CM3110
Transport Processes and Unit Operations I

Fluid Mechanics
Non-Newtonian fluids –
An Introduction

Photo by Carlos Arango Sabogal, U. Wisconsin, Madison

The *Weissenberg effect* is when a viscoelastic, non-Newtonian fluid will climb a rotating shaft.

https://www.youtube.com/watch?v=npZzlgKjs0I

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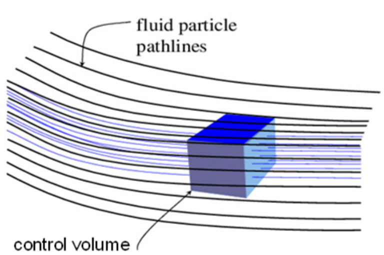
Intro to Non-Newtonian Fluid Mechanics

Continuum Modeling—Newtonian



- μ (viscosity) constant
- ρ (density) constant often a good assumption
- $\underline{\underline{\tilde{\tau}}} = \mu(\nabla \underline{v} + (\nabla \underline{v})^T)$

The form of the function $\underline{\underline{\tilde{\tau}}}(\underline{v})$ is known

—Viscosity differs for different materials
—Is a function of T, perhaps P

This is **MAJOR.** Predictions seem to be right in a wide variety of situations (water, oil, air)

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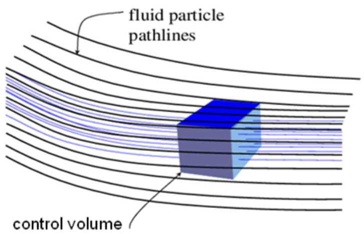
Continuum Modeling—Non-Newtonian



- μ (viscosity) **NOT** constant
- ρ (density) constant is often a good assumption
- $\underline{\underline{\tilde{\tau}}} = \cdots ?$ The form of the function $\underline{\underline{\tilde{\tau}}}(\underline{v})$ is NOT known

For non-Newtonian Fluids, we need a new form of $\underline{\underline{\tilde{\tau}}}(\underline{v})$ that matches material observations.

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When did we separate out non-Newtonian fluids?

Lecture 2:

How do Fluids Behave?

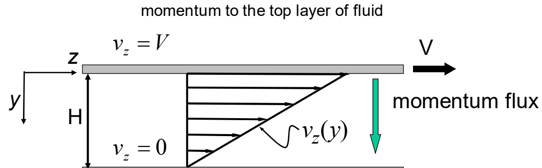
Momentum Flux

Momentum ($\mathfrak{p}$) = mass * velocity

$$\boxed{\underline{\mathfrak{p}} = m\underline{v}}$$ **vectors**

Viscosity determines the magnitude of momentum flux

top plate has momentum, and it transfers this momentum to the top layer of fluid



Each fluid layer transfers the momentum downward

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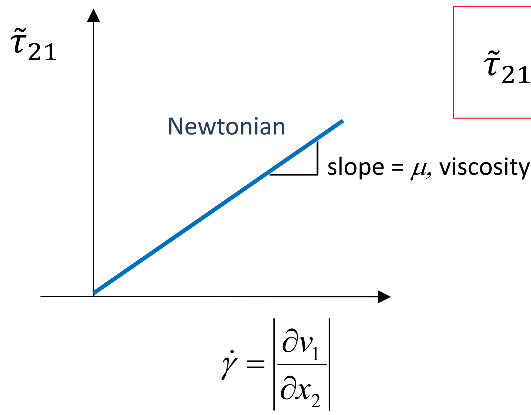
Newtonian Fluids

Newton's Law of Viscosity

(unidirectional flow)

$$\boxed{\tilde{\tau}_{21} = \mu\left(\frac{\partial v_1}{\partial x_2}\right)}$$



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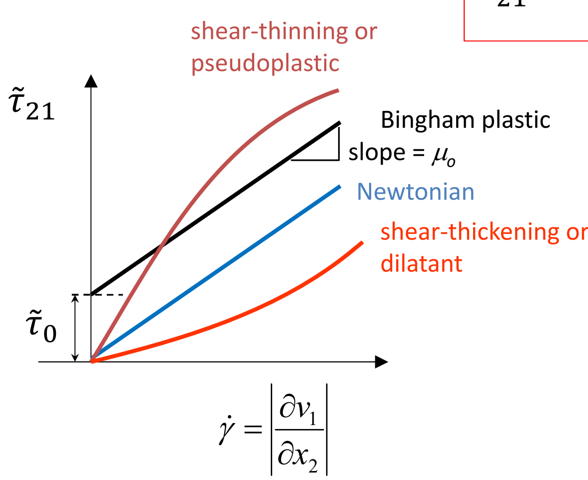
Non-Newtonian Fluids

Non-Newtonian behavior
(unidirectional flow)

$$\boxed{\tilde{\tau}_{21} = \text{function of } \left|\frac{\partial v_1}{\partial x_2}\right|}$$



Many different behaviors are observed.

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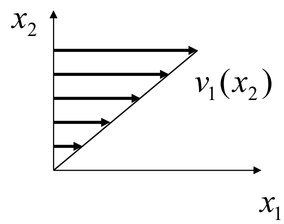
How can we define viscosity for Non-Newtonian Fluids?

- **Perform this:**



$$\dot{\gamma} = \left|\frac{\partial v_1}{\partial x_2}\right|$$

- **Measure $\tilde{\tau}_{21}$**
- **Calculate this:**

$$\boxed{\eta \equiv \frac{\tilde{\tau}_{21}}{\dot{\gamma}}}$$

Non-Newtonian viscosity (unidirectional flow)

(NOTE on coordinate system: Viscosity definition is written for shear flow in x_1 direction and gradient in x_2 direction)

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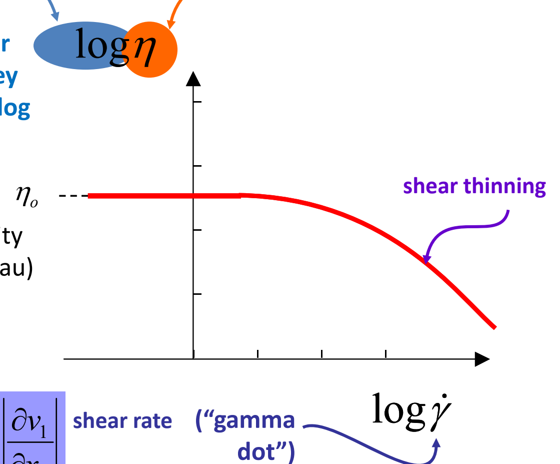
Typical polymeric behavior

Non-Newtonian behavior (unidirectional flow)

Viscosity (Greek letter eta)

The changes in viscosity with shear rate are so large they must be plotted log-log



zero-shear viscosity (Newtonian plateau)

$$\boxed{\dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right|}$$ **shear rate** **("gamma dot")**

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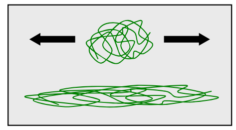
In addition, for many polymers there are shear-induced **NORMAL** (perpendicular) forces.

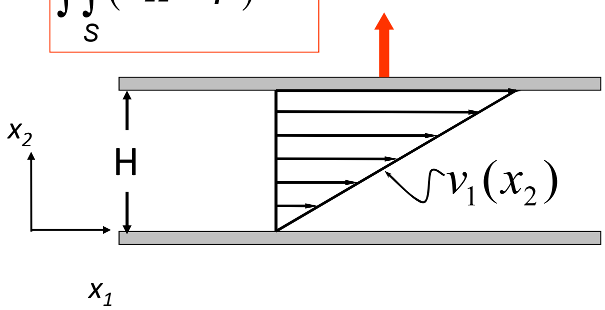


force on 2-surface in 2-direction

$$\iint_S \left(\tilde{\tau}_{22} - p\right) dS$$

force on 2-surface in 1-direction

$$\iint_S \tilde{\tau}_{21}\, dS$$



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How to deal with this?

Recall, for Newtonian fluids:

$$\underline{\underline{\tilde{\tau}}} = \mu \underline{\underline{\dot{\gamma}}} = \mu\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Newtonian Constitutive Equation

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \mu \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{xyz} \Rightarrow \boxed{\tilde{\tau}_{xz} = \mu\left(\frac{dv_z}{dx}\right)}$$

Newton's law of viscosity is a ***special case*** *of the Newtonian Constitutive equation.*

(Unidirectional flow)

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$$\underline{\underline{\tilde{\tau}}} = \cancel{\mu}\overset{\eta(\dot{\gamma})}{}\underline{\underline{\dot{\gamma}}} = \cancel{\mu}\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

Non-Newtonian ~~Newtonian~~ Constitutive Equation

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tilde{\tau}_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \cancel{\mu}\,\eta(\dot{\gamma}) \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

Generalized Newtonian Fluid

(non-Newtonian)

$$\boxed{\eta(\dot{\gamma})}$$

We pick the form of this function that works best with our data.

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Non-Newtonian behavior (unidirectional flow)

$$\tilde{\tau}_{21} = \text{function of}\left(\frac{\partial v_1}{\partial x_2}\right)$$

Power-Law Model

$$\eta(\dot{\gamma}) = m\dot{\gamma}^{n-1}$$

(does **not** model normal stresses)

$$\boxed{\tilde{\tau}_{21} = m\left|\frac{dv_1}{dx_2}\right|^{n-1}\frac{dv_1}{dx_2}}$$

m or *K* = consistency index (m = μ for Newtonian)
n = power-law index (*n* = 1 for Newtonian)

$$\boxed{\dot{\gamma} \equiv \left|\frac{\partial v_1}{\partial x_2}\right| = \textit{shear rate}}$$

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What does the power-law model predict for viscosity?

$$\boxed{\tilde{\tau}_{21} = m\left|\frac{dv_1}{dx_2}\right|^{n-1}\frac{dv_1}{dx_2}}$$

$$\eta \equiv \frac{\tilde{\tau}_{21}}{\dot{\gamma}} = \frac{\tilde{\tau}_{21}}{\left|\frac{dv_1}{dx_2}\right|} = m\left|\frac{dv_1}{dx_2}\right|^{n-1}$$

On a log-log plot, this would give
a straight line:

$$\underbrace{\log\eta}_{Y} = \underbrace{\log m}_{B} + \underbrace{(n-1)}_{M}\underbrace{\log\left|\frac{dv_1}{dx_2}\right|}_{X}$$

$$Y = B + M\,X$$

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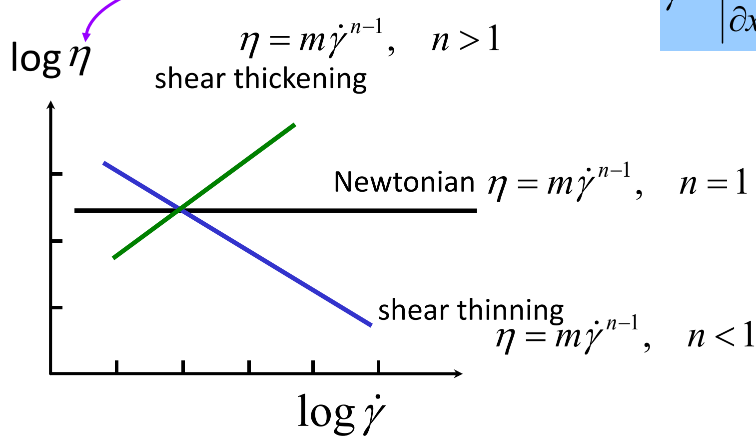
Power-Law Fluid

Non-Newtonian viscosity

$$\eta \equiv \frac{\tilde{\tau}_{21}}{\dot{\gamma}}$$

$$\dot{\gamma} = \left|\frac{\partial v_1}{\partial x_2}\right|$$



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Where do we use the power-law expression?

The Equation of Continuity and the Equation of Motion in Cartesian, cylindrical, and spherical coordinates

Continuity Equation, Cartesian coordinates

$$\frac{\partial\rho}{\partial t} + \left(v_x\frac{\partial\rho}{\partial x} + v_y\frac{\partial\rho}{\partial y} + v_z\frac{\partial\rho}{\partial z}\right) + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0$$

Continuity Equation, cylindrical coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

$$\frac{\partial\rho}{\partial t} + \underline{v}\cdot\nabla\rho = -\rho(\nabla\cdot\underline{v})$$

Continuity Equation, spherical coordinates

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial(\rho r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial(\rho v_\phi)}{\partial\phi} = 0$$

Equation of Motion for an incompressible fluid, 3 components in Cartesian coordinates

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}\right) = -\frac{\partial P}{\partial x} + \left(\frac{\partial\tilde{\tau}_{xx}}{\partial x} + \frac{\partial\tilde{\tau}_{yx}}{\partial y} + \frac{\partial\tilde{\tau}_{zx}}{\partial z}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} + v_z\frac{\partial v_y}{\partial z}\right) = -\frac{\partial P}{\partial y} + \left(\frac{\partial\tilde{\tau}_{xy}}{\partial x} + \frac{\partial\tilde{\tau}_{yy}}{\partial y} + \frac{\partial\tilde{\tau}_{zy}}{\partial z}\right) + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x\frac{\partial v_z}{\partial x} + v_y\frac{\partial v_z}{\partial y} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \left(\frac{\partial\tilde{\tau}_{xz}}{\partial x} + \frac{\partial\tilde{\tau}_{yz}}{\partial y} + \frac{\partial\tilde{\tau}_{zz}}{\partial z}\right) + \rho g_z$$

The one with "$\tilde{\tau}$" is for non-Newtonian fluids

www.chem.mtu.edu/~fmorriso/cm310/Navier.pdf

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We have a handout for that:

The Power-Law, Generalized Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

Prof. Faith A. Morrison, Michigan Technological University

Cartesian Coordinates

$$\begin{pmatrix}\tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz}\\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz}\\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz}\end{pmatrix}_{xyz} = \eta\underline{\underline{\dot{\gamma}}}$$

$$\eta = m\left(\frac{1}{2}\cdot\begin{matrix}\textit{sum of squares}\\ \textit{of each term in } \underline{\underline{\dot{\gamma}}}\end{matrix}\right)^{\frac{n-1}{2}} = m\left(\frac{1}{2}\sum_{p=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{pj}^2\right)^{\frac{n-1}{2}}$$

$$\underline{\underline{\dot{\gamma}}} \equiv \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x}\\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\\ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} & 2\frac{\partial v_z}{\partial z}\end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix}\tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz}\\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z}\\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz}\end{pmatrix}_{r\theta z} = \eta\underline{\underline{\dot{\gamma}}}$$

$$\eta = m\left(\frac{1}{2}\cdot\begin{matrix}\textit{sum of squares}\\ \textit{of each term in } \underline{\underline{\dot{\gamma}}}\end{matrix}\right)^{\frac{n-1}{2}} = m\left(\frac{1}{2}\sum_{p=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{pj}^2\right)^{\frac{n-1}{2}}$$

$$\underline{\underline{\dot{\gamma}}} \equiv \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta} & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial\theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z}\\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial\theta} + \frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z}\end{pmatrix}_{r\theta z}$$

http://www.chem.mtu.edu/~fmorriso/cm310/stpl.pdf

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Where do we use the power-law expression?

e.g., Poiseuille flow in a tube:

$$\tilde{\tau}_{rz} = \mu\left(\frac{dv_z}{dr}\right)$$

Newtonian

$$\tilde{\tau}_{rz} = \left(\frac{L\rho g + (P_o - P_L)}{2L}\right) r$$

non-Newtonian, power-law

$$\tilde{\tau}_{rz} = m\left|\frac{dv_z}{dr}\right|^{n-1}\frac{dv_z}{dr}$$

$\Rightarrow$ **solve for $v_z(r)$**

1-direction = r
2-direction = z

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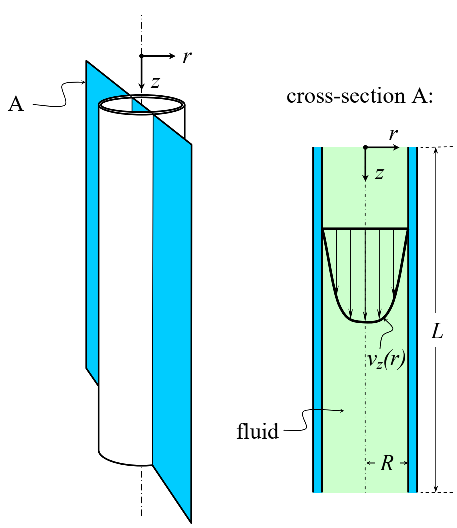
EXAMPLE III: Pressure-driven flow of a Power-law fluid in a tube



- steady state
- incompressible
- well developed
- long tube

Calculate velocity and stress profiles

$\underline{g}$

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Non-Newtonian behavior
(unidirectional flow)

$$\tilde{\tau}_{21} = \text{PL function of}\left(\frac{\partial v_1}{\partial x_2}\right)$$

Calculate the velocity field for Pressure-driven flow of a power-law (PL) fluid:

$$\tilde{\tau}_{rz} = m\underbrace{\left|\frac{dv_z}{dr}\right|}_{=-\frac{\partial v_z}{\partial r}\ \forall r}^{n-1}\frac{dv_z}{dr} = \underbrace{\left(\frac{L\rho g + (P_o - P_L)}{2L}\right)}_{\equiv \alpha} r$$

$$m\left(-\frac{dv_z}{dr}\right)^{n-1}\frac{dv_z}{dr} = \boxed{-m\left(-\frac{dv_z}{dr}\right)^{n} = \alpha r}$$

Solve for $v_z(r)$

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Boundary Conditions: ?

(same as before in the Newtonian case)

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Non-Newtonian behavior
(unidirectional flow)

$$\tilde{\tau}_{21} = \text{PL function of}\left(\frac{\partial v_1}{\partial x_2}\right)$$

Velocity field
Poiseuille flow of a power-law fluid:

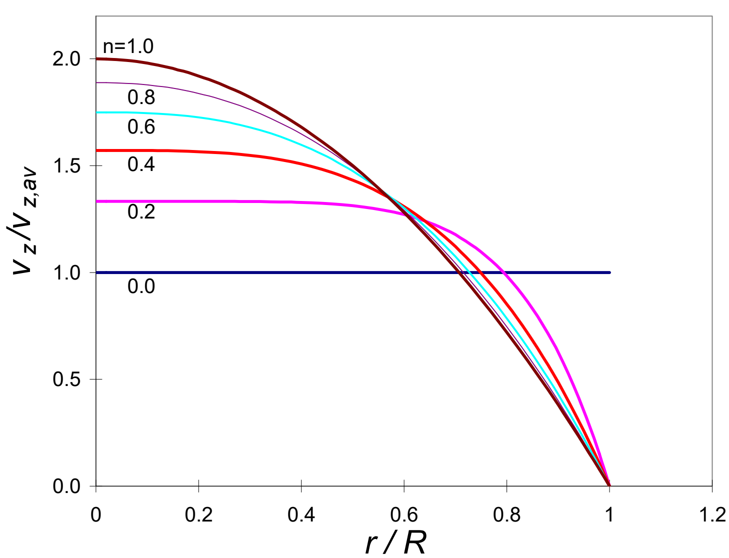
$$v_z(r) = \left(\frac{R\left(L\rho g + P_o - P_L\right)}{2Lm}\right)^{\frac{1}{n}} \left(\frac{R}{\frac{1}{n}+1}\right)\left(1-\left(\frac{r}{R}\right)^{\frac{1}{n}+1}\right)$$

$$\langle v \rangle = \frac{1}{\pi R^2}\int_0^{2\pi}\int_0^R v_z(r)\,r\,dr\,d\theta = R\left(\frac{n}{1+3n}\right)\left[\frac{R(P_o - P_L)}{2mL}\right]^{\frac{1}{n}}$$

Non-Newtonian behavior
(unidirectional flow)

$$\tilde{\tau}_{21} = \text{PL function of}\left(\frac{\partial v_1}{\partial x_2}\right)$$

Solution to Poiseuille flow in a tube
incompressible, power-law fluid



Non-Newtonian behavior (all flows)

$\tilde{\tau}_{21}$ = nonlinear function of $\nabla \underline{v}$

Rheology (Non-Newtonian Fluid Mechanics)

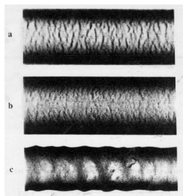
Rheology affects:

•Processing (design, costs, production rates)

www.corrugatorman.com/pic/akron%20extruder.JPG

www.math.utwente.nl/mpcm/aamp/examples.html

•End use (food texture, product pour, motor-oil function)

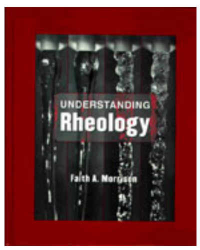


Pomar et al. JNNFM 54 143 1994

•Product quality (surface distortions, anisotropy, strength, structure development)

Rheology (Non-Newtonian Fluid Mechanics)

At Michigan Tech:
CM4650 Polymer Rheology (Even years spring)
www.chem.mtu.edu/~fmorriso/cm4650/cm4650.html



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Next:

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CM3110
Transport I
Part I: Fluid Mechanics

More Complicated Flows
(Dimensional Analysis, rough pipes, hydraulic diameter, porous media)

Professor Faith Morrison

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