CM3120: Module 2

Unsteady State Heat Transfer

I. Introduction
II. Unsteady Microscopic Energy Balance—(slash and burn)
III. Unsteady Macroscopic Energy Balance
IV. Dimensional Analysis (unsteady)—Biot number, Fourier number
V. Low Biot number solutions—Lumped parameter analysis
VI. Short Cut Solutions—(initial temperature T_0; finite h), Gurney and Lurie charts (as a function of position, $m = 1/\mathrm{Bi}$, and Fo); Heissler charts (center point only, as a function of $m = 1/\mathrm{Bi}$, and Fo)
VII. Full Analytical Solutions (stretch)

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CM3120: Module 2

Module 2 Lecture VI:

Short Cut Solutions

(Gurney and Lurie/Heisler Charts)

Professor Faith A. Morrison
Department of Chemical Engineering
Michigan Technological University

www.chem.mtu.edu/~fmorriso/cm3120/cm3120.html

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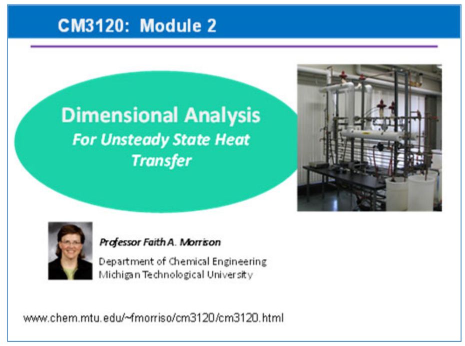
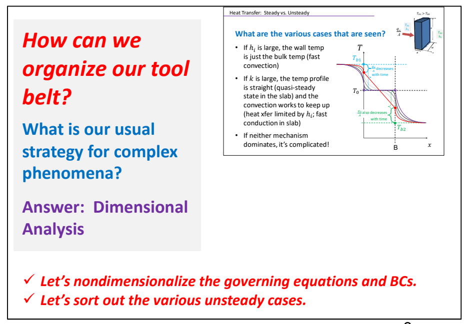
In a previous lecture, we found that **Dimensional Analysis** helped us to organize our “tool belt” for engineering problem solving.



For **Unsteady Heat Transfer** problems, we added two dimensionless numbers, the **Biot number** (*bee oh*) Bi and the **Fourier number** Fo



Unsteady State Heat Transfer in a Body

Two Additional Dimensionless Numbers

$$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$$

Quantifies the tradeoffs between the internal resistance D/k (due to conduction) and $1/h$, the resistance at the phase boundary (due to convection)

Dimensionless Numbers (momentum, energy, mass)

- $\text{Re} - \text{Reynolds} = \frac{\rho VD}{\mu} = \frac{VD}{\nu}$
- $\text{Fr} - \text{Froude} = \frac{V^2}{gD}$
- $\text{Pe} - \text{Péclet}_h = \text{RePr} = \frac{\hat{C}_p \rho VD}{k} = \frac{VD}{\alpha}$
- $\text{Pe} - \text{Péclet}_m = \text{ReSc} = \frac{VD}{D_{AB}}$

These numbers from the governing equations tell us about the relative importance of the terms they precede in the microscopic balances (***scenario*** properties).

- $\text{Pr} - \text{Prandtl} = \frac{\hat{C}_p \mu}{k} = \frac{\nu}{\alpha}$
- $\text{Sc} - \text{Schmidt} = \text{LePr} = \frac{\mu}{\rho D_{AB}} = \frac{\nu}{D_{AB}}$
- $\text{Le} - \text{Lewis} = \frac{\alpha}{D_{AB}}$

These numbers compare the magnitudes of the diffusive transport coefficients ν, α, D_{AB} (***material*** properties).

- $f - \text{Friction Factor} = \frac{\mathcal{F}_{drag}}{\left(\frac{1}{2}\rho V^2\right)A_c}$
- $\text{Nu} - \text{Nusselt} = \frac{hD}{k}$
- $\text{Sh} - \text{Sherwood} = \frac{k_m D}{D_{AB}}$

These numbers are defined to help us build transport data correlations based on the fewest number of grouped (dimensionless) variables (***scenario*** properties).

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

Scales the time evolution of the temperature profile relative to the material's thermal properties, $\alpha = k/\rho\hat{C}_p$ (thermal diffusion time).

We indicated that there are three ranges of Biot number to consider:

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

$\text{Bi} = \frac{D_{char}/k}{1/h}$

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

We have been exploring these ranges

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Summary of *low* Biot number scenarios:

Unsteady State Heat Transfer: Ranges of Biot Number

Lumped parameter analysis

(negligible internal resistance)

Low Bi: high k, low h

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

$$D_{char} \equiv \frac{\text{volume}}{\text{area}} = \frac{V_{sys}}{A_{sys}}$$

This is always the D_{char} we use for the Biot number in lumped parameter analysis. We use different D_{char} in other cases, however.

$$\text{Bi}_{LP} = \frac{hV_{sys}}{kA_{sys}}$$

The test for applicability of the lumped parameter analysis is:

$$\text{Bi}_{LP} = \frac{hV_{sys}}{kA_{sys}} < 0.1$$

Two things to remember about lumped parameter analysis:

1. $D_{char} \equiv \frac{V}{A}$
2. Only valid for $\text{Bi}_{LP} < 0.1$

Low Biot number ⇒ temperature is uniform in the body; resistance is all ***external***; solve for $T_{body}(t)$

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Now: *Moderate* and *High* Biot number behavior

We indicated that there are three ranges of Biot number to consider:

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

$\text{Bi} = \frac{D_{char}/k}{1/h}$

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.	**High Bi:** low k, high h
At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.	**Moderate Bi:** nether process dominates
At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).	**Low Bi:** high k, low h

We now explore these ranges

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$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

$\text{Bi} = \frac{D/k}{1/h}$

Quantifies the tradeoffs between the resistance to heat flow (due to conductivity, D/k) and the resistance to heat flow at the boundary ($1/h$)

When both processes affect the outcomes, the full solution may be necessary. For uniform starting temperatures, the solutions are published.

the body. high h

$T = T(x, y, z, t)$

hard BC

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.	**Moderate Bi:** nether process dominates
At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).	**Low Bi:** high k, low h

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All solutions for the temperature *profile* (temperature *distribution*, temperature *field*, $T(t,\underline{x})$) begin with the

Microscopic Energy Balance

Bi – Biot Number $= \frac{hD}{k}$

$\text{Bi} = \frac{D/k}{1/h}$

Quantifies the tradeoffs between the resistance to heat flow (due to conductivity, D/k) and the resistance to heat flow at the boundary ($1/h$)

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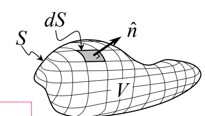
Low Bi: high k, low h

1D Heat Transfer: Unsteady State

Microscopic Energy Balance

As for the derivation of the microscopic momentum balance, the microscopic energy balance is derived on an arbitrary volume, *V*, enclosed by a surface, *S*.



Gibbs notation:

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S$$

see handout for component notation

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1D Heat Transfer: Unsteady State

Microscopic Energy Balance

(General Energy Transport Equation)

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S$$

- rate of change: $\frac{\partial T}{\partial t}$
- convection: $\underline{v}\cdot\nabla T$
- conduction (all directions): $k\nabla^2 T$
- source: S (energy generated per unit volume per time)

velocity must satisfy equation of motion, equation of continuity

see handout for component notation

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Equation of energy for Newtonian fluids of constant density, ρ, and thermal conductivity, k, with source term (source could be viscous dissipation, electrical energy, chemical energy, etc., with units of energy/(volume time)).

Source: R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Processes*, Wiley, NY, 1960, page 319.

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

thermal diffusivity $\alpha \equiv \frac{k}{\rho \hat{C}_p}$

Gibbs notation (vector notation)

$$\left(\frac{\partial T}{\partial t} + \underline{v} \cdot \nabla T\right) = \frac{k}{\rho \hat{C}_p} \nabla^2 T + \frac{S}{\rho \hat{C}_p}$$

Cartesian (xyz) coordinates:

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

Cylindrical (rθz) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

Spherical (rθϕ) coordinates:

$$\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial T}{\partial \phi} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2}\right.$$

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1D Heat Transfer: Unsteady State

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$$ thermal diffusivity

what are the boundary conditions? initial conditions?

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1D Unsteady Heat Transfer: Moderate and High Biot Number

Boundary Conditions:

- Nonzero resistance to heat transfer: $\left|\frac{q_x}{A}\right| = |h(T - T_b)|$
- No resistance to heat transfer: $h \rightarrow \infty$, or, equivalently, temperature known at the boundary
- Insulated boundary: $\left|\frac{q_x}{A}\right| = 0$, or, equivalently, the temperature field is symmetrical at the boundary

1D Heat Transfer: Unsteady State

Microscopic Energy Equation in Cartesian Coordinates

$$\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} = \frac{k}{\rho \hat{C}_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{S}{\rho \hat{C}_p}$$

$\alpha \equiv \frac{k}{\rho \hat{C}_p} =$ thermal diffusivity

what are the boundary conditions? initial conditions?

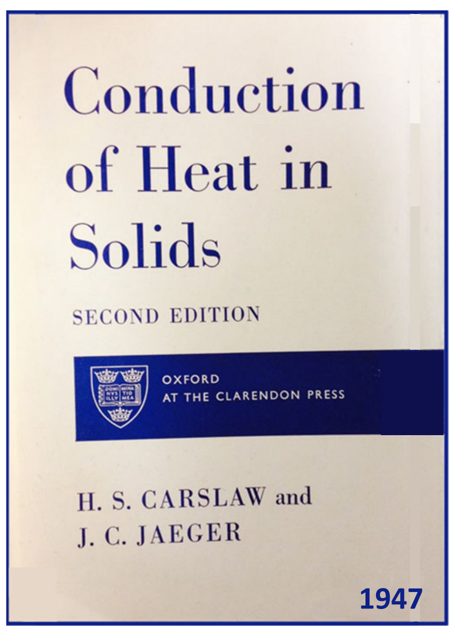
Initial Conditions:

- Initial temperature distribution uniform: T_0
- Initial temperature distribution known: $T(o, \underline{x})$

1D Unsteady Heat Transfer: Moderate and High Biot Number

For the most common geometries, initial conditions, and boundary conditions, the models have been solved and may be looked-up.

For quick, "back of the envelope" calculations, researchers (Gurney and Lurie, Heisler) have created easy-to-use plots of the predictions.



1D Unsteady Heat Transfer: Finite Bodies, Short Cut Solutions

Finite 1D Unsteady Heat Transfer, $T = T(t, x)$ or $T = T(t, r)$

Initial: Uniform initial temperature T_0; BC: exposed to bulk temperature T_1; h known

- Flat plate long, wide, thickness $= 2x_1$ $T = T(t, x)$ $Y = Y(X, n)$
- Cylinder long, radius $= x_1$ $T = T(t, r)$ $Y = Y(X, n)$
- Sphere radius $= x_1$ $T = T(t, r)$ $Y = Y(X, n)$

Note: $D_{char} = x_1$, <u>*NOT*</u> V/A

Heisler Charts $(T(\mathrm{Fo}, 0))$ and **Gurney-Lurie Charts** $(T(\mathrm{Fo}, n))$ are graphical representations of solutions of a *particular* unsteady heat-transfer problem for various values of $m = 1/\mathrm{Bi}$.

$$\mathrm{Bi} = \frac{hD_{char}}{k} = \frac{hx_1}{k} = \frac{1}{m}$$

$$\mathrm{Fo} = \frac{\alpha t}{x_1^2} = X$$

$$\frac{x}{x_1} = \frac{r}{x_1} = n$$

$$\frac{T_1 - T}{T_1 - T_0} = Y \quad \left(\frac{T - T_0}{T_1 - T_0} = 1 - Y\right)$$

1D Unsteady Heat Transfer: Finite Bodies

Gurney and Lurie Charts

Ref: Geankoplis, 4th Ed, 2003

Initial: Uniform initial temperature T_0; BC: bulk temperature T_1; $T\left(m = \frac{1}{\mathrm{Bi}}, n = \frac{x}{x_1}, \mathrm{Fo}\right)$

- Flat plate long, wide, thickness $= 2x_1$ $T = T(t, x)$ $Y = Y(X, n)$
- Cylinder long, radius $= x_1$ $T = T(t, r)$ $Y = Y(X, n)$
- Sphere radius $= x_1$ $T = T(t, r)$ $Y = Y(X, n)$

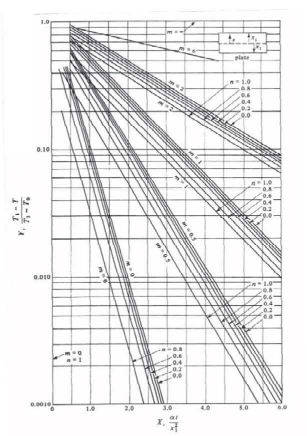


FIGURE 5.3-5. Unsteady-state heat conduction in a large flat plate. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., 15, 1170 (1923).]

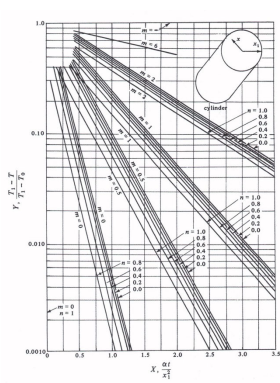


FIGURE 5.3-7. Unsteady-state heat conduction in a long cylinder. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., 15, 1170 (1923).]

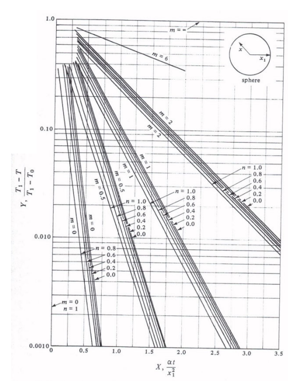


FIGURE 5.3-9. Unsteady-state heat conduction in a sphere. [From H. P. Gurney and J. Lurie, Ind. Eng. Chem., 15, 1170 (1923).]

$\mathrm{Fo} = \frac{\alpha t}{x_1^2} = X$

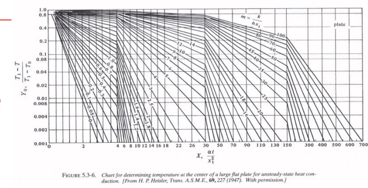
1D Unsteady Heat Transfer: Finite Bodies

Heisler Charts

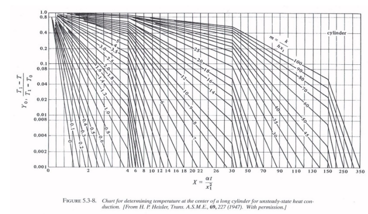
Initial: Uniform initial temperature T_0
BC: Exposed to bulk temperature T_1
h known
Plots of temperature at the center

$$T\left(m = \frac{1}{\mathrm{Bi}}, x = 0, \mathrm{Fo}\right)$$

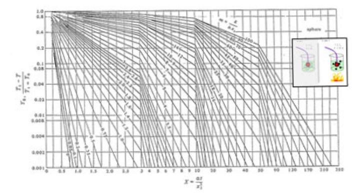
Flat plate



Cylinder



Sphere



Ref: Geankoplis, 4th Ed, 2003

$$\mathrm{Fo} = \frac{\alpha t}{x_1^2} = X$$

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Heisler Chart for Unsteady State Heat Transfer to a Sphere

(Geankoplis; see also Wikipedia)

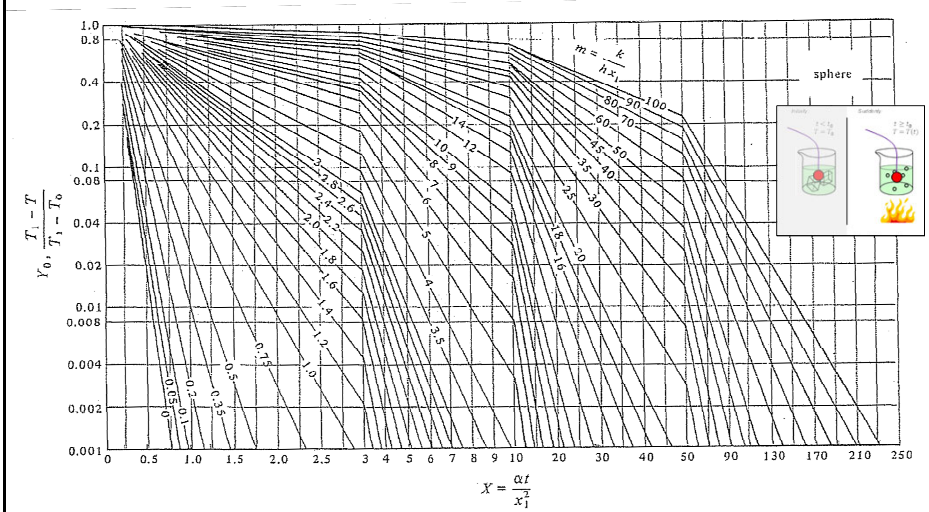

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]*

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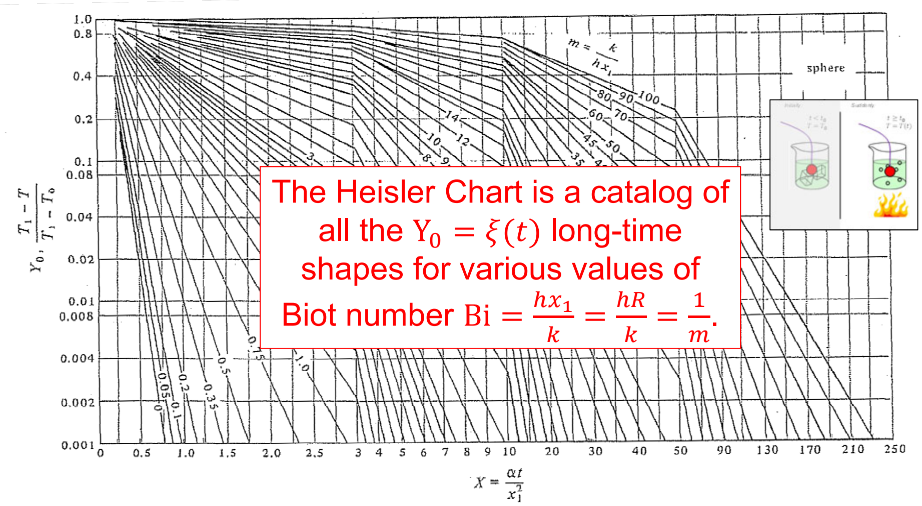
From Geankpolis, 4th edition, page 374

Heisler Chart for Unsteady State Heat Transfer to a Sphere

(Geankoplis; see also Wikipedia)



The Heisler Chart is a catalog of all the $Y_0 = \xi(t)$ long-time shapes for various values of Biot number $\mathrm{Bi} = \frac{hx_1}{k} = \frac{hR}{k} = \frac{1}{m}$.

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]*

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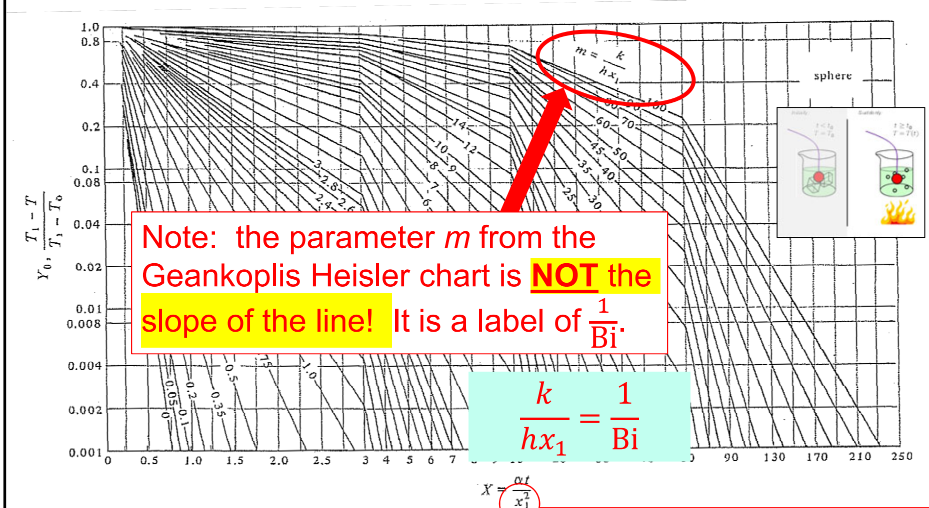
From Geankpolis, 4th edition, page 374

Heisler Chart for Unsteady State Heat Transfer to a Sphere

(Geankoplis; see also Wikipedia)



Note: the parameter *m* from the Geankoplis Heisler chart is **NOT** the slope of the line! It is a label of $\frac{1}{\mathrm{Bi}}$.

$$\frac{k}{hx_1} = \frac{1}{\mathrm{Bi}}$$

Also, x_1 is the sphere radius

FIGURE 5.3-10. *Chart for determining the temperature at the ... heat conduction. [From H. P. Heisler, Trans. ... permission.]*

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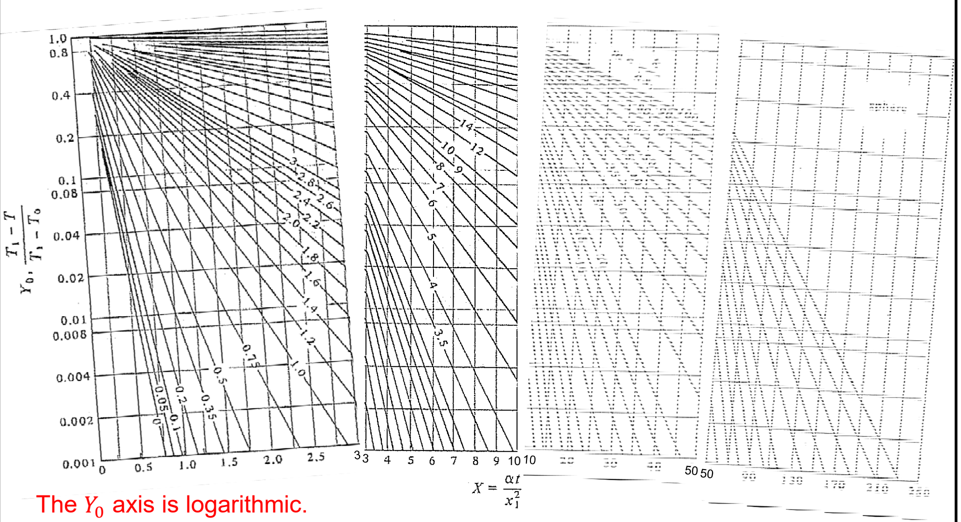
From Geankpolis, 4th edition, page 374

Heisler charts (Geankoplis; see also Wikipedia)

Note also: think of it as 4 separate graphs



The Y_0 axis is logarithmic.

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction.* [*From H. P. Heisler, Trans. A.S.M.E.*, **69**, *227 (1947). With permission.*]

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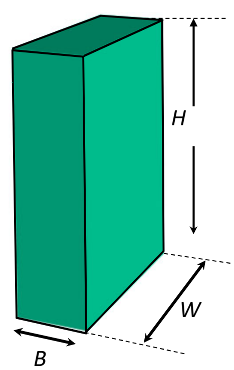
From Geankpolis, 4th edition, page 374

1D Unsteady Heat Transfer: In a Slab

Example: A tall, wide rectangular ~~copper~~ 304 stainless steel slab, ~~five~~ ten centimeters thick, uniformly at a temperature of 17^oC, is suddenly exposed on all sides to ~~air~~ water ($h = 1380\ W/m^2K$) at 45^oC. After ~~30~~ 20 minutes, what is the temperature at the middle of the slab?

(numbers were changed in 2021 to improve the problem; the old numbers appear in the video)



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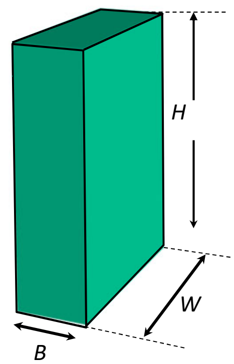
1D Unsteady Heat Transfer: In a Slab

Example: A tall, wide rectangular 304 stainless steel slab, ten centimeters thick, uniformly at a temperature of 17^oC, is suddenly exposed on all sides to water ($h = 1380\ W/m^2K$) at 45^oC. After 20 minutes, what is the temperature at the middle of the slab?

Let's try



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1D Unsteady Heat Transfer: In a Slab

Example: A long, wide rectangular slab of butter ($46\ mm$ thick) at 4.4^oC is removed from refrigeration and placed on a table at room temperature. After five hours, what are the butter temperatures at the middle of the slab and at the bottom of the slab (in contact with the table)?

Properties of butter:

$$k_{butter} = 0.197 \frac{W}{mK}$$

$$\hat{C}_{p\,butter} = 2.30 \frac{kJ}{kg\ K}$$

$$\rho_{butter} = 998 \frac{kg}{m^3}$$

Conditions of the room:

$$T_{bulk} = 24^oC$$

$$h_{conv} = 8.5 \frac{W}{m^2K}$$

Handnotes: https://pages.mtu.edu/~fmorriso/cm3120/2021_solve_Gurney-Lurie_problem_butter.pdf

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1D Unsteady Heat Transfer: In a Slab

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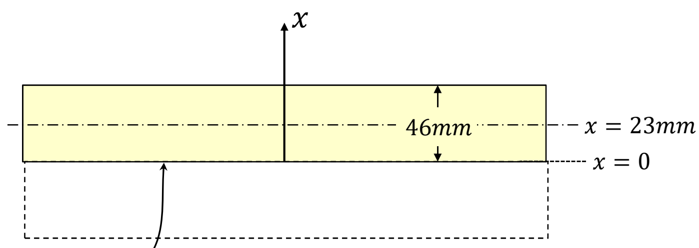

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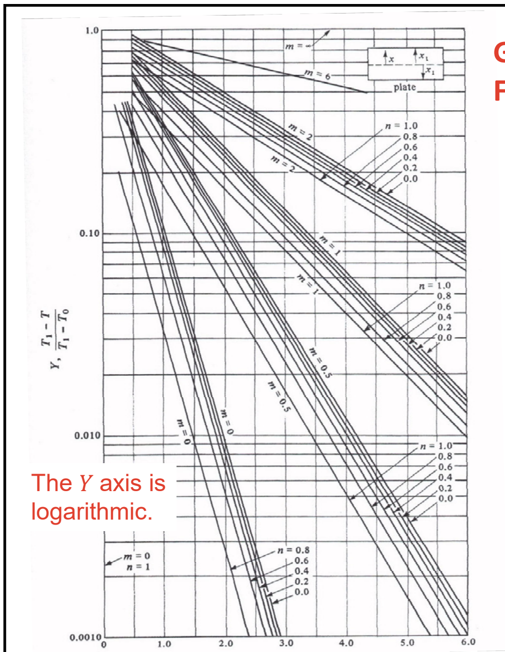

Gurney and Lurie Chart
Finite 1D Unsteady Heat Transfer

Initial: Uniform initial temperature T_0

BC: Exposed to bulk temperature T_1

h known

- Flat plate, long, wide
- thickness $2x_1$
- $T = T(t, x)$
- $Y = Y\left(\text{Fo}, n = \frac{x}{x_1}\right)$

$$T\left(m = \frac{1}{\text{Bi}}, n = \frac{x}{x_1}, \text{Fo}\right)$$

Ref: Geankoplis, 4th Ed, 2003

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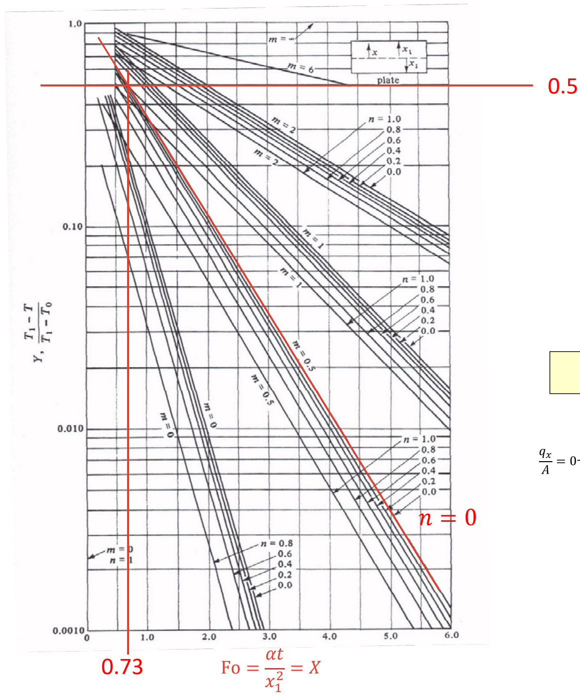
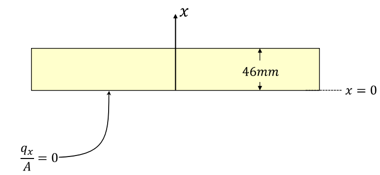


At the bottom,
$x = 0$ after 5h:

$$\frac{T_1 - T}{T_1 - T_0} = 0.5$$

$T = 14^oC$



Ref: Geankoplis, 4th Ed, 2003

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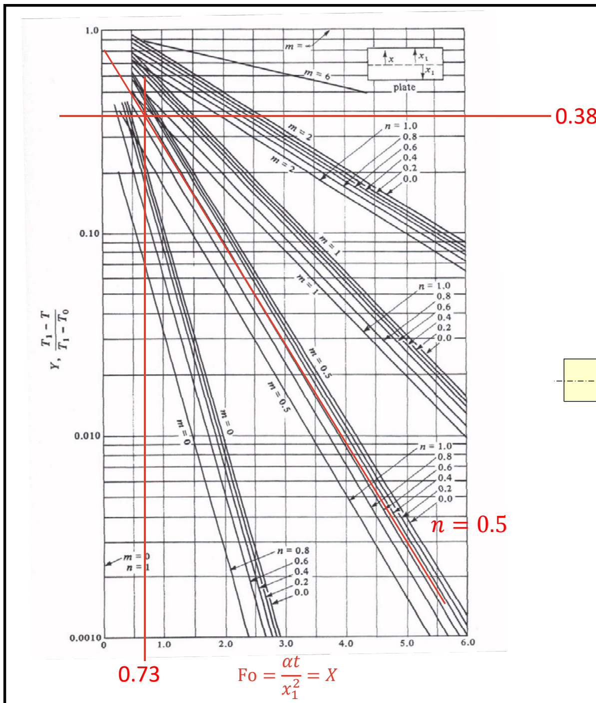
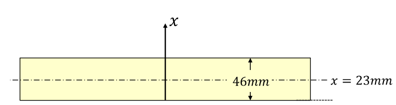


At the middle,
$x = 23mm$
after 5 h:

$$\frac{T_1 - T}{T_1 - T_0} = 0.38$$

$T = 17^oC$



Ref: Geankoplis, 4th Ed, 2003

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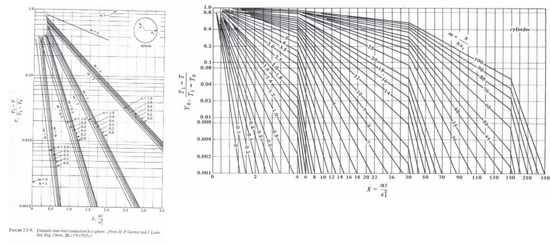
1D Unsteady Heat Transfer: Moderate and High Biot Number

Summary of low Moderate/High Biot number scenarios **(short-cut solutions):**



- Good for quick "back of the envelope" calculations
- Uniform starting temperature
- If surface temperature is known, Bi $= \infty$, that is, $m = 0$
- If boundary is insulated (flat plate), set that boundary as the center of the plate ($q_x/A = 0$) boundary condition)
- Limited sig figs (*i.e. 1-2*, due to reading charts, interpolating)

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Next: Complete solutions for *Moderate* and *High* Biot number behavior

We indicated that there are three ranges of Biot number to consider:

$$\text{Bi} = \frac{D_{char}/k}{1/h}$$

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

We now explore these ranges

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