Unsteady Heat Conduction in Spheres

(tbco, 12/04/2006, 12/03/2007)

Laboratory Objective: To obtain thermal diffusivity coefficients, α, of different materials from unsteady heat conduction of spheres.

Mathematical Models:

General Equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \qquad \alpha = \frac{k}{\rho C_p}$$

For Spherical Coordinates:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right)$$

Assumptions:

1. Symmetry around the center
2. Sphere is initially at uniform temperature: T_{initial} =constant
3. Temperature bath is constant: T_{bath} =constant
4. Surface temperature is equal to T_{bath}.

Working Model:

Let u be the non-dimensionalized temperature variable :

$$u = \frac{T - T_{\text{initial}}}{T_{\text{bath}} - T_{\text{initial}}} \quad \rightarrow \quad \boxed{\frac{\partial u}{\partial t} = \alpha \left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)\right)} \tag{1}$$

Initial Condition:	$u(r,0) = 0$	$0 \le r \le R, \quad t = 0$
Boundary Condition 1:	$\left.\frac{\partial u}{\partial r}\right\vert_{r=0} = 0$	$r = 0, \quad t > 0$
Boundary Condition 2:	$u(R,t) = 1$	$r = R, \quad t > 0$

Analytical Solution:

Using method of separation of variables, we can obtain:

$$\boxed{u(r,t) = 1 + \left(\frac{2R}{\pi r}\right)\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\left(\frac{n\pi r}{R}\right) e^{-(n\pi)^2 \phi(t)}} \tag{2}$$

where $\phi(t) = \dfrac{\alpha t}{R^2}$

A plot of equation (2) is shown in Figure 1. An equivalent representation is given by the Gurney-Lurie charts (see C. Geankoplis, "Transport Processes and Unit Operations").

(Note: this solution can be shown to satisfy the differential equation and the boundary and initial conditions by direct substitution).

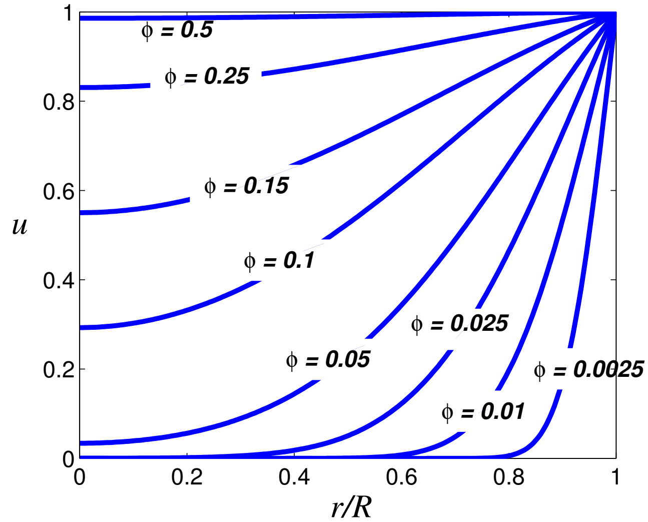

Figure 1.

When $r = 0$, we can use the l'Hospital rule to obtain

$$u(0,t) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-(n\pi)^2 \phi(t)} \tag{3}$$

Based on Figure 1, we see that $u(0,t)=0$ for $\phi<0.01$. A reasonably good finite approximation of the solution is given by:

$$u(0,t) = \begin{cases} 0 & \text{if } \left(\phi = \dfrac{\alpha t}{R^2}\right) < 0.01 \\ 1 + 2\sum_{n=1}^{10}(-1)^n e^{-(n\pi)^2 \phi(t)} & \text{if } \left(\phi = \dfrac{\alpha t}{R^2}\right) \geq 0.01 \end{cases} \tag{4}$$

A plot of equation (4) is shown in Figure 2.

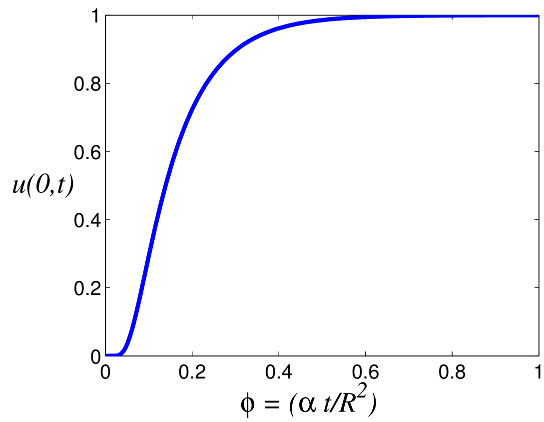

Figure 2.

Experimental Procedure:

Part I. Obtain thermocouple data

1. Prepare a beaker of warm water. Warm it to around 45^{o}C.
2. Turn on the stirrer to a setting of around 6 to 7. (This is to increase the heat transfer coefficient .)
3. Switch on the power strip.
4. Load the "Brass Sphere" program by double-clicking on the icon in the Desktop page. Let the thermocouple reading settle.
5. Stop the program and restart so that the temperature data will start at a constant initial temperature.
6. At a chosen "Elapsed time", dip the thermocouple into the warm water. Record the value of the "Elapsed time" of when the thermocouple was dipped as t_{start}.
7. Wait a few minutes until a new steady state is obtained (about 2 to 10 minutes depending on the material and size), then click on the **[STOP]** button.

Part II. Obtain thermal diffusivity α.

1. Open an Excel spreadsheet file and initialize the parameters. (see Figure 3).
2. Set-up headings for time, temperature data and other columns for calculations, then insert data into the respective columns (see Figure 4). Note: use numerical values for the different values of n, i.e. the headings: 1,2,… should not be string/text.
3. Evaluate ϕ and the summation terms for each n (see Figure 5), then evaluate u and T_{model}. Note that in the formula for ϕ in the spreadsheet, we use (t-t_{start}) instead of just t, because equation (4) was based on t_{start}=0.
4. Copy the cells up to the time range desired, then evaluate the square of the error between T_{data} and T_{model}, and add a cell to evaluate *RMS* (see Figure 6).

	A	B	C	D
1				
2		Radius:	0.375	
3				
4		Parameters:	α	0.1
5			T_{inital}	24
6			T_{bath}	59
7			t_{start}	19
8				

Figure 3.

	C	D	E	F	G	H	I	J	K	S
10								n:		
11	Err^2		Time	T_{data}	T_{model}	u	ϕ	1	2	10
12			0	24.468495						
13			7.219	24.46821						
14			7.234	24.467954						
4799			152.906	58.920294						
4800			152.938	58.920326						
4801			152.969	58.920361						
4802										

Figure 4.

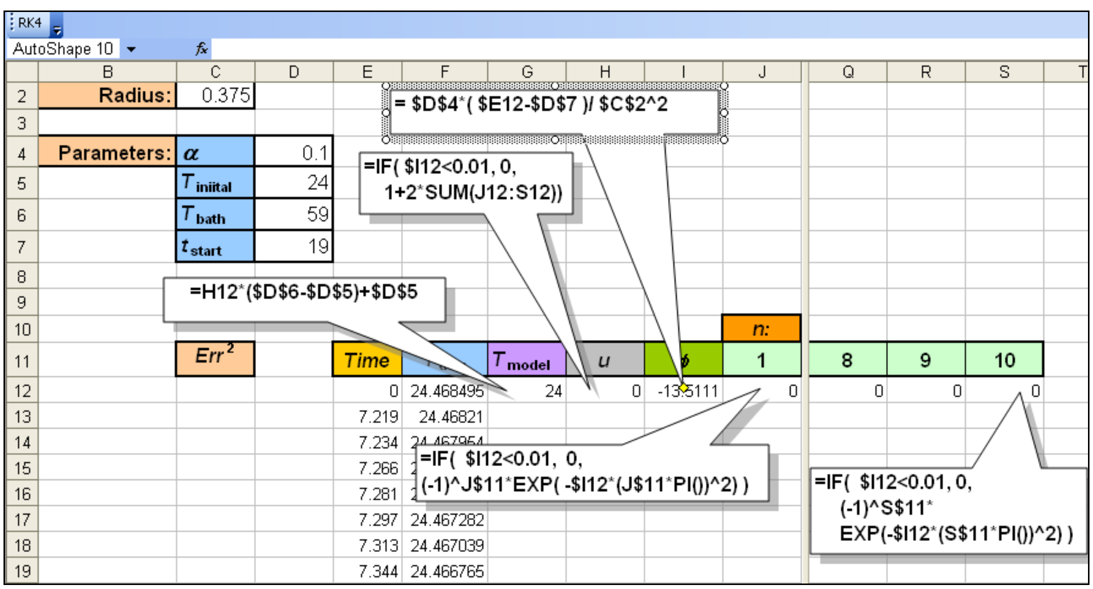

Figure 5.

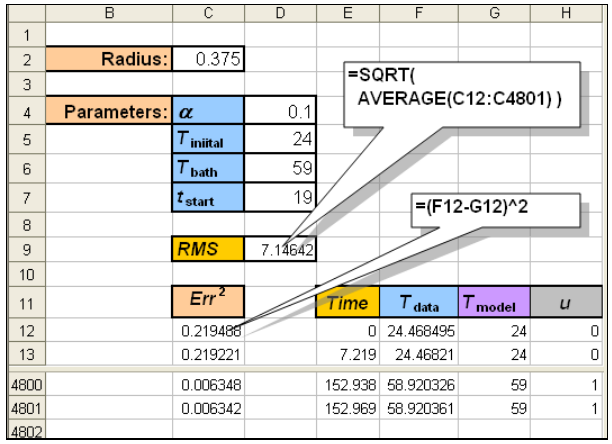

Figure 6.

5. Use **SOLVER** to search for the optimum value of α that minimizes the *RMS* error. (see Figure 7). (Note: in some cases, you may need to set the values of T_{initial} and T_{bath}, and let **SOLVER** adjust only α).

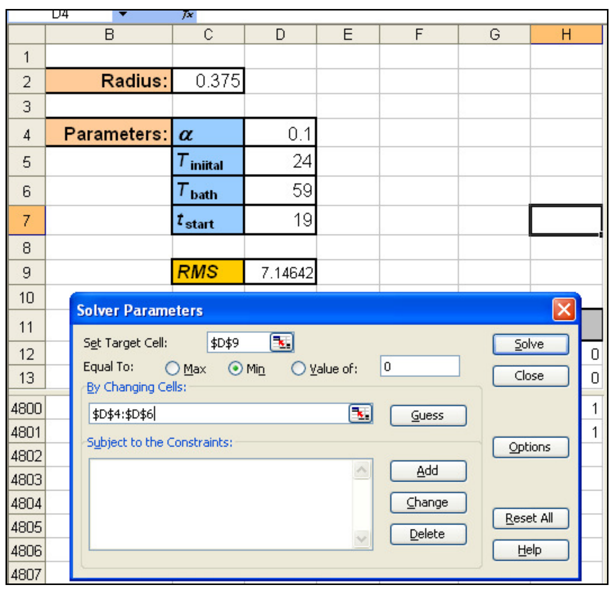

Figure 7.

6. Plot both temperature data and estimated temperature from model to check closeness of fit. If the plots are close, check if the α estimated is close to the value $\dfrac{k}{\rho C_p}$ found from literature sources. You should find that the estimated value will be an order of magnitude less, thus what you would have found using the above procedure is only the "apparent" thermal diffusivity. This is proof that the assumption that $T_{bath} \neq T_{surface}$ has to be modified (i.e. the Biot number, $N_{\text{Bi}} = \dfrac{hR}{3k}$, is not infinite). The second boundary condition should instead be given by

$$-k\frac{\partial u}{\partial r}\bigg|_{r=R} = h\,u\big|_{r=R} \qquad t > 0$$

A solution for this case is given by Heisler charts (see C. J. Geankoplis, "Transport Processes and Unit Operations" for more information about these charts). These charts can be used to determine the value of the heat transfer coefficient for the experimental run. Another method is to use numerical methods such as finite difference methods to estimate *h*.

Note: in both the Gurney-Lurie charts and Heisler charts, they use a different non-dimensionalized temperature difference, i.e.

$$Y = 1 - u = \frac{T - T_{bath}}{T_{bath} - T_{initial}}$$

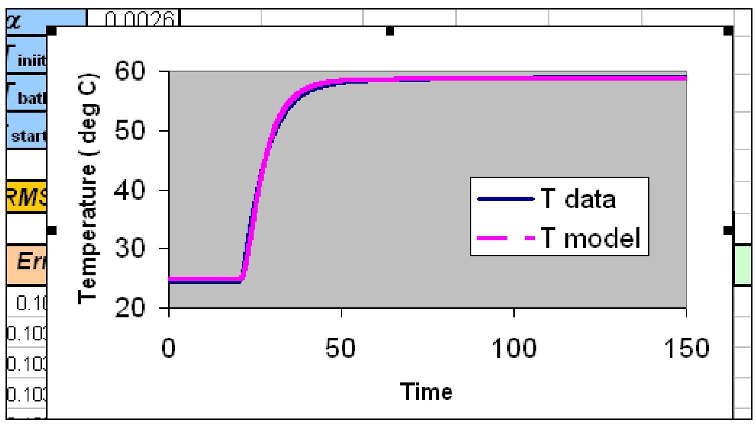

Figure 8