT perpendicular** to control surfaces \( A_i \)
**Macroscopic Mass Balance**

\[ 0 = \text{net mass out} \]

\[ 0 = \rho_1 v^{(1)} \cos \theta_1 A_1 + \rho_2 v^{(2)} \cos \theta_2 A_2 \]

**Assumptions:**
- steady state
- single-input, single output
- \( v^{(i)} \) NOT perpendicular to \( A_i \)
- \( \rho_i \) constant across surface

\( \hat{n}_i = \) outwardly pointing unit normal

This takes care of ‘out’ or ‘in’
Reminder:

θ relates to the orientation of inlet and outlet surfaces in the chosen coordinate system.

\[ \hat{n}_i = \text{outwardly pointing unit normal at in/outlet of CV} \]

**Macroscopic Mass Balance**

\[ \rho = \sum_{i=1}^{N} \int_{A_i} (\hat{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS \]

**Macroscopic Momentum Balance:**

\[ \sum F_{on\,CV} = \frac{d}{dt} \int_{CS} (\hat{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS + \int_{CS} \int (\hat{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS \]

We can specialize the \textit{convective term} for macroscopic control volumes.

\[ \int_{CS} (\hat{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS = \sum_{i=1}^{N} \left[ \int_{CS} (\hat{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS \right] \]

N bounding control surfaces

Momentum balance on fluid in a control volume

(net momentum convected out)
**Macroscopic Momentum Balance**

We can now specify for each $A_i$:

$$v^{(i)} = v^{(i)} \hat{p}^{(i)}$$

We now separate velocity magnitude from the direction.
For each inlet or outlet surface $A_i$:

\[ \mathbf{v}(i) = v(i) \mathbf{\hat{p}}(i) \]

- $v(i)$ = magnitude of velocity through $A_i$
- $\mathbf{\hat{p}}(i)$ = unit vector in direction of velocity through $A_i$
- $\mathbf{n} \cdot \mathbf{v}(i) = v(i) \cos(\theta_i)$ = component of velocity “through” $A_i$

Macroscopic Momentum Balance-Convective Term

\[ \left( \text{net momentum out of CV} \right) = \int \int_{CS} (\mathbf{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS = \sum_{i=1}^{N} \left[ \int \int_{CS} (\mathbf{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS \right]_{i} \]

Input, output surfaces $A_i$

\[ \sum_{i=1}^{N} \left[ \int \int_{CS} (\mathbf{n} \cdot \mathbf{v}) \rho \mathbf{v} \, dS \right]_{i} = \sum_{i} \left[ \int \int_{A_i} (\rho \mathbf{v}(i)) (\mathbf{n} \cdot \mathbf{v}(i)) \, dA \right] 

\[ \mathbf{v}(i) = v(i) \mathbf{\hat{p}}(i) \]

\[ \mathbf{n} \cdot \mathbf{v}(i) = v(i) \cos \theta_i \]
Macroscopic Momentum Balance-Convective Term

\[
\text{net momentum convected out} = \sum_i \int_A (\rho \nu^{(i)})(\hat{n}_i \cdot \nu^{(i)}) dA
\]

\[
= \sum_i \int_A (\rho \nu^{(i)} \hat{v}_i^{(i)}) (\nu^{(i)} \cos \theta_i) dA_i 
\]

\[
= \sum_i \rho_i \hat{v}_i^{(i)} \cos \theta_i \left( \int_A (\nu^{(i)})^2 dA_i \right)
\]

We have assumed that the direction of \( \nu^{(i)} \) does not vary across \( A_i \); only the velocity magnitudes vary across \( dA_i \); they appear as \( \nu^{(i)}^2 \)

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Assumptions:
• steady state
• single-input, single output
• $v^{(i)}$ NOT perpendicular to $A_i$
• $\rho_i$ constant across surfaces
• $v^i$ constant across surfaces

\[
0 = -\rho_1 \cos \theta_1 v^{(1)} \left[ \iiint_{A_1} (v^{(1)})^2 dA \right] - \rho_2 \cos \theta_2 v^{(2)} \left[ \iiint_{A_2} (v^{(2)})^2 dA \right] + \sum_i F_{i,\text{on}}.
\]

Macroscopic Momentum Balance

We can write these terms compactly as $\frac{(v)^2}{\beta}$, as we now show

\[
0 = -\rho_1 \cos \theta_1 v^{(1)} \left[ \iiint_{A_1} (v^{(1)})^2 dA \right] - \rho_2 \cos \theta_2 v^{(2)} \left[ \iiint_{A_2} (v^{(2)})^2 dA \right] + \sum_i F_{i,\text{on}}.
\]

Recall that the average of a function $f$ across a surface $A$ is calculated from:

\[
\langle f(x,y) \rangle = \frac{\iiint_A f \, dA}{A} = \frac{1}{A} \iiint_A f \, dA
\]

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Macroscopic Momentum Balance

\[ 0 = -\rho_1 \cos \theta_1 \langle \nu^{(1)} \rangle^2 A_1 + \rho_2 \cos \theta_2 \langle \nu^{(2)} \rangle^2 A_2 + \sum_i F_{i, on} \]

\[ = \langle \langle \nu^{(1)} \rangle^2 \rangle A_1 \]
\[ = \langle \langle \nu^{(2)} \rangle^2 \rangle A_2 \]

But what is this?

We can make this look more like other convective terms we have seen by introducing a factor relating \( \langle \nu^2 \rangle \) to average velocity squared.

\beta quantifies the variation of the true velocity profile from plug flow (flat profile). \( \beta \equiv \text{Velocity Correction Factor} \)

\[ \sum_i F_{i, on} = \rho_1 A_1 \langle \langle \nu^{(1)} \rangle^2 \rangle \cos \theta_1 \nu^{(1)} + \rho_2 A_2 \langle \langle \nu^{(2)} \rangle^2 \rangle \cos \theta_2 \nu^{(2)} \]

\[ \beta \equiv \frac{\langle \nu^2 \rangle}{\langle \nu^3 \rangle} \]

Experimental result:

\( \beta_{\text{turbulent}} = 0.95 - 0.99 \)
\( \beta_{\text{laminar}} = 0.75 \)

Result: Steady State Macroscopic Momentum Balance (convective terms)

\[ \sum_i F_{i, on} = \frac{\rho_1 A_1 \langle \langle \nu^{(1)} \rangle^2 \rangle \cos \theta_1 \nu^{(1)}}{\beta_1} + \frac{\rho_2 A_2 \langle \langle \nu^{(2)} \rangle^2 \rangle \cos \theta_2 \nu^{(2)}}{\beta_2} \]

\( \text{vector equation} \)

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**Force Terms**

\[
\sum_{i} F_{i,\text{on}} = \frac{\rho_{1}A_{i}\left(v^{(1)}\right)^{2}\cos\theta_{1}}{\beta_{1}}\hat{v}^{(1)} + \frac{\rho_{2}A_{i}\left(v^{(2)}\right)^{2}\cos\theta_{2}}{\beta_{2}}\hat{v}^{(2)}
\]

Sum of the forces on the fluid in the CV

\[
\sum_{i} F_{i,\text{on}} = \text{contact} + \text{noncontact}
\]

\[
E_{\text{contact}} = \iint_{S} \hat{n} \cdot \hat{F}_{n} \, dS = M_{CV} g
\]

Molecular forces

(viscosity and pressure)

---

**Contact Forces = pressure + viscous**

Viscous: \[R\]  
This is the force on the fluid  
(force on walls is \(-R\))

Pressure: \[E_{\text{contact}} = \iint_{S} \hat{n} \cdot (-pI) \, dS\]

\[= \sum_{i} \left[ \iint_{S} \hat{n} \cdot (-pI) \, dS \right]_{i}\]

\[= \sum_{i} \left[ (-p)\hat{n} \right] \left[ \iint dS \right]_{i}\]

\[= \sum_{i} \left[ (-p)\hat{n}A \right]_{i}\]
Macroscopic Momentum Balance

\[ \frac{d \mathbf{p}}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho \cos \theta (v)^2}{\beta} \hat{\mathbf{n}} \right]_{A_i} = \sum_{i=1}^{\text{streams}} [p A \hat{\mathbf{n}}]_{A_i} + \mathbf{R} + M_{CV} \mathbf{g} \]

\( R \) = net force on fluid due to walls
\( M_{CV} \) = mass of control volume
\( \hat{\mathbf{n}} \) = outwardly pointing unit normal of the macroscopic control volume, CV

See inside front Cover of Morrison, 2013
And the exam formula sheet

Compare with the (more familiar) Navier-Stokes

Macroscopic Momentum Balance

\[ \frac{d \mathbf{p}}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho \cos \theta (v)^2}{\beta} \hat{\mathbf{n}} \right]_{A_i} = \sum_{i=1}^{\text{streams}} [p A \hat{\mathbf{n}}]_{A_i} + \mathbf{R} + M_{CV} \mathbf{g} \]

Microscopic Momentum Balance

\[ \left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]
Macroscopic Momentum Balance

\[
\frac{d\mathbf{p}}{dt} = \sum_{i=1}^{\text{streams}} \left[ \frac{\rho A \cos(\theta)(v)^2}{\beta} \hat{\mathbf{v}} \right]_{A_i} = \sum_{i=1}^{\text{streams}} \left[ -p A \hat{\mathbf{n}} \right]_{A_i} + R + M_{CV} \mathbf{g}
\]

Microscopic Momentum Balance

\[
\left( \rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]
Rate of change of momentum with time

Macroscopic Momentum Balance

Convective terms

Pressure forces

Microscopic Momentum Balance

Rate of change of momentum with time

Macroscopic Momentum Balance

Convective terms

Pressure forces

Viscous forces

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Macroscopic Momentum Balance

\[ \frac{dP}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho A \cos \theta (v_i)^2}{\beta} \right] = \sum_{i=1}^{\text{streams}} \left[ -pA\hat{n}_i \right] A_i + R + M_{CV}g \]

Microscopic Momentum Balance

\[ \left( \rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v \right) = -\nabla p + \mu \nabla^2 v + \rho g \]

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We know how to apply this
Macroscopic Momentum Balance

\[
\frac{dP}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \right] = \sum_{i=1}^{\text{#streams}} \left[ - p A \hat{n}_{A_i} + R + M_{CV} g \right]
\]

Microscopic Momentum Balance

\[
\left( \rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v \right) = - \nabla p + \mu \nabla^2 v + \rho g
\]

Now we need to learn when and how to apply this

We know how to apply this

Macroscopic Momentum Balance

\[
\rho = \text{fluid momentum}
\]

\[
\frac{d\rho_x}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \hat{\rho}_{x_{A_i}} \right] = \sum_{i=1}^{\text{#streams}} \left[ - p A \hat{n}_{A_i} + R + M_{CV} g \right]
\]

\[
\frac{d\rho_y}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \hat{\rho}_{y_{A_i}} \right] = \sum_{i=1}^{\text{#streams}} \left[ - p A \hat{n}_{A_i} + R + M_{CV} g \right]
\]

\[
\frac{d\rho_z}{dt} + \sum_{i=1}^{\text{#streams}} \left[ \frac{\rho A \cos \theta (v)^2}{\beta} \hat{\rho}_{z_{A_i}} \right] = \sum_{i=1}^{\text{#streams}} \left[ - p A \hat{n}_{A_i} + R + M_{CV} g \right]
\]

\[
R = \text{net force on fluid due to walls}
\]

\[
M_{CV} = \text{mass of control volume}
\]

\[
\hat{n} = \text{outwardly pointing unit normal of CV}
\]

\[
\beta_{\text{laminar}} = 0.75
\]

\[
\beta_{\text{turbulent}} \approx 1
\]

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Macroscopic Momentum Balance Example:
Drag on the walls of a pipe

For steady pressure-driven turbulent flow in a horizontal pipe of circular cross section, what is the drag (force) on the walls due to the fluid?

Assume:
- steady state
- turbulent
- neglect gravity

\[ F_{wall} = -F_{fluid} \]

\[ F_x = -R_x \]

\[ \overline{R} = \text{net force on fluid due to walls} \]
Macroscopic Momentum Balance

Macroscopic Momentum Balance Example:
Calculate the force on a reducing bend

For steady pressure-driven turbulent flow in a reducing bend (shown below), what is the force on the walls due to the fluid?

Assume:
• steady state
• turbulent
• neglect gravity

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Types of Momentum Transfer

**Macroscopic**
- convection
- pressure forces
- viscous forces
- body forces

**Microscopic**
- convection
- pressure forces
- viscous forces
- body forces

After calculating the flow field with microscopic balances you can calculate wall forces. With macroscopic balances you can often calculate wall forces directly.

---

### Problem-Solving Procedure - Steady State Macroscopic Momentum Problems

\[
\frac{\partial \mathbf{u}}{\partial t} + \sum_{i=1}^{\text{streams}} \left[ \frac{p A \cos \theta (v)^2}{\beta} \right]_{A_i} = \sum_{i=1}^{\text{streams}} \left[ -p A \hat{n} \right]_{A_i} + R + M_{CV} g
\]

1. sketch system; choose CV on which you will perform balance
2. choose coordinate system
3. perform macroscopic mass balance **(Consider angles carefully)**
4. perform macroscopic momentum balance (vector equation; forces are pressure, gravity, force on the wall; all forces ON the fluid in CV)
5. solve (usually for force on the wall)
Solution to force on a reducing bend:
www.chem.mtu.edu/~fmorriso/cm310/reducing_bend.pdf

Dr. Morrison doing a Macro-Momentum Balance on YouTube:
(DrMorrisonMTU)
www.youtube.com/watch?v=jXNkN7NM1NM
(note that there is a sign error in the gravity term; sorry about that; gravity is negligible)

Many useful handouts:
www.chem.mtu.edu/~fmorriso/cm310/handouts.html

(just need to practice; see HW4)
**Bonus:** When to use which?

**Mechanical Energy Balance**

\[
\frac{P_2 - P_1}{\rho} + \frac{(v_2^2 - v_1^2)}{2\alpha} + g(z_2 - z_1) + F_{21} = \frac{W_{s, on}}{m}
\]

**Macroscopic Momentum Balance**

\[
\frac{dP}{dt} + \sum_{i=1}^{\text{streams}} \left[ \frac{\rho \cos \theta (v)^2}{\beta} \right]_{A_i} = \sum_{i=1}^{\text{streams}} \left[ -pA_i \hat{n}_i + R + M_{CV} g \right]
\]

**Microscopic Momentum Balance**

\[
\left( \rho \frac{\partial v}{\partial t} + \rho v \cdot \nabla v \right) = -\nabla P + \mu \nabla^2 v + \rho g
\]

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