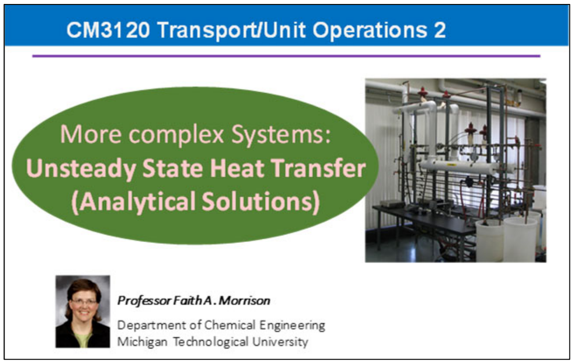
Last time, Unsteady State Heat Transfer

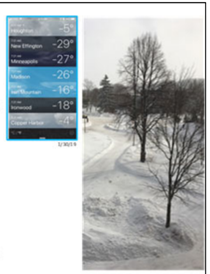


 1

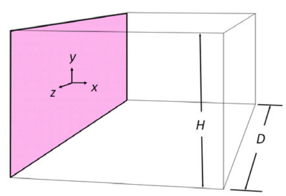
Solution Summary:

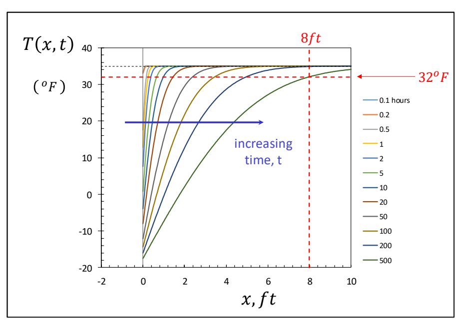
Example:
When will my pipes freeze?

The temperature has been 35°F for a while now, sufficient to chill the ground to this temperature for many tens of feet below the surface. Suddenly the temperature drops to -20°F. How long will it take for freezing temperatures (32°F) to reach my pipes, which are 8 *ft* under ground?



Answer:
$t = 509\ hours \approx 21\ days$





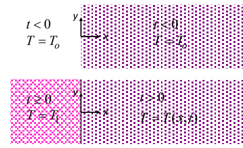
 2

Dimensional Analysis, Unsteady State Convection

Non-dimensionalize (eqns, BCs)

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2}\right)$$

$$q_x = -k\frac{\partial T}{\partial x} = hA(T_1 - T)$$



non-dimensional variables:

position: $x^* \equiv \frac{x}{D}$

temperature: $Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$

time: $t^* \equiv \frac{\alpha t}{D^2}$

This dimensionless time is called Fourier number Fo.

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D^2}$$

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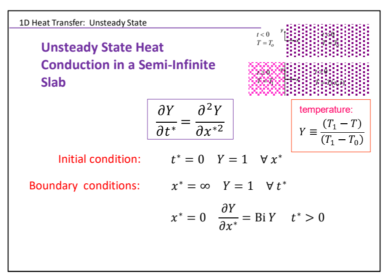
In dimensionless form, we see that this problem reduces to

$$Y = Y\left(\frac{x}{D}, \text{Fo}, \text{Bi}\right)$$


1D Heat Transfer: Unsteady State

Unsteady State Heat Conduction in a Semi-Infinite Slab

$$\frac{\partial Y}{\partial t^*} = \frac{\partial^2 Y}{\partial x^{*2}}$$

temperature: $Y \equiv \frac{(T_1 - T)}{(T_1 - T_0)}$

Initial condition: $t^* = 0 \quad Y = 1 \quad \forall\, x^*$

Boundary conditions: $x^* = \infty \quad Y = 1 \quad \forall\, t^*$

$x^* = 0 \quad \frac{\partial Y}{\partial x^*} = \text{Bi}\, Y \quad t^* > 0$

Dimensionless quantities:

$Y = \frac{(T_1 - T)}{(T_1 - T_0)}$ — **Y** (dimensionless temperature interval)

$t^* = \text{Fo} = \frac{\alpha t}{D^2}$ — **Fourier number** (dimensionless time)

$x^* = \frac{x}{D}$

$\text{Bi} = \frac{hD}{k}$ — **Biot number** (pronounced BEE-OH) Ratio of heat transfer at the boundary to heat transfer within the solid. This is a transport issue.

4

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

5

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat

High Bi: low k,

When the temperature is uniform in the body, we can do a macroscopic energy balance to solve many problems of interest. This is called a "**lumped parameter analysis**."

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

6

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

High Bi: low k, high h

At moderate Bi, heat transfer is affected by ... body and t... the surface...

Moderate Bi:

At low Bi, t... uniform in ... transfer is limited by rate of heat transfer to the surface (h).

high k, low h

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

 7

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

When both processes affect the outcomes, the full solution may be necessary. For uniform starting temperatures, the solutions are published.

the body.

high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

Moderate Bi: nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

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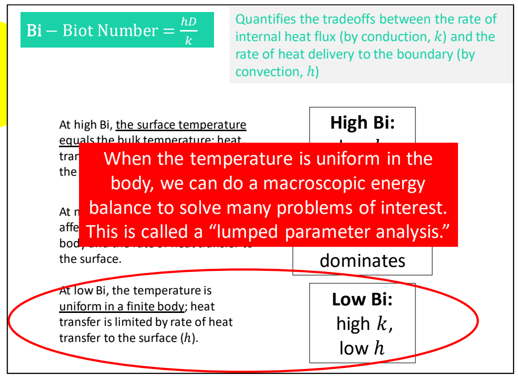
1. Lumped parameter analysis

Low Bi:
high k,
low h

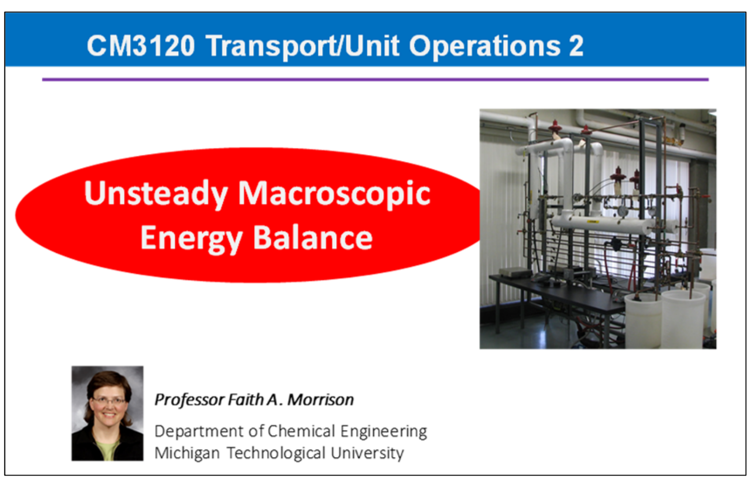


At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

$$D_{char} \equiv \frac{\text{volume}}{\text{area}} = \frac{V_{sys}}{A}$$

This is always the D_{char} we use for the lumped parameter analysis. We use different D_{char} in other cases, however.

This topic is part of a more general subject:

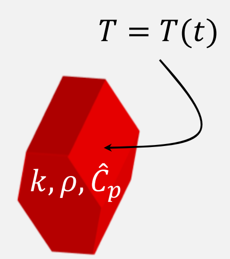


Unsteady State Heat Transfer: Low Biot Number

Example: Quench cooling of a manufactured part.

If a piece of steel with $T = T_0$ is dropped into a large, well stirred reservoir of fluid at bulk temperature T_∞, what is the temperature of the steel as a function of time?

- k = large, which means that there is no internal resistance to heat transfer in the part
- Therefore, we are NOT calculating a temperature profile (internal T is uniform)
- **⇒ Use Unsteady, Macroscopic Energy Balance**



Fluid temperature= T_∞

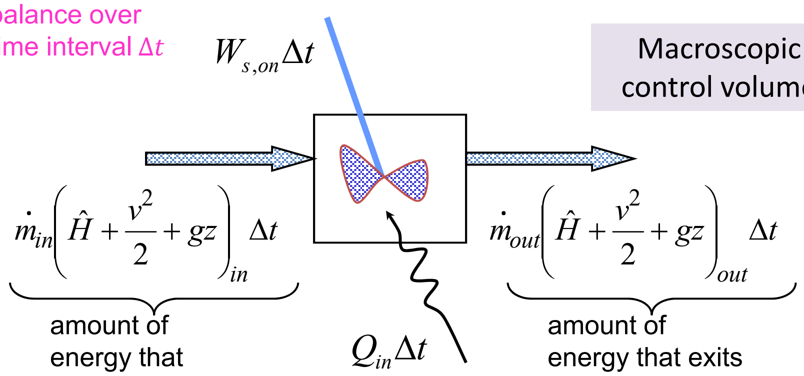
Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance

see Felder and Rousseau or Himmelblau

balance over time interval Δt

Macroscopic control volume



amount of energy that enters with the flow between t and $t + \Delta t$

amount of energy that exits with the flow between t and $t + \Delta t$

Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance

accumulation = input − output

$$\frac{d}{dt}\left(U_{sys}+E_{k,sys}+E_{p,sys}\right)=-\Delta H-\Delta E_k-\Delta E_p+Q_{in}+W_{s,on}$$

Background:
pages.mtu.edu/~fmorriso/cm310/IFMWeb
AppendixDMicroEBalanceMorrison.pdf

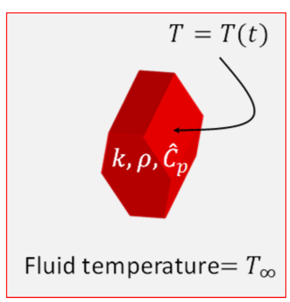
Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance

accumulation = input − output

$$\frac{d}{dt}\left(U_{sys}+E_{k,sys}+E_{p,sys}\right)=-\Delta H-\Delta E_k-\Delta E_p+Q_{in}+W_{s,on}$$

How do we apply this balance to our current problem?

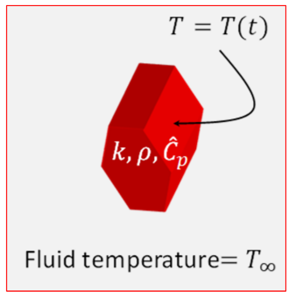


Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance

$$accumulation = input - output$$

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right) = -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$



You try.

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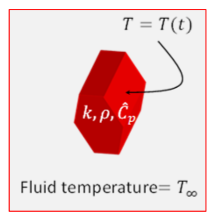
Unsteady State Heat Transfer: Low Biot Number



Unsteady Macroscopic Energy Balance

$$accumulation = input - output$$

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right) = -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$

negligible ($E_{k,sys}$, $E_{p,sys}$) — no flow (ΔH, ΔE_k, ΔE_p) — no shafts ($W_{s,on}$)

For negligible changes in E_p and E_k, no flow, no phase change, no chemical rxn, and no shafts:

$$\frac{dU_{sys}}{dt} = Q_{in}$$

$$\rho V_{sys} \hat{C}_v \frac{dT_{sys}}{dt} = Q_{in}$$

$\hat{C}_v \approx \hat{C}_p$ for liquids, solids

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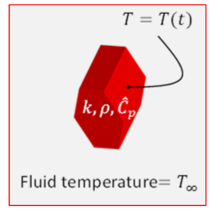
Unsteady State Heat Transfer: Low Biot Number



Unsteady Macroscopic Energy Balance

How do we quantify the heat in Q_{in}?

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right) = -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$

negligible — no flow — no shafts

For negligible changes in E_p and E_k, no flow, no phase change, no chemical rxn, and no shafts:

$$\frac{dU_{sys}}{dt} = Q_{in}$$

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$\hat{C}_v \approx \hat{C}_p$ for liquids, solids

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Unsteady Macroscopic Energy Balance

accumulation = input − output

Q_{in} = Heat ***in***to the chosen macroscopic control volume

$$\frac{d}{dt}\left(U_{sys} + E_{k,sys} + E_{p,sys}\right) = -\Delta H - \Delta E_k - \Delta E_p + Q_{in} + W_{s,on}$$

$Q_{in} = \sum_i q_{in,i}$ comes from a variety of sources:

- Thermal conduction: $q_{in} = -kA\frac{dT}{dx}$
- Convection heat xfer: $q_{in} = hA(T_b - T)$
- Radiation: $q_{in} = \varepsilon\sigma A\left(T_{surroundings}^4 - T_{surface}^4\right)$
- Electric current: $q_{in} = I^2 R_{elec} L$
- Chemical Reaction: $q_{in} = S_{rxn} V_{sys}$

$S[=]\frac{\text{energy}}{\text{time volume}}$

pages.mtu.edu/~fmorriso/cm310/IFMWebAppendixDMicroEBalanceMorrison.pdf
Incropera and DeWitt, 6th edition p18

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Unsteady Macroscopic Energy Balance

accumulation = input − output

Q_{in} = Heat *into* the chosen macroscopic control volume

$$\frac{d}{dt}\left(U_{sys}+E_{k,sys}+E_{p,sys}\right)=-\Delta H-\Delta E_k-\Delta E_p+Q_{in}+W_{s,on}$$

$Q_{in}=\sum_i q_{in,i}$ comes from a variety of sources:

Signs must match transfer from outside (bulk fluid) to inside (metal)

- Thermal conduction: $q_{in}=-kA\frac{dT}{dx}$
- Convection heat xfer: $q_{in}=hA(T_b-T)$
- Radiation: $q_{in}=\varepsilon\sigma A\left(T_{surroundings}^4-T_{surface}^4\right)$
- Electric current: $q_{in}=I^2R_{elec}L$
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Unsteady Macroscopic Energy Balance

$$\frac{d}{dt}\left(U_{sys}+E_{k,sys}+E_{p,sys}\right)=-\Delta H-\Delta E_k-\Delta E_p+Q_{in}+W_{s,on}$$

$Q_{in}=\sum_i q_{in,i}$ comes from a variety of sources:

- **Thermal conduction: $q_{in}=-kA\frac{dT}{dx}$**
  e.g. device held by bracket; a solid phase that extends through boundaries of control volume
- **Convection heat xfer: $q_{in}=hA(T_b-T)$**
  e.g. device dropped in stirred liquid; forced air stream flows past, natural convection occurs outside system; phase change at boundary
- **Radiation: $q_{in}=\varepsilon\sigma A\left(T_{surroundings}^4-T_{surface}^4\right)$**
  e.g. device at high temp. exposed to a gas/vacuum; hot enough to produce nat. conv.=possibly hot enough for radiation
- **Electric current: $q_{in}=I^2R_{elec}L$**
  e.g. if electric current is flowing within the device/control volume/ system
- **Chemical Reaction: $q_{in}=S_{rxn}V_{sys}$**
  e.g. if a homogeneous reaction is taking place <u>throughout</u> the device/ control volume/system

S-B constant: $\sigma=5.676\times10^{-8}\frac{W}{m^2K^4}$

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Unsteady Macroscopic Energy Balance

accumulation = input − output

Q_{in} = Heat ***in***to the chosen macroscopic control volume

$$\frac{d}{dt}\left(U_{sys}+E_{k,sys}+E_{p,sys}\right)=-\Delta H-\Delta E_k-\Delta E_p+Q_{in}+W_{s,on}$$

$Q_{in}=\sum_i q_{in,i}$ comes from a variety of sources:

- X Thermal conduction: $q_{in}=-kA\frac{dT}{dx}$
- ✓ Convection heat xfer: $q_{in}=hA(T_b-T)$
- X Radiation: $q_{in}=\varepsilon\sigma A\left(T_{surroundings}^4-T_{surface}^4\right)$
- X Electric current: $q_{in}=I^2R_{elec}L$
- X Chemical Reaction: $q_{in}=S_{rxn}V_{sys}$

$S[=]\frac{\text{energy}}{\text{time volume}}$

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Incropera and DeWitt, 6th edition p18

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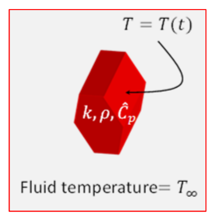
Unsteady State Heat Transfer: Low Biot Number



Unsteady Macroscopic Energy Balance Applied to cooling steel part:

$$\rho V_{sys}\hat{C}_p\frac{dT_{sys}}{dt}=Q_{in}$$

The temperature changes in the part are due to the heat loss

The heat loss depends on the heat-transfer coefficient from the part to the environment

$$Q_{in}=Ah(T_\infty-T)$$

$\hat{C}_v\approx\hat{C}_p$ for liquids, solids

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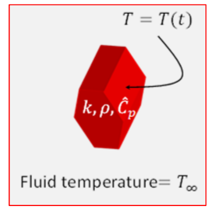
Unsteady State Heat Transfer: Low Biot Number



Unsteady Macroscopic Energy Balance Applied to cooling steel part:

$$\rho V_{sys} \hat{C}_p \frac{dT_{sys}}{dt} = Q_{in}$$

The temperature changes in the part are due to the heat loss

The heat loss depends on the heat-transfer coefficient from the part to the environment

$$Q_{in} = Ah(T_\infty - T)$$

You solve.

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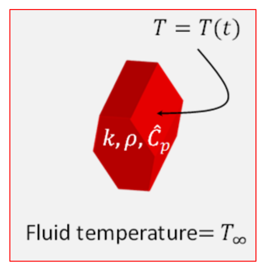
Unsteady State Heat Transfer: Low Biot Number



Unsteady Macroscopic Energy Balance Applied to cooling steel part:

$$\frac{(T_\infty - T)}{(T_\infty - T_0)} = e^{-\left(\frac{hA}{\rho \hat{C}_p V}\right) t}$$

$V_{sys} = V$

$$\ln\left(\frac{(T_\infty - T)}{(T_\infty - T_0)}\right) = -\left(\frac{hA}{\rho \hat{C}_p V}\right) t$$

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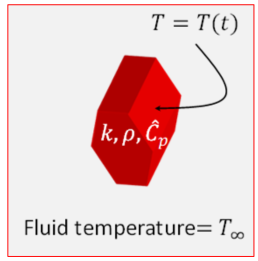
Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance Applied to cooling steel part:



$$\frac{(T_\infty - T)}{(T_\infty - T_0)} = e^{-\left(\frac{hA}{\rho \hat{C}_p V}\right)t}$$

$V_{sys} = V$

$$\ln\left(\frac{(T_\infty - T)}{(T_\infty - T_0)}\right) = -\left(\frac{hA}{\rho \hat{C}_p V}\right) t$$

In dimensionless form?

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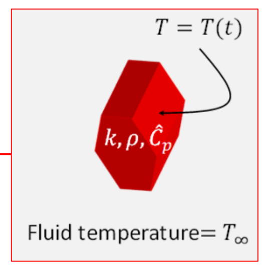
Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance Applied to cooling steel part:



$$\frac{(T_\infty - T)}{(T_\infty - T_0)} = e^{-\left(\frac{hA}{\rho \hat{C}_p V}\right)t}$$

$$\frac{hAt}{\rho \hat{C}_p V} = \left(\frac{h}{k}\right)\left(\frac{k}{\rho \hat{C}_p}\right)\left(\frac{A}{V}\right)t = \left(\frac{\text{Bi}}{D_{char}}\right)\alpha\left(\frac{t}{D_{char}}\right) = \text{Bi Fo}$$

$$D_{char} \equiv \frac{\text{volume}}{\text{area}} = \frac{V}{A}$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

$$\text{Bi} - \text{Biot Number} = \frac{hD_{\text{char}}}{k}$$

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D_{\text{char}}^2}$$

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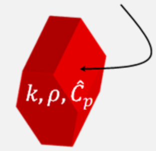
Unsteady State Heat Transfer: Low Biot Number

Unsteady Macroscopic Energy Balance
Applied to cooling steel part:

$T = T(t)$



Fluid temperature= T_∞

$$Y = \frac{(T_\infty - T)}{(T_\infty - T_0)} = e^{-\text{Bi Fo}}$$

WRF p279

Lumped parameter analysis

https://en.wikipedia.org/wiki/Lumped_element_model

$$D_{char} \equiv \frac{\text{volume}}{\text{area}} = \frac{V}{A}$$

thermal diffusivity

$$\alpha \equiv \frac{k}{\rho \hat{C}_p}$$

Lumped parameter analysis:

$$\text{Bi} - \text{Biot Number} = \frac{hD_{\text{char}}}{k} = \frac{hV}{kA}$$

$$\text{Fo} - \text{Fourier Number} = \frac{\alpha t}{D_{\text{char}}^2}$$

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Unsteady State Heat Transfer

Summary

$\text{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat tran... the...

High Bi:

When the temperature is uniform in the body, we can do a macroscopic energy balance to solve many problems of interest. This is called a "lumped parameter analysis."

At ... affe... bod... the surface.

dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h).

Low Bi: high k, low h

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\text{Bi} = hV/kA < 0.1$

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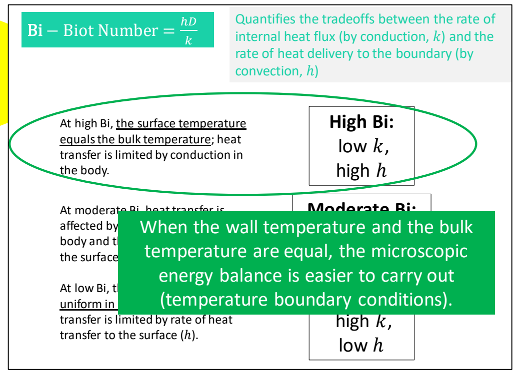
2. Negligible Surface Resistance



High Bi:
low k,
high h

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

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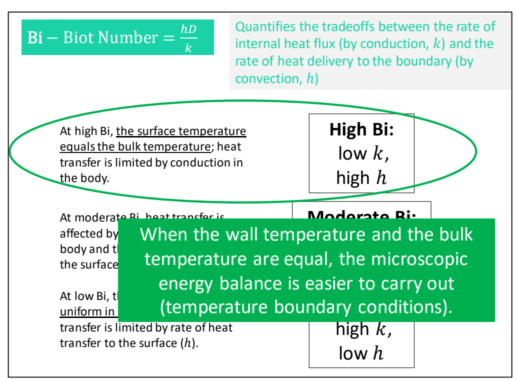
Negligible Surface Resistance



High Bi:
low k,
high h

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body.

$$\lim_{h\to\infty} |T_{bulk} - T_{wall}| = 0$$
$$\Rightarrow T_{wall} = T_{bulk}$$

We have done many examples with constant temperature boundary conditions.

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Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\mathrm{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body. — **High Bi:** low k, high h

When the wall temperature and the bulk temperature are equal, the microscopic energy balance is easier to carry out (temperature boundary conditions).

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h). — **Low Bi:** high k, low h

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\mathrm{Bi} = hV/kA < 0.1$

3. No Mechanism Dominates

Moderate Bi: nether process dominates

Bi – Biot Number $= \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

When both processes affect the outcomes, the full solution may be necessary; for uniform starting temperatures there are charts.

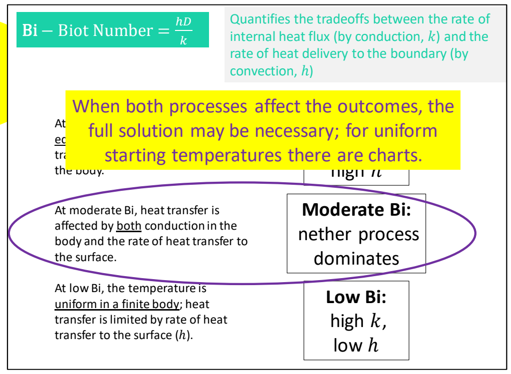
At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface. — **Moderate Bi:** nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h). — **Low Bi:** high k, low h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface.

3. No Mechanism Dominates

Moderate Bi: nether process dominates



At moderate Bi, heat transfer is affected by <u>both</u> conduction in the body and the rate of heat transfer to the surface.

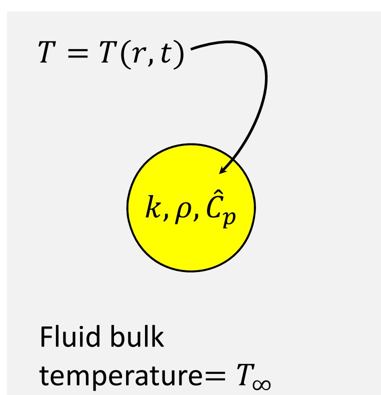
This is the most complicated set of cases.

Unsteady State Heat Transfer: Intermediate Biot Number

No Mechanism Dominates

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (k) and external (h) resistances are important
- We need to match *measurable* quantities with *calculable* quantities
- ⇒ ***Microscopic*** Energy Balance
- ⇒*Uncertainty considerations*



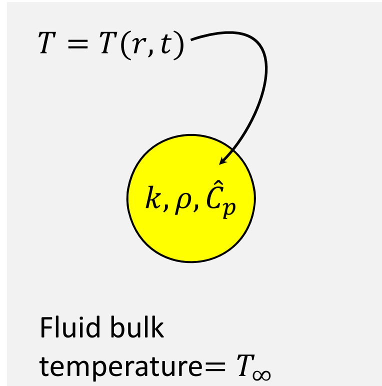
Unsteady State Heat Transfer: Intermediate Biot Number

No Mechanism Dominates

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (k) and external (h) resistances are important
- We need to match *measurable* quantities with *calculable* quantities
- ⇒ ***Microscopic*** Energy Balance
- ⇒ *Uncertainty considerations*

?



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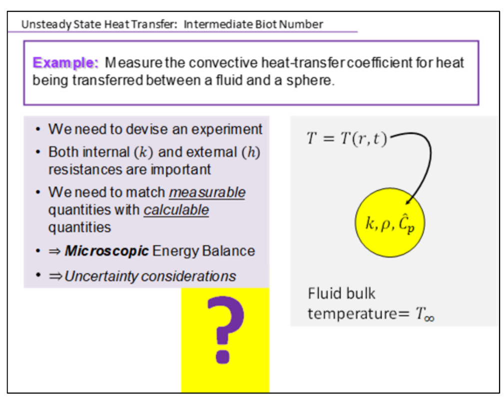
Unsteady State Heat Transfer: Intermediate Biot Number
Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.
We need to devise an experiment
Both internal (k) and external (h) resistances are important
We need to match measurable quantities with calculable quantities
⇒ Microscopic Energy Balance
⇒ Uncertainty considerations
$T = T(r,t)$
$k, \rho, \hat{C}_p$
Fluid bulk temperature= T_∞

Thinking

- Create an unsteady state heat transfer situation…
- Measure . . .?
- Compare . . .?
- Consider uncertainty in measurements . . . ?

You try.

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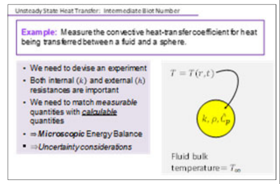
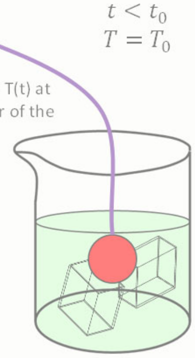
Unsteady State Heat Transfer: Intermediate Biot Number



Experiment: Measure $T(t)$ at the center of a sphere $(r = 0)$:

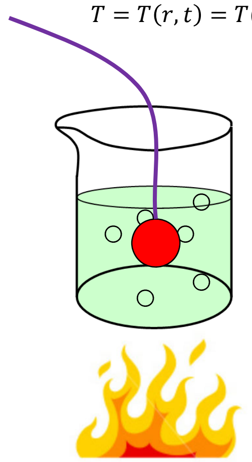
Initially:

$t < t_0$
$T = T_0$



Suddenly:

$t \geq t_0$
$T = T(r,t) = T(0,t)$



Unsteady state heat transfer takes place.

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Unsteady State Heat Transfer: Intermediate Biot Number

Experiment: Measure $T(t)$ at the center of a sphere $(r = 0)$:

Initially:

$t < t_0$
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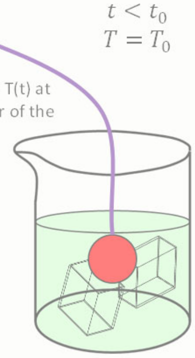

Suddenly:

$t \geq t_0$
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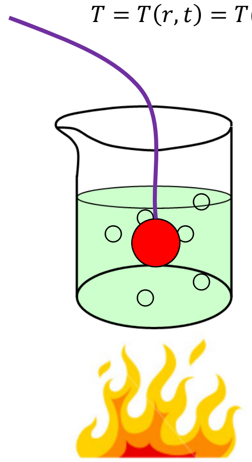

Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
2.11E-01	7.44E+00
3.09E-01	7.44E+00
4.09E-01	7.57E+00
5.24E-01	7.46E+00
6.23E-01	7.49E+00
7.39E-01	7.53E+00
8.37E-01	7.46E+00
9.54E-01	7.59E+00
1.05E+00	7.53E+00
1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
⋯	⋯

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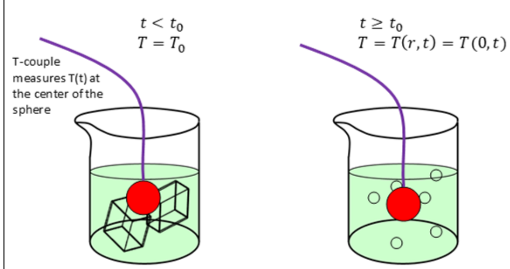
Unsteady State Heat Transfer: Intermediate Biot Number

Modeling

What are the modeling equations?

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):



Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
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8.37E-01	7.46E+00
9.54E-01	7.59E+00
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...	...

?

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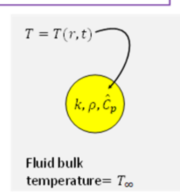
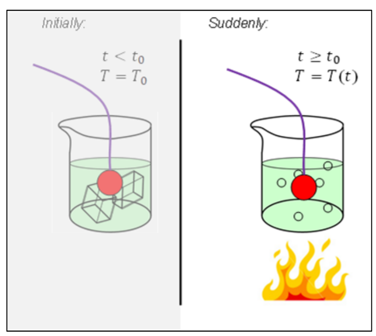
Unsteady State Heat Transfer: Intermediate Biot Number

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

- We need to devise an experiment
- Both internal (k) and external (h) resistances are important
- We need to match *measurable* quantities with *calculable* quantities
- $\Rightarrow$ ***Microscopic*** Energy Balance
- $\Rightarrow$ *Uncertainty considerations*

?





Can we meet our objective?

To determine h:

- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

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Unsteady State Heat Transfer: Intermediate Biot Number

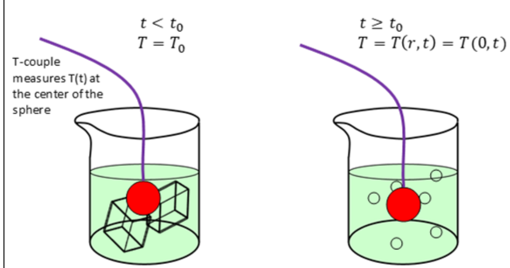
Modeling

What are the modeling equations?

Experiment: Measure $T(t)$ at the center of a sphere ($r = 0$):

Initially: $t < t_0$, $T = T_0$

Suddenly: $t \geq t_0$, $T = T(r,t) = T(0,t)$



Excel:

$t(s)$	$T(^oC)$
9.50E-02	7.46E+00
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1.15E+00	7.58E+00
1.27E+00	7.48E+00
1.37E+00	7.57E+00
...	...

You try.

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Unsteady State Heat Transfer: Intermediate Biot Number

Microscopic Energy Balance

Microscopic energy balance, constant thermal conductivity; Gibbs notation

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + \underline{v}\cdot\nabla T\right) = k\nabla^2 T + S$$

Microscopic energy balance, constant thermal conductivity; Cartesian coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_x\frac{\partial T}{\partial x} + v_y\frac{\partial T}{\partial y} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; cylindrical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z\frac{\partial T}{\partial z}\right) = k\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right) + S$$

Microscopic energy balance, constant thermal conductivity; spherical coordinates

$$\rho\hat{C}_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial T}{\partial \phi}\right) = k\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right) + S$$

www.chem.mtu.edu/~fmorriso/cm310/energy2013.pdf

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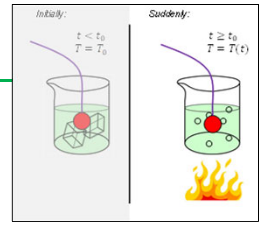
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$\equiv \alpha$

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right)$$

- Unsteady
- Solid ($\underline{v} = 0$)
- θ, ϕ symmetry
- No current, no rxn

Boundary conditions:

$r = R,\quad \frac{q_r}{A} = -k\frac{\partial T}{\partial r} = h(T(r) - T_{bulk}) \quad t > 0$

$r = 0,\quad \frac{q_r}{A} = 0 \quad \forall t$

Initial condition:

$t = 0,\quad T = T_{initial} \quad \forall r$

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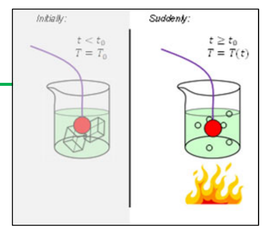
Unsteady State Heat Transfer to a Sphere



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Initial condition:

$t = 0,\quad T = T_{initial} \quad \forall r$

("∀" means "for all")

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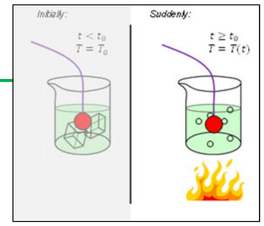
Unsteady State Heat Transfer to a Sphere



Microscopic energy balance in the sphere:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho \hat{C}_p}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right)\right)$$

$\equiv \alpha$

- Unsteady
- Solid ($\underline{v} = 0$)

Bc

Now, Solve

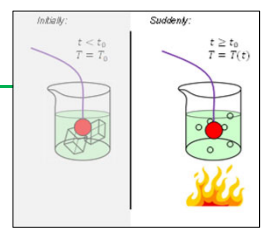
$r = 0$, $\qquad \frac{q}{A} = 0 \qquad \forall t$

Initial condition:

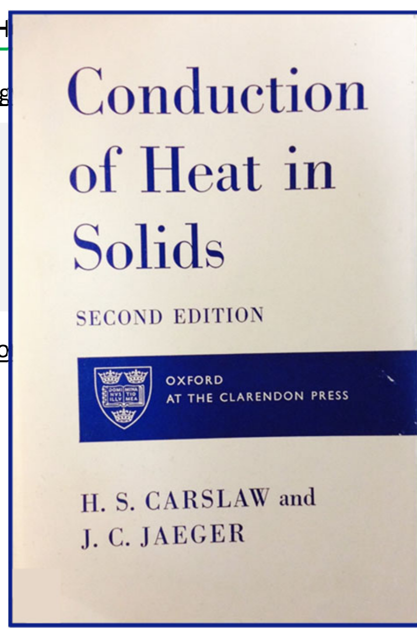
$t = 0$, $\qquad T = T_{initial} \qquad \forall r$

("∀" means "for all")

Unsteady State H



Microscopic energ



- Unsteady
- Solid ($\underline{v} = 0$)
- θ, ϕ symmetry
- No current, no rxn

Boundary conditio

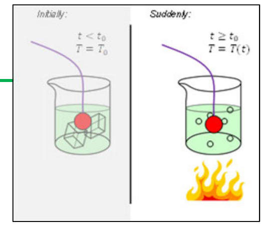
$r = R$, $\qquad t > 0$

$r = 0$, $\qquad \forall t$

Initial condition:

$t = 0$, $\qquad \forall r$

Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Characteristic Equation

(Carslaw and Yeager, 1959, eqn 10, p238)
Incopera and DeWitt, 7th ed, eqn 5.51a, p303

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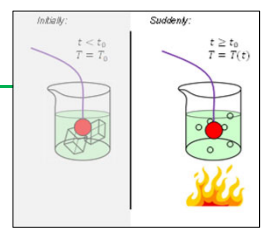
Unsteady State Heat Transfer to a Sphere



Solution:

$$\xi(r,t) \equiv \frac{T(r,t) - T_{bulk}}{T_{initial} - T_{bulk}}$$

Depends on material ($\alpha = k/\rho\hat{C}_p$), and heat transfer processes at surface (h)

$$\text{Bi} \equiv \frac{hR}{k} \qquad \text{Fo} \equiv \frac{\alpha t}{R^2}$$

Bi = Biot number;
Fo = Fourier number

$$\xi = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin r\lambda_n}{r\lambda_n}\right)\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

We're interested in $T(r,t)$ at the center of the sphere, $r = 0$.

Characteristic Equation

(Carslaw and Yeager, 1959, eqn 10, p238)
Incopera and DeWitt, 7th ed, eqn 5.51a, p303

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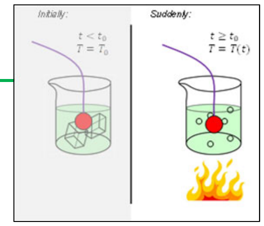
Unsteady State Heat Transfer to a Sphere



What does *this* look like?

$$\xi(0, \mathrm{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\mathrm{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\mathrm{Bi}-1)^2}{(R\lambda_n)^2 + \mathrm{Bi}(\mathrm{Bi}-1)}\right)$$

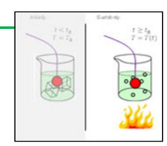
where the eigenvalues λ_n satisfy this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1 = 0$$

Characteristic Equation

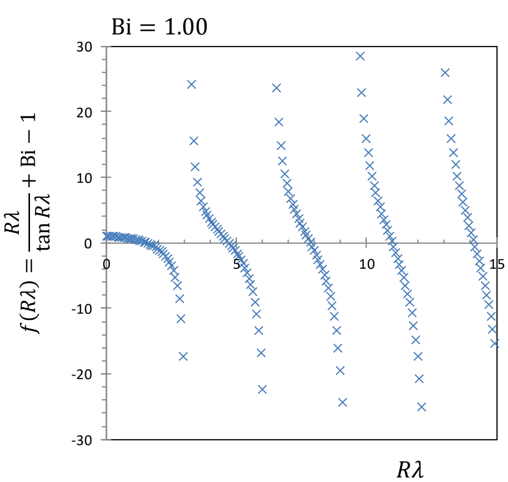
Let's plot it to find out. (Excel)

Unsteady State Heat Transfer to a Sphere



Eigenvalues are the roots of the characteristic equation

$$\mathrm{Bi} \equiv \frac{hR}{k}$$

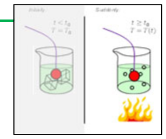


Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1$$

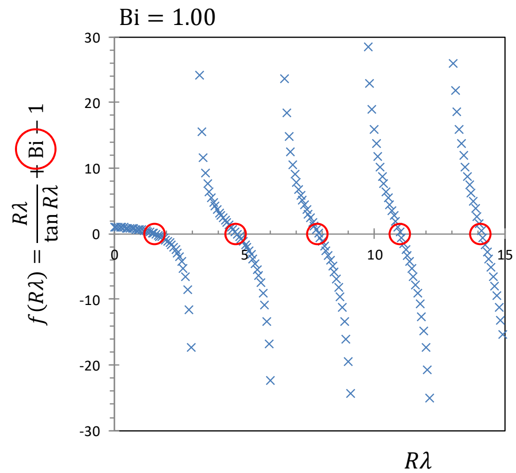
- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

Unsteady State Heat Transfer to a Sphere



$\mathrm{Bi} \equiv \frac{hR}{k}$

Eigenvalues are the roots of the characteristic equation



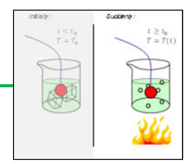
Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1$$

- The λ_n are the roots (zero crossings) of the characteristic equation
- They depend on Biot number Bi

Let's plot it to find out: what are the variables?

$\mathrm{Bi} \equiv \frac{hR}{k}$ $\quad$ $\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$



Solution:

$$\xi(0,\mathrm{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\mathrm{Bi}\sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2}\left(\frac{\sin R\lambda_n}{R\lambda_n}\right)\left(\frac{(R\lambda_n)^2 + (\mathrm{Bi}-1)^2}{(R\lambda_n)^2 + \mathrm{Bi}(\mathrm{Bi}-1)}\right)$$

Exponential decay with Fo (scaled time)

$$\xi(0,\mathrm{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\mathrm{Bi}\sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2}\left(\begin{array}{c}\text{bunch of terms}\\ \text{that vary with Bi and } \lambda_n(\mathrm{Bi})\end{array}\right)$$

$\lambda_n(\mathrm{Bi})$ varies only with Bi and n:

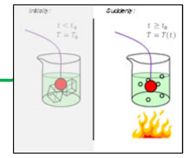
$$\frac{R\lambda_n}{\tan R\lambda_n} + \mathrm{Bi} - 1 = 0$$

Characteristic Equation

If we choose a fixed Bi, then ξ only varies with Fo

If we choose a **fixed Bi**,
then ξ only varies with Fo

$\text{Bi} \equiv \frac{hR}{k}$ $\quad \text{Fo} \equiv \frac{\alpha t}{R^2}$



For a **fixed Bi:**

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-Fo(\lambda_n R)^2} \left(\frac{\sin R\lambda_n}{R\lambda_n}\right) \left(\frac{(R\lambda_n)^2 + (\text{Bi}-1)^2}{(R\lambda_n)^2 + \text{Bi}(\text{Bi}-1)}\right)$$

Exponential decay with Fo *(scaled time)*

$$\xi(0,\text{Fo}) = \frac{T - T_b}{T_i - T_b} = 2\text{Bi} \sum_{n=1}^{\infty} e^{-(Fo)(\lambda_n R)^2} \left(\begin{array}{c}\text{bunch of terms} \\ \text{that vary with Bi and } \lambda_n(\text{Bi})\end{array}\right)$$

An infinite sum of decaying **exponentials**

- whose *argument* is Fourier number scaled by something that depends on Biot number and n
- with a prefactor that also depends on Biot number and n

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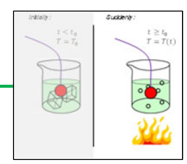
If we choose a **fixed Bi**,
then ξ only varies with Fo

$\text{Bi} \equiv \frac{hR}{k}$ $\quad \text{Fo} \equiv \frac{\alpha t}{R^2}$



For a **fixed Bi:**

$$\xi(0,\text{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 Fo}$$

An infinite sum of decaying **exponentials**

- $\tilde{C}_n$ depends on n through λ_n
- λ_n are calculated (numerically) from the roots of this equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \text{Bi} - 1 = 0$$

Let's plot $\xi(0,\text{Fo})$

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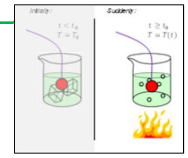
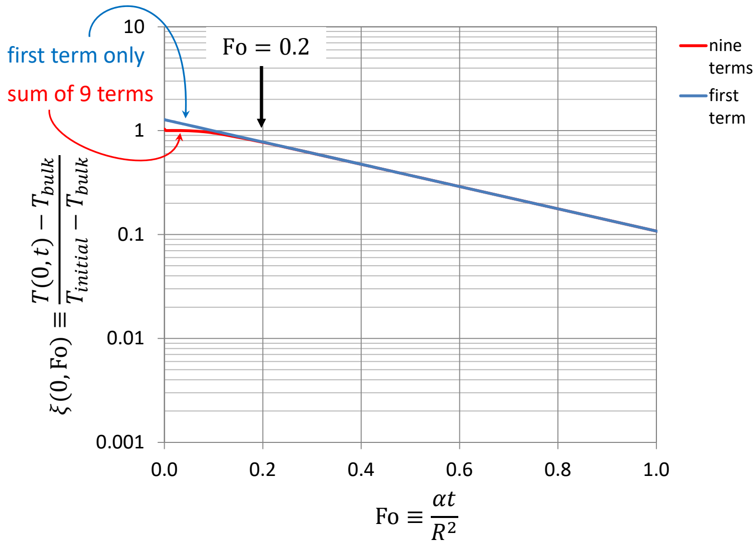
Unsteady State Heat Transfer to a Sphere



Solution:



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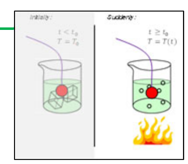
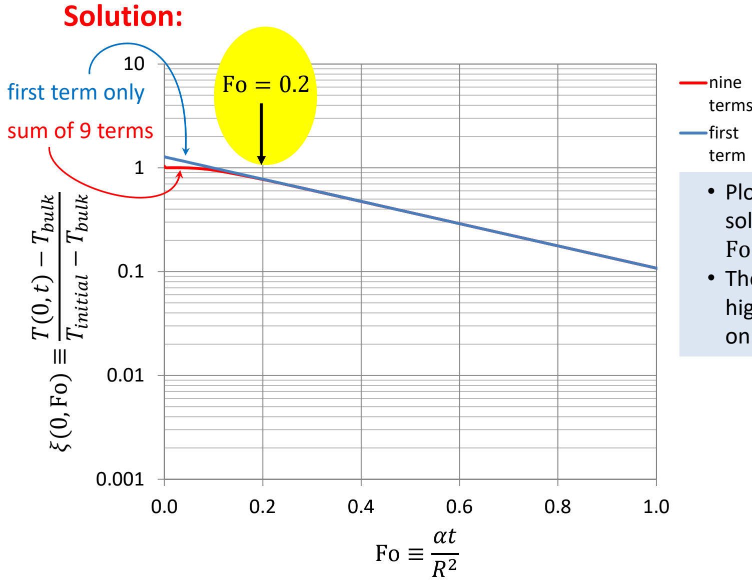
Unsteady State Heat Transfer to a Sphere



Solution:

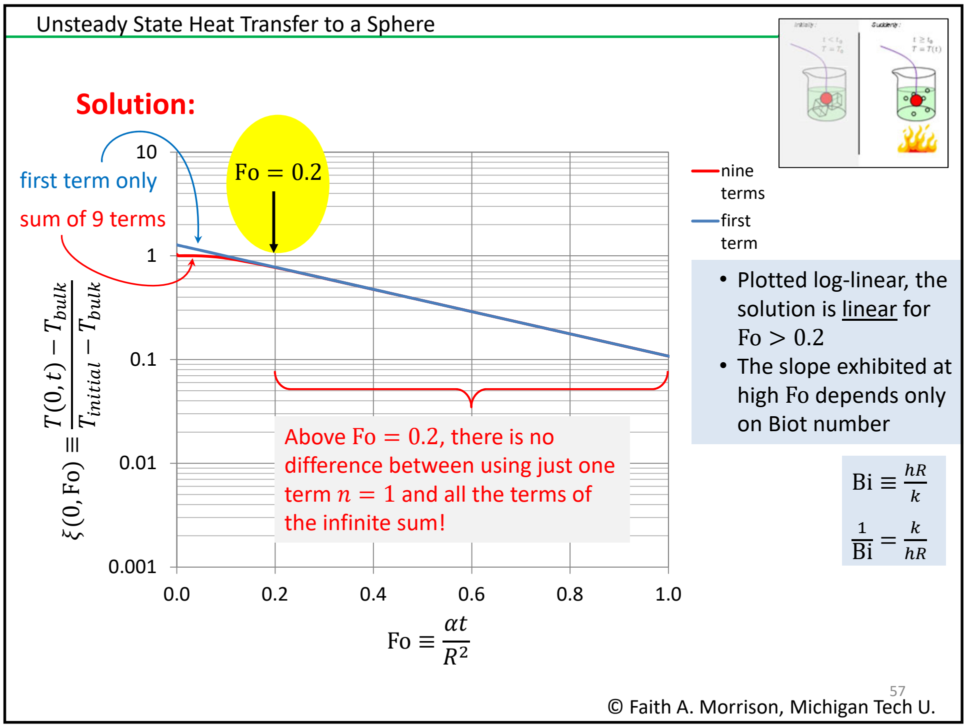


- Plotted log-linear, the solution is <u>linear</u> for $\text{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$$\text{Bi} \equiv \frac{hR}{k}$$

$$\frac{1}{\text{Bi}} = \frac{k}{hR}$$

 56


Unsteady State Heat Transfer to a Sphere
Solution:
first term only
sum of 9 terms
Fo = 0.2
nine terms
first term
• Plotted log-linear, the solution is linear for Fo > 0.2
• The slope exhibited at high Fo depends only on Biot number
Above Fo = 0.2, there is no difference between using just one term n = 1 and all the terms of the infinite sum!
Bi ≡ hR/k
1/Bi = k/hR
ξ(0, Fo) ≡ (T(0,t) − T_bulk)/(T_initial − T_bulk)
Fo ≡ αt/R²
10
1
0.1
0.01
0.001
0.0
0.2
0.4
0.6
0.8
1.0
57
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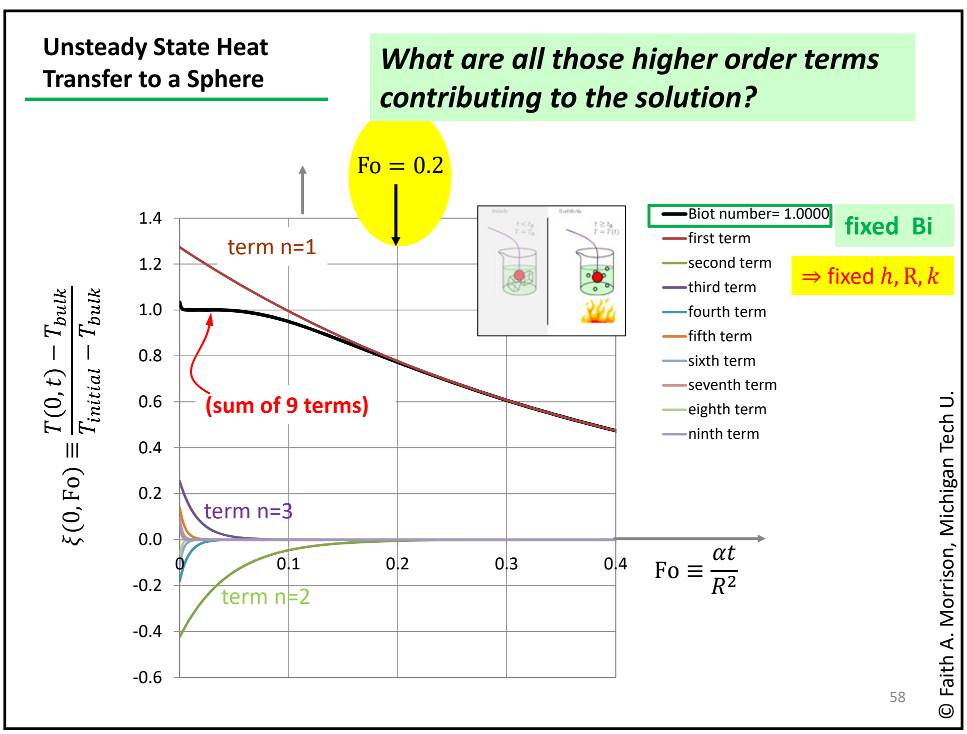
Unsteady State Heat Transfer to a Sphere
What are all those higher order terms contributing to the solution?
Fo = 0.2
term n=1
(sum of 9 terms)
term n=3
term n=2
Biot number= 1.0000
first term
second term
third term
fourth term
fifth term
sixth term
seventh term
eighth term
ninth term
fixed Bi
⇒ fixed h, R, k
ξ(0, Fo) ≡ (T(0,t) − T_bulk)/(T_initial − T_bulk)
Fo ≡ αt/R²
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
-0.6
0
0.1
0.2
0.3
0.4
58
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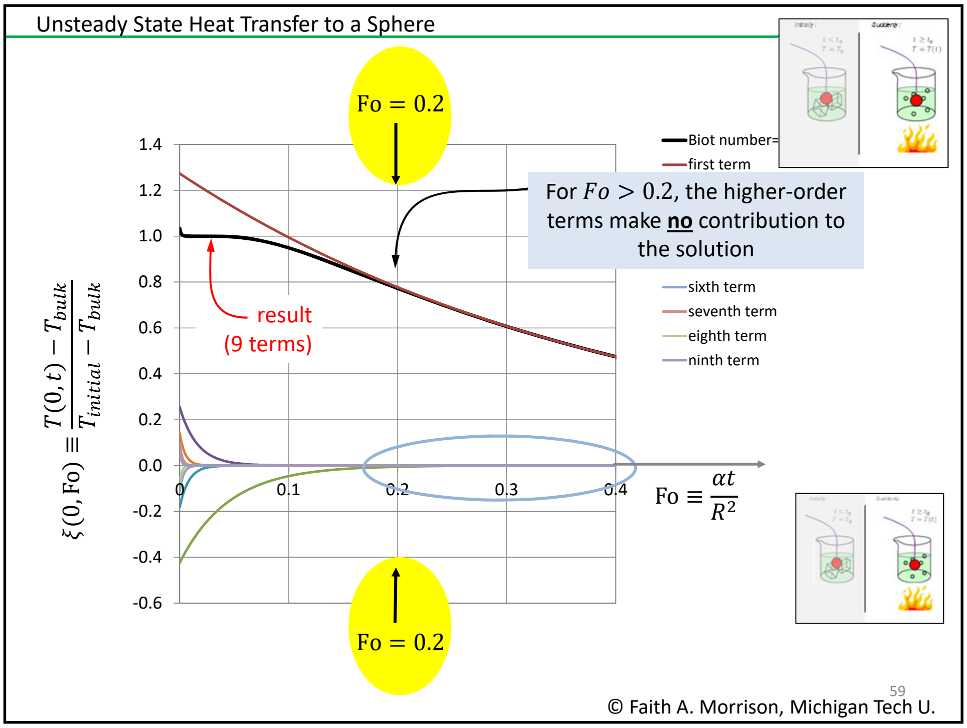
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
result (9 terms)
Biot number=
first term
sixth term
seventh term
eighth term
ninth term
$\xi(0, \text{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
$\text{Fo} \equiv \frac{\alpha t}{R^2}$
Fo = 0.2
59
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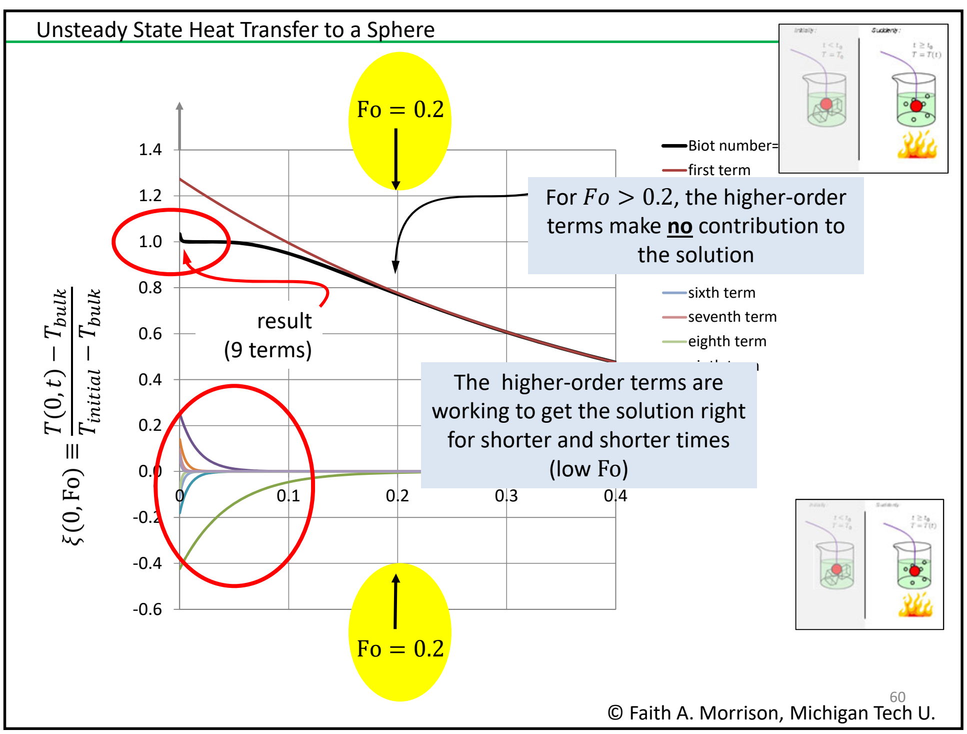
Unsteady State Heat Transfer to a Sphere
Fo = 0.2
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution
result (9 terms)
Biot number=
first term
sixth term
seventh term
eighth term
The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)
$\xi(0, \text{Fo}) \equiv \frac{T(0,t) - T_{bulk}}{T_{initial} - T_{bulk}}$
Fo = 0.2
60
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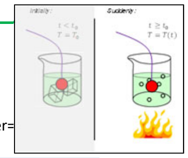
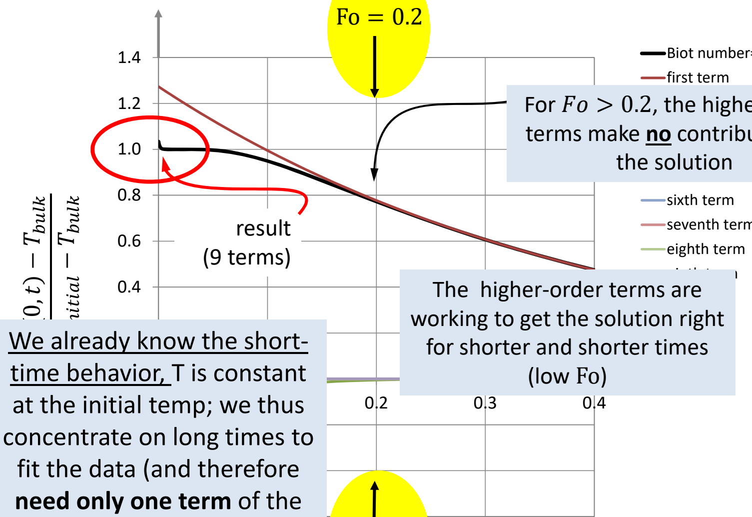
Unsteady State Heat Transfer to a Sphere

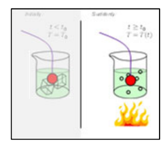




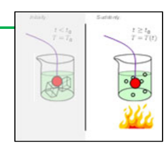
For $Fo > 0.2$, the higher-order terms make **no** contribution to the solution

The higher-order terms are working to get the solution right for shorter and shorter times (low Fo)

We already know the short-time behavior, T is constant at the initial temp; we thus concentrate on long times to fit the data (and therefore **need only one term** of the summation)

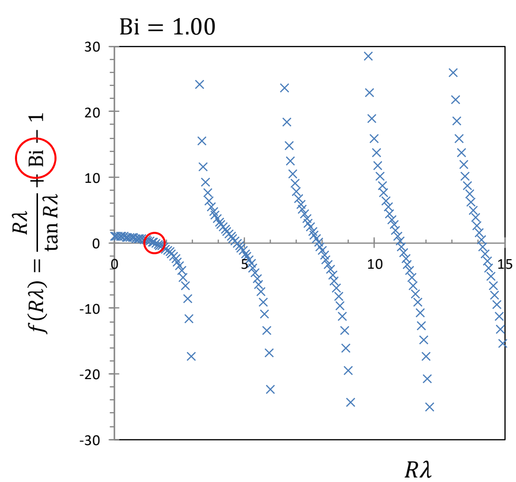


Unsteady State Heat Transfer to a Sphere



Eigenvalues are the roots of the characteristic equation

$$\mathrm{Bi} \equiv \frac{hR}{k}$$

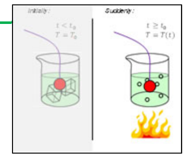


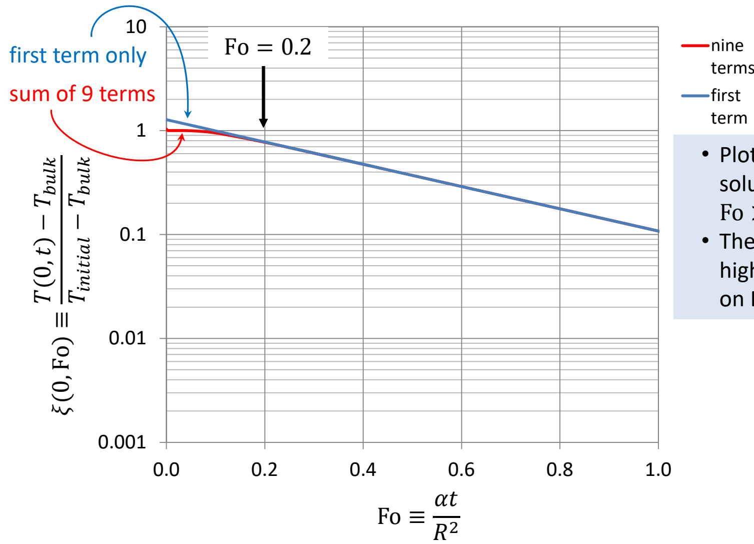
Characteristic Equation:

$$f(R\lambda) = \frac{R\lambda}{\tan R\lambda} + \mathrm{Bi} - 1$$

- The λ_1 eigenvalue is the one that dominates at long time
- The value of λ_1 depends on Biot number Bi

Unsteady State Heat Transfer to a Sphere





- Plotted log-linear, the solution is linear for $\mathrm{Fo} > 0.2$
- The slope exhibited at high Fo depends only on Biot number

$\mathrm{Bi} \equiv \frac{hR}{k}$

$\frac{1}{\mathrm{Bi}} = \frac{k}{hR}$

Unsteady State Heat Transfer to a Sphere

No Mechanism Dominates

$\mathrm{Bi} \equiv \frac{hR}{k}$

$\mathrm{Fo} \equiv \frac{\alpha t}{R^2}$

Summary

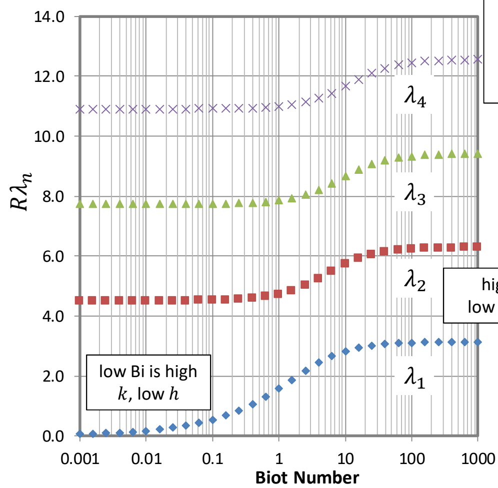
- For a fixed Bi the results are only a function of Fo.
- Solution is an infinite sum of terms.
- Each term corresponds to one eigenvalue, λ_n
- The first term $n = 1$ (λ_1) is the dominant term
- The $n > 1$ terms alternate in sign (positive and negative)
- Higher terms are "fixing" the short time behavior
- At fixed Biot number, the time-dependence is an exponential decay (for $\mathrm{Fo} > 0.2$); this is linear on a log-linear plot versus Fo

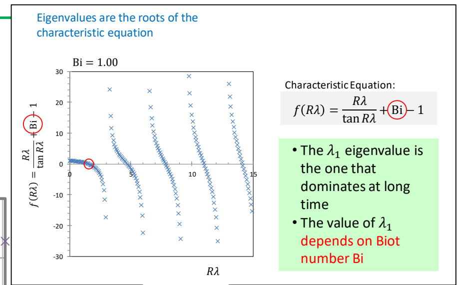
So, actually, it turns out all we need are those slopes as a function of Biot number.

Question: How do various values of Biot number affect the heat transfer that occurs?

Unsteady State Heat Transfer to a Sphere

$$\text{Bi} \equiv \frac{hR}{k}$$

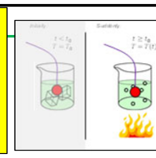




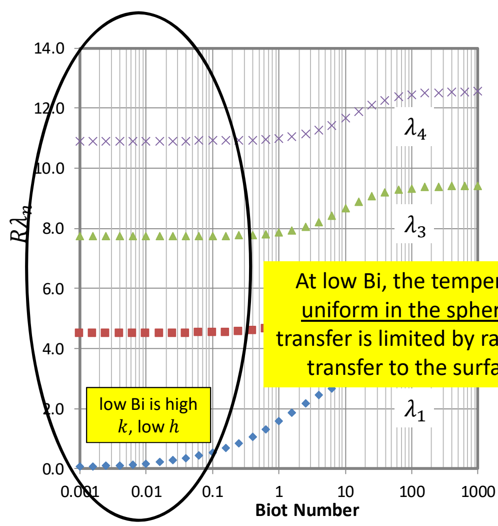
- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

Unsteady State Heat Transfer to a Sphere

Low Bi: high k, low h



$$\text{Bi} \equiv \frac{hR}{k}$$



At low Bi, the temperature is <u>uniform in the sphere</u>; heat transfer is limited by rate of heat transfer to the surface (h).

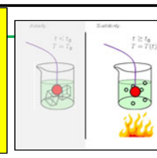
Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

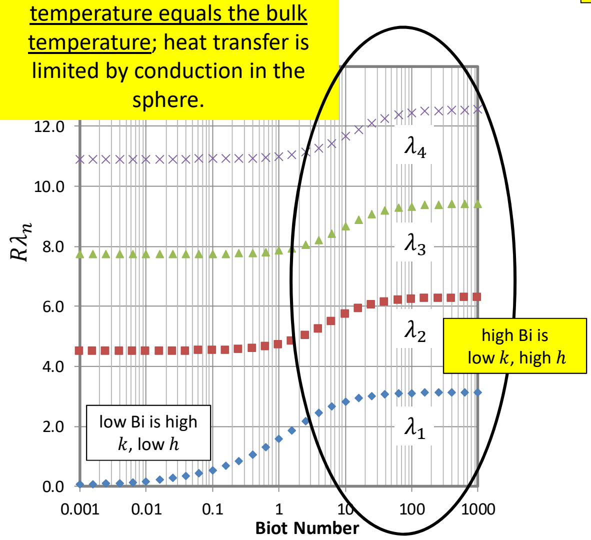
- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

Unsteady State Heat Transfer to a Sphere

High Bi: low k, high h



At high Bi, <u>the surface temperature equals the bulk temperature</u>; heat transfer is limited by conduction in the sphere.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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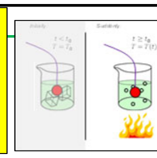
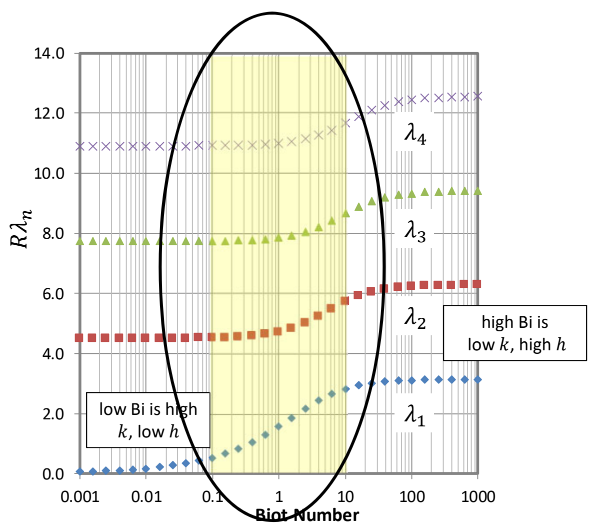
Unsteady State Heat Transfer to a Sphere

Moderate Bi: nether process dominates



At moderate Bi, heat transfer is affected by <u>both</u> conduction in the sphere and the rate of heat transfer to the surface.



Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

- The λ_n are the roots of the characteristic equation
- They depend on Biot number Bi
- At low $(\text{Bi} < 0.1)$ and high Bi $(\text{Bi} > 10)$ the solution becomes independent of Bi

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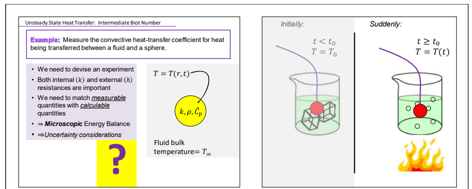
What were we trying to do?

Example: Measure the convective heat-transfer coefficient for heat being transferred between a fluid and a sphere.

Where are we in the process?

✓ We have the model

- We need the measured center-point temperature as a function of time
- We need to compare the two to deduce h.



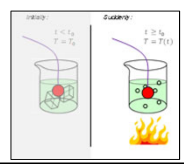
Can we meet our objective?

To determine h:
- Measure center-point temperature as a function of time
- Compare with model predictions, accounting for uncertainty in measurements
- Deduce h

The solution of the model:

$$\xi(0, \mathrm{Fo}) = \sum_{n=1}^{\infty} \tilde{C}_n e^{-\lambda_n^2 R^2 Fo}$$

$$\mathrm{Fo} \equiv \frac{\alpha t}{R^2} \qquad \mathrm{Bi} \equiv \frac{hR}{k}$$



Use to interpret data.

For a **fixed Bi, Fo > 0.2:**

$$\xi(0, \mathrm{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 Fo}$$

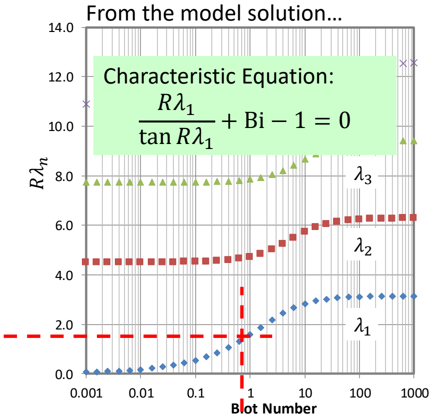
$$\ln \xi(0, \mathrm{Fo}) = \ln(\tilde{C}_1) - \lambda_1^2 R^2 \mathrm{Fo}$$

From experiments…

Plot: $\ln \xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> slope $= -\lambda_1^2 R^2$

From the model solution…



Once we know Bi, we can calculate h from Bi

$$\mathrm{Bi} \equiv \frac{hR}{k}$$

Experimental Data

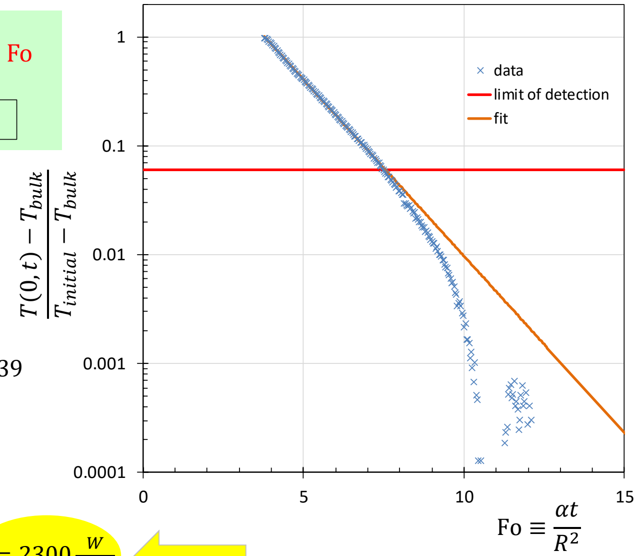
$k = 109 \frac{W}{mK}$

$R = 0.0127\ m$

From experiments...

Plot: $\ln \xi = \ln\left(\frac{T - T_b}{T_i - T_b}\right)$ vs Fo

=> slope $= -\lambda_1^2 R^2$



Slope= -0.74656

$\lambda_1 R = \sqrt{0.74656} = 0.864039$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

$\Rightarrow \text{Bi} = \frac{hR}{k} = 0.2621919,\ h = 2300 \frac{W}{m^2 K}$

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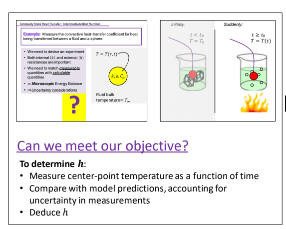
Recap:

Measure convective heat-transfer coefficient h for heat being transferred between a fluid and a sphere.

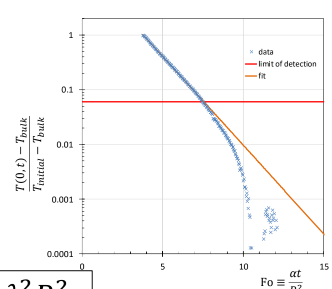
What was the process?

1. Create the scenario and the model of the scenario



2. Take data of center-point $T(t)$

3. Plot the data in a way that we can match it to the model to deduce Bi

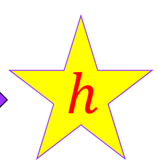


slope $= -\lambda_1^2 R^2$

Characteristic Equation:

$$\frac{R\lambda_1}{\tan R\lambda_1} + \text{Bi} - 1 = 0$$

$\text{Bi} \equiv \frac{hR}{k}$



4. Calculate h.

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Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\mathrm{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

Moderate Bi: The limits for "moderate" are $0.1 \le \mathrm{Bi} \le 10$. When Bi is in this range, a more complete solution may be necessary; $\mathrm{Bi} = hD_{char}/k$. ($D_{char}$ varies with the problem)

Low Bi: no internal temperature variation ⇒ Lumped parameter analysis (macroscopic energy balance, unsteady); $\mathrm{Bi} = hV/kA < 0.1$

$\mathrm{Bi} - \text{Biot Number} = \frac{hD}{k}$

Quantifies the tradeoffs between the rate of internal heat flux (by conduction, k) and the rate of heat delivery to the boundary (by convection, h)

At high Bi, the surface temperature equals the bulk temperature; heat transfer is limited by conduction in the body. — **High Bi:** low k, high h

At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface. — **Moderate Bi:** nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h). — **Low Bi:** high k, low h

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Unsteady State Heat Transfer

Summary

High Bi: dominated by internal temperature variation ⇒ solve with temperature boundary conditions; $\mathrm{Bi} = hD_{char}/k$ (D_{char} varies with the problem)

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At moderate Bi, heat transfer is affected by both conduction in the body and the rate of heat transfer to the surface. — **Moderate Bi:** nether process dominates

At low Bi, the temperature is uniform in a finite body; heat transfer is limited by rate of heat transfer to the surface (h). — **Low Bi:** high k, low h

D_{char}=characteristic lengthscale

We use $D_{char} = V/A$ **only** for the lumped parameter analysis. We use different D_{char} in other cases.

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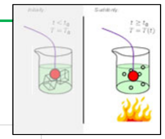
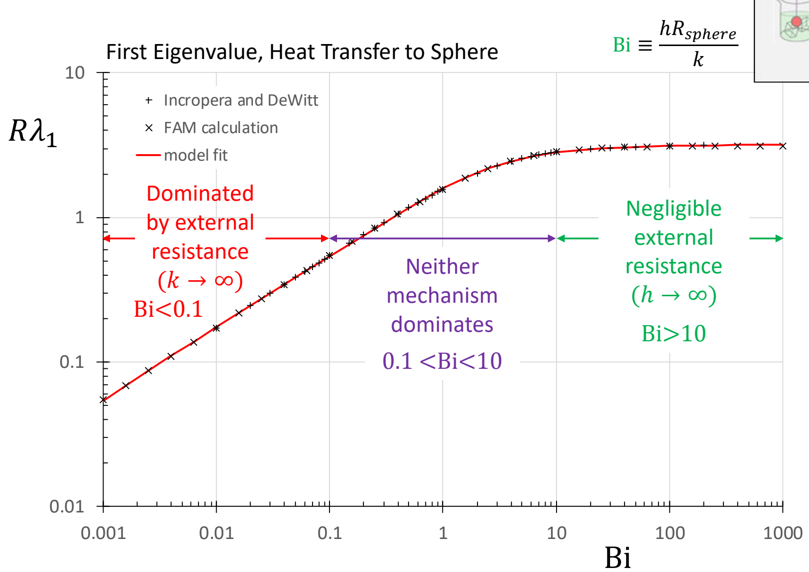
Unsteady State Heat Transfer to a Sphere





Fit by Faith A. Morrison 31 Jan 2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

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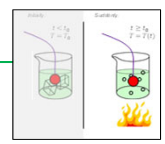
Unsteady State Heat Transfer to a Sphere



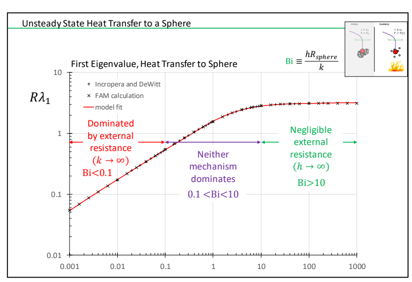
If we know $R\lambda_n$ and we're determining Bi (i.e. h), we use the characteristic equation directly.

If we know h (and hence, Bi) we need to find $\lambda_1 R$ from an iterative solution of the characteristic equation.

Characteristic Equation:

$$\frac{R\lambda_n}{\tan R\lambda_n} + \text{Bi} - 1 = 0$$

Or use a table or correlation for the calculated roots.



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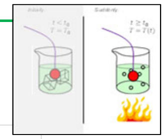
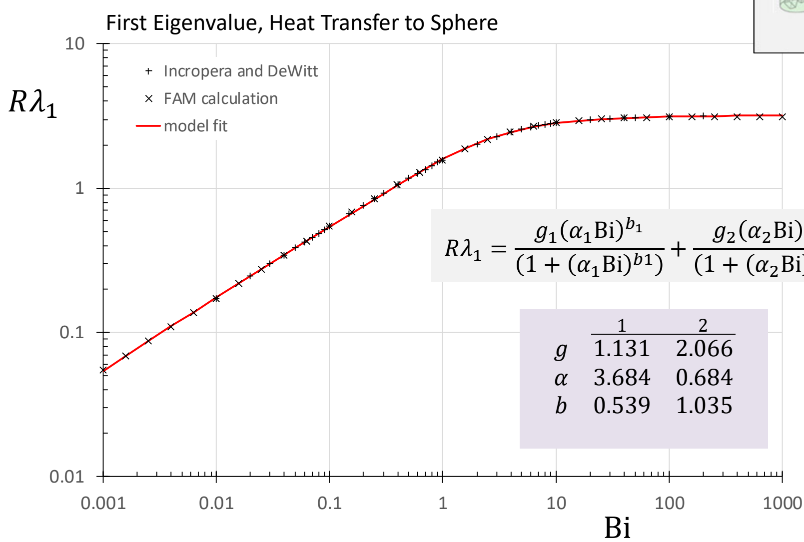
Unsteady State Heat Transfer to a Sphere





$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

Fit by Faith A. Morrison 1/31/2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

(matches true within 1.2%)

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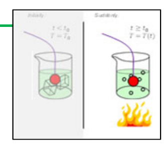
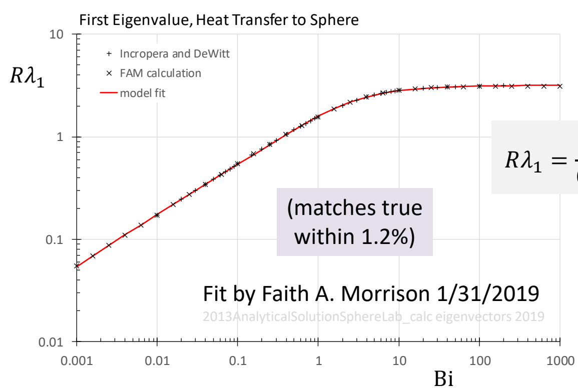
Unsteady State Heat Transfer to a Sphere





(matches true within 1.2%)

Fit by Faith A. Morrison 1/31/2019

2013AnalyticalSolutionSphereLab_calc eigenvectors 2019

$$R\lambda_1 = \frac{g_1(\alpha_1 \mathrm{Bi})^{b_1}}{(1+(\alpha_1 \mathrm{Bi})^{b1})} + \frac{g_2(\alpha_2 \mathrm{Bi})^{b_2}}{(1+(\alpha_2 \mathrm{Bi})^{b2})}$$

	1	2
g	1.131	2.066
α	3.684	0.684
b	0.539	1.035

For a **fixed Bi**, $\mathrm{Fo} > \mathbf{0.2}$:

$$\xi(0, \mathrm{Fo}) \approx \tilde{C}_1 e^{-\lambda_1^2 R^2 Fo}$$

$$\frac{2\mathrm{Bi}[(R\lambda_1)^2 + (\mathrm{Bi}-1)^2]}{(R\lambda_1)^2 + \mathrm{Bi}(\mathrm{Bi}-1)}\left(\frac{\sin R\lambda_1}{R\lambda_1}\right) = \tilde{C}_1 = \frac{4[\sin(\lambda_1 R) - \lambda_1 R\cos(\lambda_1 R)]}{2\lambda_1 R - \sin 2\lambda_1 R}$$

The two expressions may be shown to be equivalent by substituting the characteristic equation; see FAM notebook v.53 2019; the second version is more robust due to avoiding multiplying by zero when $R\lambda_1 \rightarrow \pi$.

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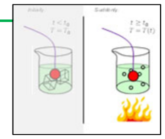
Unsteady State Heat Transfer to a Sphere



Reasonable estimates of the sphere (slab, cylinder) solutions may also be obtained from the "Heisler Charts"

Heisler charts

See:

Geankoplis

Wikipedia

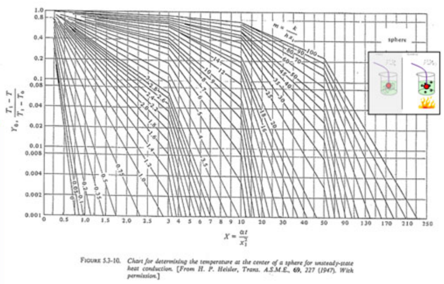
Welty, Rorrer, and Foster (appendix F)

D_{char}=characteristic lengthscale

For the Heisler chart for spheres, we use $D_{char} = R$. Note this is not the same as what is used in lumped parameter analysis

$$\mathrm{Bi} = \frac{hD_{char}}{k}$$

$D_{char} = x_1 =$ R (sphere)
R (cylinder)
B (slab of thickness $2B$)



From Geankoplis, 4th edition, page 374

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Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)

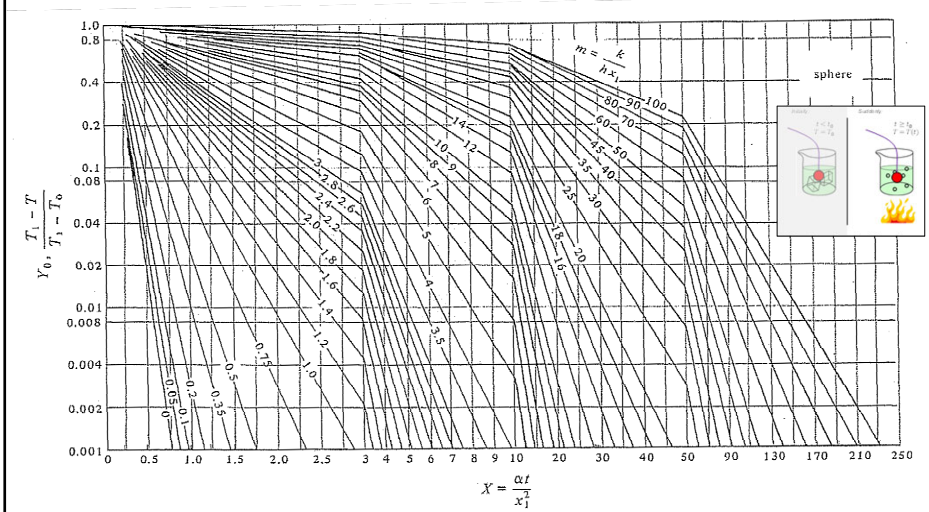

FIGURE 5.3-10. Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]

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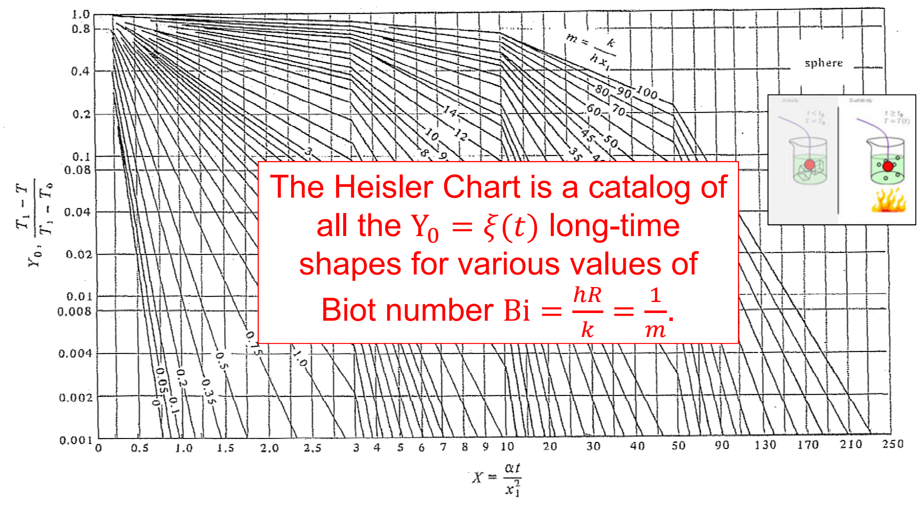
Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)



The Heisler Chart is a catalog of all the $Y_0 = \xi(t)$ long-time shapes for various values of Biot number $\mathrm{Bi} = \frac{hR}{k} = \frac{1}{m}$.

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]*

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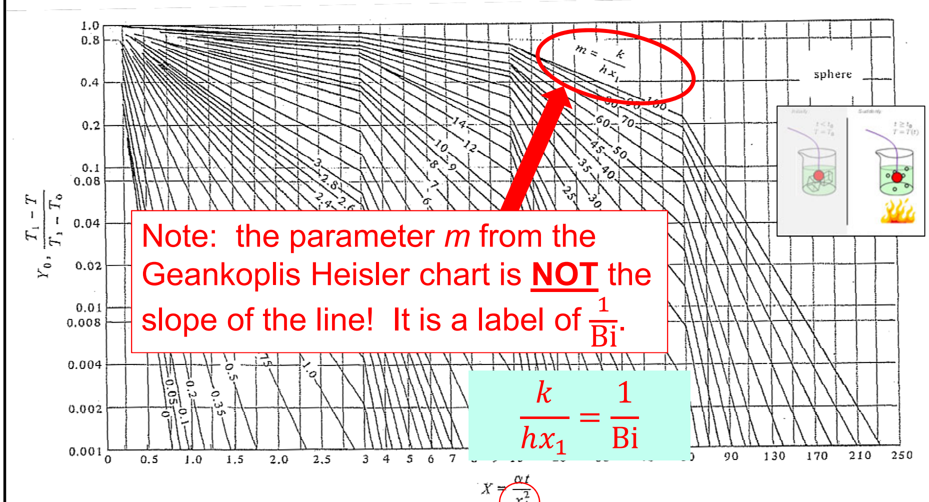
Literature solutions to Unsteady State Heat Transfer to a Sphere

Heisler charts (Geankoplis; see also Wikipedia)



Note: the parameter *m* from the Geankoplis Heisler chart is **NOT** the slope of the line! It is a label of $\frac{1}{\mathrm{Bi}}$.

$$\frac{k}{hx_1} = \frac{1}{\mathrm{Bi}}$$

Also, x_1 is the sphere radius

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]*

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Heisler charts (Geankoplis; see also Wikipedia)

Note also: think of it as 4 separate graphs

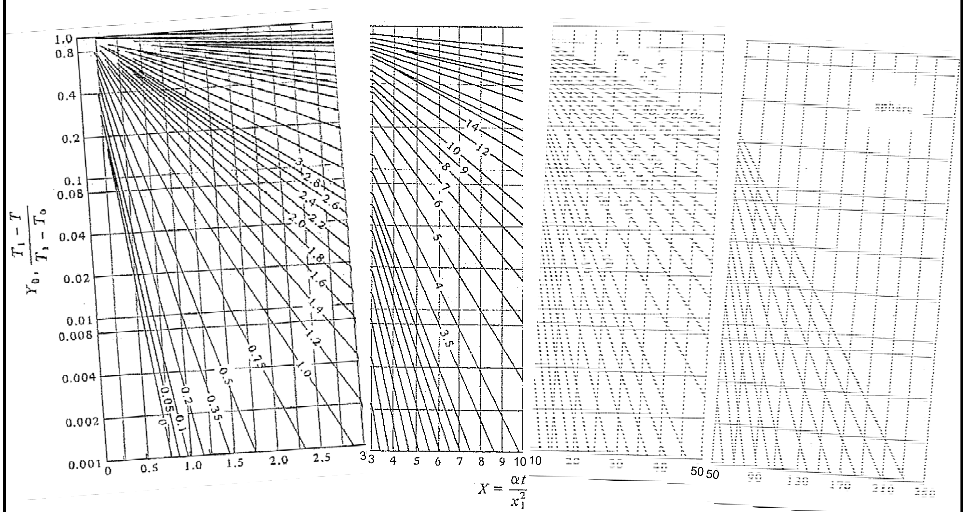

FIGURE 5.3-10. *Chart for determining the temperature at the center of a sphere for unsteady-state heat conduction. [From H. P. Heisler, Trans. A.S.M.E., 69, 227 (1947). With permission.]* 83

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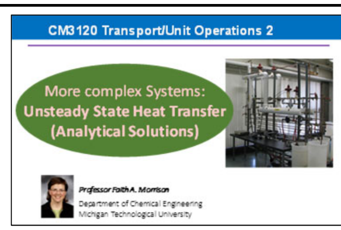
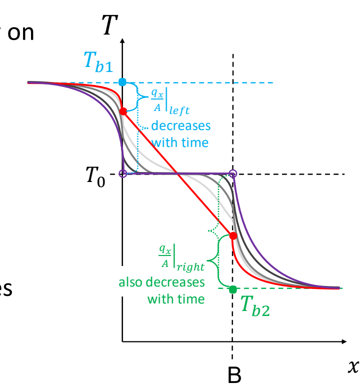
Unsteady State Heat Transfer



Summary

- Unsteady state heat transfer is very common in the chemical process industries
- Temperature distributions depend strongly on what initiates the heat transfer (usually something at the boundary)
- ***Internal resistance*** $(1/k)$ can be limiting, irrelevant, or one among many resistances
- ***External resistance*** $(1/h)$ can be limiting irrelevant, or one among many resistances
- **Dimensional analysis**, once again, organizes the impacts of various influences $(\mathrm{Bi}, \mathrm{Fo})$



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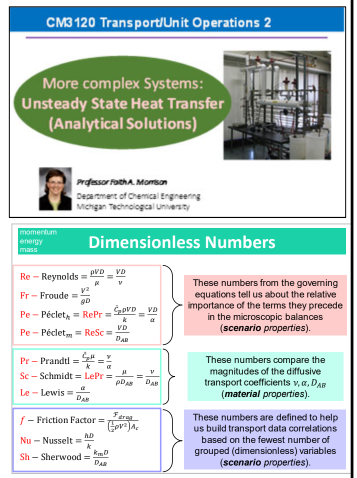
Unsteady State Heat Transfer



Summary *(continued)*

- In transport phenomena, we have dimensionless numbers that represent three important aspects of situations that interest us:
    1. The relative importance of individual terms in the equations of change
    2. The relative magnitudes of the diffusive transport coefficients ν, α, D_{AB}
    3. Scaled values of quantities of interest, e.g. wall forces, heat transfer coefficients, and mass transfer coefficients (data correlations)

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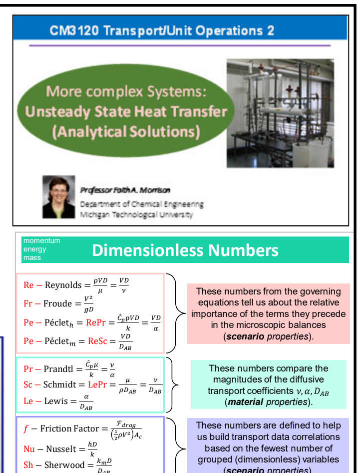
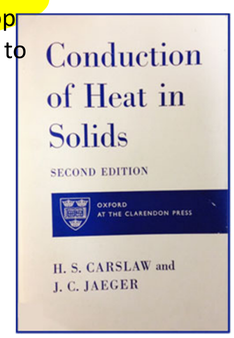
Unsteady State Heat Transfer



Summary *(continued)*

- If we can develop a model situation for questions of interest, the solutions of the models are often in the literature
- **Our responsibility in 21st century:** Learn to develop models that will allow us to estimate or determine answers to the questions that interest us



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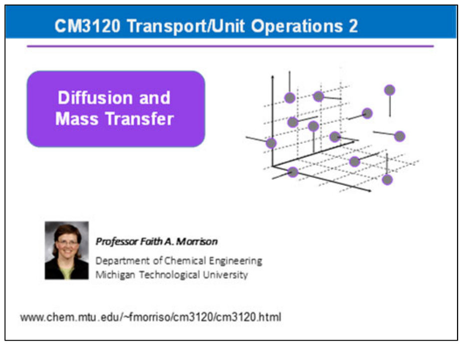
NEXT: Diffusion and Mass Transfer



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