The Power-Law, Generalized Newtonian Constitutive Equation in Cartesian, Cylindrical, and Spherical coordinates

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Cartesian Coordinates

$$\begin{pmatrix} \tilde{\tau}_{xx} & \tilde{\tau}_{xy} & \tilde{\tau}_{xz} \\ \tilde{\tau}_{yx} & \tilde{\tau}_{yy} & \tilde{\tau}_{yz} \\ \tilde{\tau}_{zx} & \tau_{zy} & \tilde{\tau}_{zz} \end{pmatrix}_{xyz} = \eta \dot{\underline{\underline{\gamma}}}$$

$$\eta = m\left(\frac{1}{2}\cdot \begin{matrix} sum\ of\ squares \\ of\ each\ term\ in\ \dot{\underline{\underline{\gamma}}} \end{matrix}\right)^{\frac{n-1}{2}} = m\left(\frac{1}{2}\sum_{p=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{pj}^{2}\right)^{\frac{n-1}{2}}$$

$$\dot{\underline{\underline{\gamma}}} \equiv \begin{pmatrix} 2\frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x} & \frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x} \\ \frac{\partial v_x}{\partial y}+\frac{\partial v_y}{\partial x} & 2\frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y} \\ \frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x} & \frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz}$$

Cylindrical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{rz} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta z} \\ \tilde{\tau}_{zr} & \tau_{z\theta} & \tilde{\tau}_{zz} \end{pmatrix}_{r\theta z} = \eta \dot{\underline{\underline{\gamma}}}$$

$$\eta = m\left(\frac{1}{2}\cdot \begin{matrix} sum\ of\ squares \\ of\ each\ term\ in\ \dot{\underline{\underline{\gamma}}} \end{matrix}\right)^{\frac{n-1}{2}} = m\left(\frac{1}{2}\sum_{p=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{pj}^{2}\right)^{\frac{n-1}{2}}$$

$$\dot{\underline{\underline{\gamma}}} \equiv \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right) & \frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z} & 2\frac{\partial v_z}{\partial z} \end{pmatrix}_{r\theta z}$$

Spherical Coordinates

$$\begin{pmatrix} \tilde{\tau}_{rr} & \tilde{\tau}_{r\theta} & \tilde{\tau}_{r\phi} \\ \tilde{\tau}_{\theta r} & \tilde{\tau}_{\theta\theta} & \tilde{\tau}_{\theta\phi} \\ \tilde{\tau}_{\phi r} & \tau_{\phi\theta} & \tilde{\tau}_{\phi\phi} \end{pmatrix}_{r\theta\phi} = \eta \dot{\underline{\underline{\gamma}}}$$

$$\eta = m\left(\frac{1}{2}\cdot \begin{matrix} sum\ of\ squares \\ of\ each\ term\ in\ \dot{\underline{\underline{\gamma}}} \end{matrix}\right)^{\frac{n-1}{2}} = m\left(\frac{1}{2}\sum_{p=1}^{3}\sum_{j=1}^{3}\dot{\gamma}_{pj}^{2}\right)^{\frac{n-1}{2}}$$

$$\dot{\underline{\underline{\gamma}}} \equiv \begin{pmatrix} 2\frac{\partial v_r}{\partial r} & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta} & 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi}+r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) & \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\phi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} & 2\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right) \end{pmatrix}_{r\theta\phi}$$

These expressions are general and are applicable to three-dimensional flows.

Reference: Faith A. Morrison, *An Introduction to Fluid Mechanics* (Cambridge University Press: New York, 2013)

16 April 2014

An Introduction to Fluid Mechanics

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A caution about sign convention

In *Understanding Rheology* (Morrison, 2001) and *An Introduction to Fluid Mechanics* (Morrison, 2012) two different stress sign conventions are used. In the rheology text we follow Bird, Armstrong, and Hassager, *Dynamics of Polymeric Fluids* (Wiley, 1986) ($\underline{\underline{\Pi}} = -\underline{\underline{\tilde{\Pi}}}$, $\underline{\underline{\tau}} = -\underline{\underline{\tilde{\tau}}}$),), while in the fluids text we follow the usual engineering sign convention ($\underline{\underline{\tilde{\Pi}}}$, $\underline{\underline{\tilde{\tau}}}$). Any express that contains $\underline{\underline{\Pi}}$ or $\underline{\underline{\tau}}$ is affected.[§]

$$\underline{\underline{\tilde{\Pi}}} = -p\underline{\underline{I}} + \underline{\underline{\tilde{\tau}}}$$

$$\underline{\underline{\Pi}} = p\underline{\underline{I}} + \underline{\underline{\tau}} \quad \text{(Bird et al.)}$$

$$\underline{\underline{\tilde{\tau}}} = +\mu\left(\nabla\underline{v} + \left(\nabla\underline{v}\right)^T\right)$$

$$\underline{\underline{\tau}} = -\mu\left(\nabla\underline{v} + \left(\nabla\underline{v}\right)^T\right) \quad \text{(Bird et al.)}$$

Force on surface with unit normal $\hat{n}$ and area S:

$$\underline{F} = \iint_S \left[\hat{n}\cdot\underline{\underline{\tilde{\Pi}}}\right]_{surface} dS = \iint_S \left[\hat{n}\cdot\left(-\underline{\underline{\Pi}}\right)\right]_{surface} dS$$

Torque on surface with unit normal $\hat{n}$ and area S:

$$\underline{T} = \iint_S \underline{R}\times\left[\hat{n}\cdot\underline{\underline{\tilde{\Pi}}}\right]_{surface} dS = \iint_S \underline{R}\times\left[\hat{n}\cdot\left(-\underline{\underline{\Pi}}\right)\right]_{surface} dS$$

Sorry about that.