explicitly in terms of the basis vectors $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$, and we use the expressions in equations 1.270-1.272 to algebraically convert the basis vectors.

$$\underline{\underline{A}} \;=\; 2\hat{e}_1\hat{e}_1 \tag{1.317}$$

To write $\hat{e}_1$ in terms of $\hat{e}_r$, $\hat{e}_\theta$, and $\hat{e}_z$ we solve equations 1.270 -1.271 for $\hat{e}_1$ explicitly.

$$\hat{e}_r \;=\; \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2 \tag{1.318}$$

$$\hat{e}_\theta \;=\; -\sin\theta\hat{e}_1 + \cos\theta\hat{e}_2 \tag{1.319}$$

Solving for $\hat{e}_1$,

$$\hat{e}_1 \;=\; \cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta \tag{1.320}$$

Substituting this result into equation 1.317 twice and carrying out the distributive law we obtain,

$$\underline{\underline{A}} \;=\; 2\hat{e}_1\hat{e}_1 \tag{1.321}$$

$$=\; (\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta)\,(\cos\theta\hat{e}_r - \sin\theta\hat{e}_\theta) \tag{1.322}$$

$$=\; \cos^2\theta\hat{e}_r\hat{e}_r - \sin\theta\cos\theta\hat{e}_r\hat{e}_\theta - \sin\theta\cos\theta\hat{e}_\theta\hat{e}_r + \sin^2\theta\hat{e}_\theta\hat{e}_\theta \tag{1.323}$$

$$\underline{\underline{A}} \;=\; \begin{pmatrix} \cos^2\theta & -\sin\theta\cos\theta & 0 \\ -\sin\theta\cos\theta & \sin^2\theta & 0 \\ 0 & 0 & 0 \end{pmatrix}_{r\theta z} \tag{1.324}$$

The same tensor $\underline{\underline{A}}$ is expressed in equations 1.316 and 1.324 – they are just expressed with respect to different coordinate systems.

1.3.3 Substantial Derivative

The mass, momentum, and conservation equations introduced in section 1.3.2 are written in equations 1.244-1.246 in a way that emphasizes the similarity of the left-hand terms. Notice that on the left side of those equations, the following pattern recurs:

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f \tag{1.325}$$

where, depending on which equation we look at, f is density, velocity, or energy. This pattern is given a name, the substantial derivative. The notation for substantial derivative is a derivative written with a capital D.

$$\begin{array}{c} \text{Substantial} \\ \text{derivative} \\ \text{(Gibbs notation)} \end{array} \qquad \boxed{\dfrac{Df}{Dt} \equiv \dfrac{\partial f}{\partial t} + \underline{v}\cdot\nabla f} \tag{1.326}$$

$$\begin{array}{c} \text{Cartesian coordinates} \\ \text{(see Table B.2)} \end{array} \qquad \dfrac{Df}{Dt} \equiv \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial x_1}v_1 + \dfrac{\partial f}{\partial x_2}v_2 + \dfrac{\partial f}{\partial x_3}v_3 \tag{1.327}$$

The substantial derivative has a physical meaning: the rate of change of a quantity (mass, energy, momentum) as experienced by an observer that is moving along with the flow. The observations made by a moving observer are affected by the stationary time-rate-of-change of the property ($\partial f/\partial t$), but what is observed also depends on where the observer goes as it floats along with the flow ($\underline{v} \cdot \nabla f$). If the flow takes the observer into a region where, for example, the local energy is higher, then the observed amount of energy will be higher due to this change in location. The rate of change from the point of view of an observer floating along with a flow appears naturally in the equations of change.

The physical meaning of the substantial derivative is discussed more completely in the sidebar below and in a National Committee of Fluid Mechanics film available on the internet[136]. The chapter concludes with some practical mathematical advice in section 1.3.4. In Chapter 2 we turn to a tour through fluid behavior as a first step to fluid modeling.

SIDEBAR *Substantial Derivative in Fluid Mechanics*

In fluid mechanics and in other branches of physics, we often deal with properties that vary in space and that also change with time. Thus, we need to consider the differentials of multivariable functions. Consider such a multivariable function, $f(t, x_1, x_2, x_3)$, associated with a particle of fluid, where t is time, and x_1, x_2, and x_3 are the three spatial coordinates. The function f might be, for example, the density of flowing material as a function of time and position. The expression Δf is the change in f when comparing the value of the function f at two nearby points, (t, x_1, x_2, x_3) and $(t + \Delta t, x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$.

$$
\begin{aligned}
f &= f(t, x_1, x_2, x_3) & (1.328) \\
\Delta f &= f(t + \Delta t, x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3) - f(t, x_1, x_2, x_3) & (1.329)
\end{aligned}
$$

In the limit that the two points are close together, Δf becomes the differential df:

$$
df = \lim_{\substack{\Delta x_1 \longrightarrow 0 \\ \Delta x_2 \longrightarrow 0 \\ \Delta x_3 \longrightarrow 0 \\ \Delta t \longrightarrow 0}} \Delta f \qquad (1.330)
$$

We can write Δf in terms of partial derivatives, functions that give the rates of change of f (slopes) in the three coordinate directions x_1, x_2, x_3 (see web appendix [124] for a review).

$$
\Delta f = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{\partial f}{\partial x_3}\Delta x_3 \qquad (1.331)
$$

Since the differential df is the limit of Δf as all changes of variable go to zero, we can take the limit of equation 1.331 to obtain df in terms of dx_1, dx_2, and dx_3.

$$df \quad = \quad \lim_{\substack{\Delta x_1 \longrightarrow 0 \\ \Delta x_2 \longrightarrow 0 \\ \Delta x_3 \longrightarrow 0 \\ \Delta t \longrightarrow 0}} \Delta f \tag{1.332}$$

$$df \quad = \quad \lim_{\substack{\Delta x_1 \longrightarrow 0 \\ \Delta x_2 \longrightarrow 0 \\ \Delta x_3 \longrightarrow 0 \\ \Delta t \longrightarrow 0}} \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{\partial f}{\partial x_3}\Delta x_3 \tag{1.333}$$

$$df \quad = \quad \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 \tag{1.334}$$

This is the familiar chain rule. The direction in going from (t, x_1, x_2, x_3) to $(t + \Delta t, x_1 + \Delta x_1, x_2 + \Delta x_2, x_3 + \Delta x_3, t + \Delta t)$ is not specified in the definition of df; equation 1.334 applies to any path between any two nearby points.

There is, however, a particular path and set of neighboring particles that is of recurring interest in fluid mechanics, and that is the path that fluid particles take. Fluid particles are discussed in detail in Chapter 3, but briefly, a fluid particle is an infinitesimally small piece of fluid. If we choose one such piece of fluid, its motion describes a path through three-dimensional space (Figure 1.46). These paths are called pathlines of the flow.
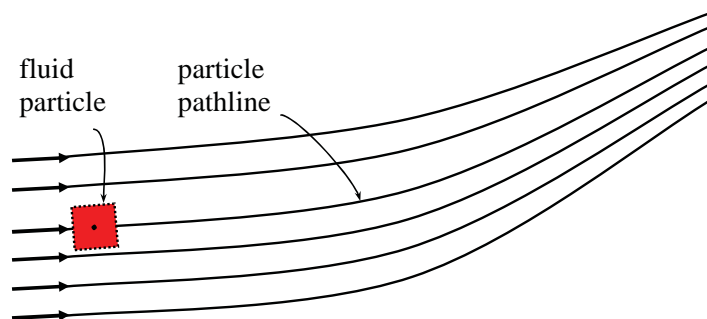

Figure 1.46: A fluid particle consists of the same molecules at all times. The path that a particle follows through a flow is called a pathline.

Consider variation in the function f along a particular path, the path that a fluid particle traces out as it travels through a flow. The function f might be density as a function of position and time, or temperature as a function of position and time, for example. Beginning at an arbitrary point in the flow, we

compare the value of f at the original point and the value of f at the nearby point $f + \Delta f$. For an arbitrary path as we just discussed, Δf is given by equation 1.333.

$$\Delta f|_{along\ ANY\ path} \quad = \quad \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \frac{\partial f}{\partial x_3}\Delta x_3 \quad (1.335)$$

If we now choose to follow fluid particles along a particular path, the particle pathline, then we can relate the directions $\Delta x_1, \Delta x_2, \Delta x_3$ to the local fluid velocity components, v_1, v_2, and v_3.

$$\text{Along a pathline:} \quad \begin{cases} \Delta x_1 = v_1 \Delta t \\ \Delta x_2 = v_2 \Delta t \\ \Delta x_3 = v_3 \Delta t \end{cases} \quad (1.336)$$

Substituting these expressions into equation 1.335, we obtain

$$\Delta f|_{along\ particle\ pathline} \quad = \quad \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x_1}v_1\Delta t + \frac{\partial f}{\partial x_1}v_2\Delta t + \frac{\partial f}{\partial x_1}v_3\Delta t \quad (1.337)$$

$$= \quad \Delta t \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_1}v_2 + \frac{\partial f}{\partial x_1}v_3 \right) \quad (1.338)$$

Dividing through by Δt and taking the limit as Δt goes to zero, we arrive at the expression below, which has been named the substantial derivative.

$$\frac{\Delta f}{\Delta t}\bigg|_{along\ particle\ pathline} \quad = \quad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_1}v_2 + \frac{\partial f}{\partial x_1}v_3 \quad (1.339)$$

$$\frac{Df}{Dt} \equiv \frac{df}{dt}\bigg|_{along\ particle\ pathline} \quad = \quad \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}\bigg|_{along\ particle\ pathline} \quad (1.340)$$

Substantial derivative or rate-of-change of f along a pathline
$$\boxed{\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_2}v_2 + \frac{\partial f}{\partial x_3}v_3} \quad (1.341)$$

Thus, the substantial derivative gives the time rate of change of a function f as the observer floats along a pathline in a flow, attached to a fluid particle.

Why does the "time rate of change of a function as the observer floats along in the flow" matter in fluid mechanics? One reason is that sometimes measurements are made in just such a way, by floating an instrument in a flow as is done in, for example, a weather balloon (Figure 1.47). The density, velocity, or temperature as a function of time recorded this way would be the substantial derivative along the pathline traveled. In meteorology and oceanography it is common to make measurements of the substantial derivative.

The main reason, however, that the substantial derivative is important is because it appears in the mass, momentum, and energy conservation equations. Returning to the equations of change for mass, momentum, and energy given in
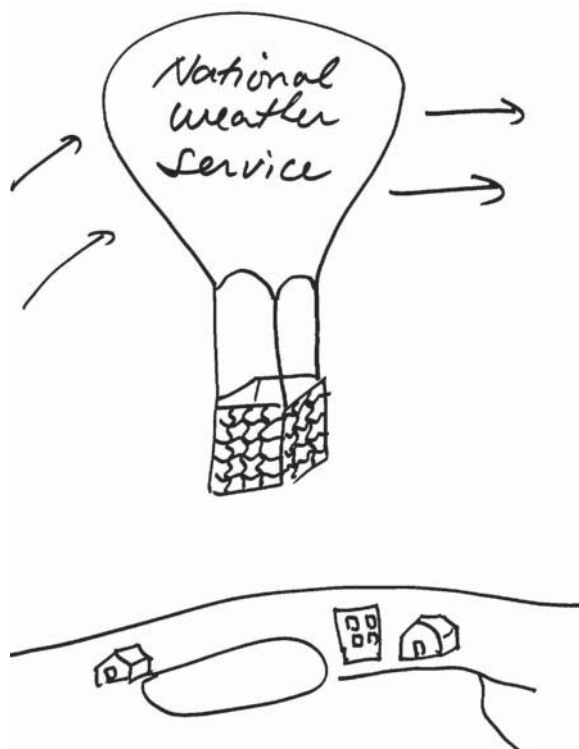
Figure 1.47: Weather balloons float along at the average velocity of the fluid and measure velocity, temperature, and other variables of interest to weather forcasters. This is an example of a measurement of properties along a pathline.

equations 1.244-1.246 (discussed more fully in Chapter 6 and reference [17, 120]), we see that the substantial derivatives of the density, velocity, and energy appear.

$$\text{Mass conservation} \qquad \frac{D\rho}{Dt} = -\rho\left(\nabla \cdot \underline{v}\right) \tag{1.342}$$

$$\text{Momentum conservation} \qquad \rho\frac{D\underline{v}}{Dt} = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g} \tag{1.343}$$

$$\text{Energy conservation} \qquad \rho\frac{D\hat{E}}{Dt} = -\nabla \cdot \underline{q} - \nabla \cdot (p\underline{v}) + \nabla \cdot \underline{\tilde{\tau}} \cdot \underline{v} + \mathcal{S}_e \tag{1.344}$$

The substantial derivative appears because each of these equations is written in terms of the properties of a field (written in terms of the field variables ρ, $\underline{v}$, $\hat{E}$) and not in terms of the properties of a single, isolated body (Figure 1.48). The mass of a body is conserved in the sense that if the mass changes, if a piece is shaved off, for example, it is not the same body. The momentum of a body is conserved (Newton's second law, Chapter 3), and the energy of a body is conserved (first law of thermodynamics, Chapter 6). For a body, the conservation laws contain the usual rates of change of mass (dm/dt), momentum $(d(m\underline{v})/dt)$,

rate-of-change associated with a body	rate-of-change of a field variable as recorded by an observer moving with the flow

mass	$\dfrac{d(m)_{body}}{dt}$	$\dfrac{D\rho}{Dt}$
momentum	$\dfrac{d(m\underline{v})_{body}}{dt}$	$\rho\dfrac{D\underline{v}}{Dt}$
energy	$\dfrac{d(E)_{body}}{dt}$	$\rho\dfrac{D\hat{E}}{Dt}$

Figure 1.48: The balance equations for mass, momentum, and for energy may be written for a body or for a position in space, that is for a field.

and energy (dE/dt). When one is concerned with the properties characteristic of a location in a *field* rather than of a chosen body, the correct expression for the rate of change of the field variable at a fixed point can be shown to be the substantial derivative (Chapters 3, 6). The rate of change of a property – mass, momentum, energy – for a given position in a field depends both on the instantaneous rate of change of the property at that location $(\partial/\partial t)$ as well as on the rate at which the property is convected to that location by the fluid motion $(\underline{v} \cdot \nabla)$. In Chapter 6 we derive the mass, momentum, and energy balances for a position in a field, and the substantial derivative appears naturally. The concepts outlined here are discussed fully in Chapter 3 and 6. We present two examples below to build familiarity with the substantial derivative.

EXAMPLE 1.29 *Using equation 1.247 to write ∇f, use matrix multiplication to verify the equality of the two expressions below for the substantial derivative.*

Substantial derivative (Gibbs notation)
$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f \tag{1.345}$$

Cartesian coordinates (see Table B.2)
$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1}v_1 + \frac{\partial f}{\partial x_2}v_2 + \frac{\partial f}{\partial x_3}v_3 \tag{1.346}$$

SOLUTION To show the equality of these two equation, we write the Gibbs notation expressions $\underline{v}$ and ∇f in Cartesian coordinates and matrix multiply.

$$\underline{v} \; = \; \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \tag{1.347}$$

$$\nabla f \; = \; \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\[6pt] \dfrac{\partial f}{\partial x_2} \\[6pt] \dfrac{\partial f}{\partial x_3} \end{pmatrix}_{123} \tag{1.348}$$

$$\underline{v} \cdot \nabla f \; = \; \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}_{123} \cdot \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\[6pt] \dfrac{\partial f}{\partial x_2} \\[6pt] \dfrac{\partial f}{\partial x_3} \end{pmatrix}_{123} \tag{1.349}$$

$$= \; v_1 \frac{\partial f}{\partial x_1} + v_2 \frac{\partial f}{\partial x_2} + v_3 \frac{\partial f}{\partial x_3} \tag{1.350}$$

Thus,

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_1} v_1 + \frac{\partial f}{\partial x_2} v_2 + \frac{\partial f}{\partial x_3} v_3 \tag{1.351}$$

EXAMPLE 1.30 *What is the substantial derivative $D\underline{v}/Dt$ of the steady state velocity field represented by the velocity vector below? Note that the answer is a vector.*

$$\underline{v}(x, y, z, t) \; = \; \begin{pmatrix} -3.0x \\ -3.0y \\ 6z \end{pmatrix}_{xyz} \tag{1.352}$$

SOLUTION We begin with the definition of the substantial derivative in equation 1.326 and substitute $\underline{v}$ for f.

$$\frac{D\underline{v}}{Dt} \; = \; \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \tag{1.353}$$

Now we consult Table B.2 to determine the components of $\underline{v} \cdot \nabla \underline{v}$ in Cartesian coordinates, and we construct the Cartesian expression for $D\underline{v}/Dt$.

$$\underline{v} \cdot \nabla \underline{v} = \begin{pmatrix} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz} \tag{1.354}$$

$$\frac{D\underline{v}}{Dt} = \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \tag{1.355}$$

$$= \begin{pmatrix} \frac{\partial v_x}{\partial t} \\ \frac{\partial v_y}{\partial t} \\ \frac{\partial v_z}{\partial t} \end{pmatrix}_{xyz} + \begin{pmatrix} \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{pmatrix}_{xyz} \tag{1.356}$$

Finally, we carry out the partial derivatives on the various terms of the velocity field an substitute these into equation 1.356.

$$\frac{D\underline{v}}{Dt} = \begin{pmatrix} 0 + (-3)(-3x) + 0 + 0 \\ 0 + 0 + (-3)(-3y) + 0 \\ 0 + 0 + 0 + 6(6z) \end{pmatrix}_{xyz} \tag{1.357}$$

$$= \begin{pmatrix} 9x \\ 9y \\ 36z \end{pmatrix}_{xyz} \tag{1.358}$$

1.3.4   Practical Advice

The analysis of flows often means solving for density, velocity, and stress fields. The equations that we encounter in these analyses are ordinary differential equations (ODEs) and partial differential equations (PDEs). The solutions of differential equations give the complete density field, velocity field, and stress field for the problem, from which many engineering quantities can be calculated. In this text it is assumed that students have taken multivariable calculus, linear algebra, and a first course in solving differential equations; we shall be applying these skills and other mathematics skills in studying fluid mechanics.

To aid students in preparing to study fluid mechanics, the web appendix[124] contains a review of solution methods for differential equations. Also, several exercises below provide problem-solving practice that some may find helpful. For instructional videos on mathematics through differential equations, see reference [88]. For more on solving ODEs and PDEs see the web appendix [124] and reference[76]. We move on to modeling flows in general in the next chapter.