Homework 3

CM4650

Solution

1. $$\underline{\underline{A}} + \underline{\underline{A}}^T = A_{ij}\hat{e}_i\hat{e}_j + \left(A_{pk}\hat{e}_p\hat{e}_k\right)^T$$

$$= A_{ij}\hat{e}_i\hat{e}_j + A_{pk}\hat{e}_k\hat{e}_p$$

$i \to m$ $\quad$ $k \to m$

$j \to s$ $\quad$ $p \to s$

$$= A_{ms}\hat{e}_m\hat{e}_s \; A_{sm}\hat{e}_m\hat{e}_s$$

$$= \left(A_{ms} + A_{sm}\right)\hat{e}_m\hat{e}_s$$

which is symmetric (switching m & s gives the same tensor.)

$$\underline{\underline{A}} - \underline{\underline{A}}^T = A_{ij}\,\hat{e}_i\,\hat{e}_j - A_{pk}\,\hat{e}_k\,\hat{e}_p$$

$i \to m$ $\qquad$ $k \to m$

$j \to s$ $\qquad$ $p \to s$

$$= A_{ms}\,\hat{e}_m\,\hat{e}_s - A_{sm}\,\hat{e}_m\,\hat{e}_s$$

$$= \left(A_{ms} - A_{sm}\right)\hat{e}_m\,\hat{e}_s$$

Which is antisymmetric
(switching m & s
gives the
negative of the
result.)

2. Text 2.23

$$\underline{\underline{A}} = 5\hat{e}_1\hat{e}_1 + 3\hat{e}_1\hat{e}_2 - 3\hat{e}_1\hat{e}_3 - \hat{e}_2\hat{e}_1 - \hat{e}_2\hat{e}_2 + 2e_2\hat{e}_3 - 3\hat{e}_3\hat{e}_1$$

$$= \begin{pmatrix} 5 & 3 & -3 \\ -1 & -1 & 2 \\ -3 & 0 & 0 \end{pmatrix}_{123}$$

$$|\underline{\underline{A}}| = \sqrt{\frac{\underline{\underline{A}}:\underline{\underline{A}}}{2}}$$

δ_{pj} "p becomes j"

$$\underline{\underline{A}}:\underline{\underline{A}} = A_{mp}\hat{e}_m\hat{e}_p : A_{jk}\hat{e}_j\hat{e}_k$$

δ_{mk} "m becomes k"

$$= A_{kj}A_{jk}$$

$$= \sum_{k=1}^{3}\sum_{j=1}^{3} \underbrace{A_{kj}A_{jk}}_{\text{pairs across diagonal}}$$

$$\underline{\underline{A}}:\underline{\underline{A}} = (5)^2 + (-1)(3) + (-3)(-3) + (-1)(3) + (-1)^2 + 2(0) + (-3)(-3) + 0(2) + 0^2$$

$$= 25 - 3 + 9 - 3 + 1 + 9$$

$$= 38$$

$$\sqrt{\frac{A:A}{2}} = \sqrt{\frac{38}{2}} = \boxed{\sqrt{19}}$$

3. $$I_{\underline{\underline{A}}} = \operatorname{trace} \underline{\underline{A}} = (3) + 0 + (-3) = \boxed{0}$$

$$II_{A} = \operatorname{trace} \underline{\underline{A}} \cdot \underline{\underline{A}}$$

$$\underline{\underline{A}} \cdot \underline{\underline{A}} = \begin{pmatrix} 3 & 10.2 & 0 \\ 10.2 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}_{123} \cdot \begin{pmatrix} 3 & 10.2 & 0 \\ 10.2 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}_{123}$$

$$= \begin{pmatrix} 113.04 & 30.6 & 0 \\ 30.6 & 104.04 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$I_{\underline{\underline{A}}} \ \operatorname{trace} \underline{\underline{A}} \cdot \underline{\underline{A}} = \boxed{226.08}$$

$$\mathrm{III}_{\underline{\underline{A}}} = \operatorname{trace} \underline{\underline{A}}\cdot\underline{\underline{A}}\cdot\underline{\underline{A}}$$

$$\underline{\underline{A}}\cdot\underline{\underline{A}}\cdot\underline{\underline{A}} = \begin{pmatrix} 113.04 & 30.6 & 0 \\ 30.6 & 104.04 & 0 \\ 0 & 0 & 9 \end{pmatrix}_{123} \begin{pmatrix} 3 & 10.2 & 0 \\ 10.2 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix}_{123}$$

$$= \begin{pmatrix} 651.24 & 1153.008 & 0 \\ 1153.008 & 312.12 & 0 \\ 0 & 0 & 27 \end{pmatrix}_{123}$$

$$\operatorname{trace} \underline{\underline{A}}\cdot\underline{\underline{A}}\cdot\underline{\underline{A}} = \mathrm{III}_{\underline{\underline{A}}} = \boxed{990.36}$$

We can also use Excel (or other program). I used Excel for $\underline{\underline{B}}$.

We can more easily do these matrix calculations in Excel (Matlab, Mathcad, Mathematica). I used Excel, using the matrix command MMULT(array1,array2) (must enter with "CTRL+SHIFT+ENTER). I did not use the MDETERM function as it seems to give the wrong sign for the determent.

					Trace	From definition p453:	
A	3	10.2	0	I=	0	Θ=	0
	10.2	0	0			Φ=	-113.04
	0	0	-3			Ψ=	312.12
A.A	113.04	30.6	0	II=	226.08	from relations on p476:	
	30.6	104.04	0			Θ=	0
	0	0	9			Φ=	-113.04
						Ψ=	312.12
A.A.A	651.24	1153.008	0	III=	936.36		
	1153.008	312.12	0				
	0	0	-27				

					Trace	From definition p453:	
B	4	0	0	I=	0	Θ=	0
	0	4	0			Φ=	-48
	0	0	-8			Ψ=	-128
B.B	16	0	0	II=	96	from relations on p476:	
	0	16	0			Θ=	0
	0	0	64			Φ=	-48
						Ψ=	-128
B.B.B	64	0	0	III=	-384		
	0	64	0				
	0	0	-512				

4. Fake-O Model®

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\,\underline{\underline{\dot{\gamma}}}$$

$$M(\dot{\gamma}_0) = \begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

shear start up:

$$\underline{v} = \begin{pmatrix} \dot{\zeta}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\zeta}(t) & 0 \\ \dot{\zeta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

<u>ASIDE:</u>

$$\dot{\gamma} = \sqrt{\frac{\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}}{2}} = \sqrt{\frac{\left(\dot{\zeta}(t)\right)^2(2)}{2}}$$

$$= |\dot{\zeta}(t)| = \dot{\zeta}(t)$$

$$\dot{\zeta}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

$$\underline{\underline{\tau}} = -M(\dot{\gamma}_0)\,\underline{\underline{\dot{\gamma}}}$$

$$= -M(\dot{\gamma}_0)\begin{pmatrix} 0 & \dot{\zeta}(t) & 0 \\ \dot{\zeta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\eta^+(t) = -\frac{\tau_{21}}{\dot{\gamma}_0}$$

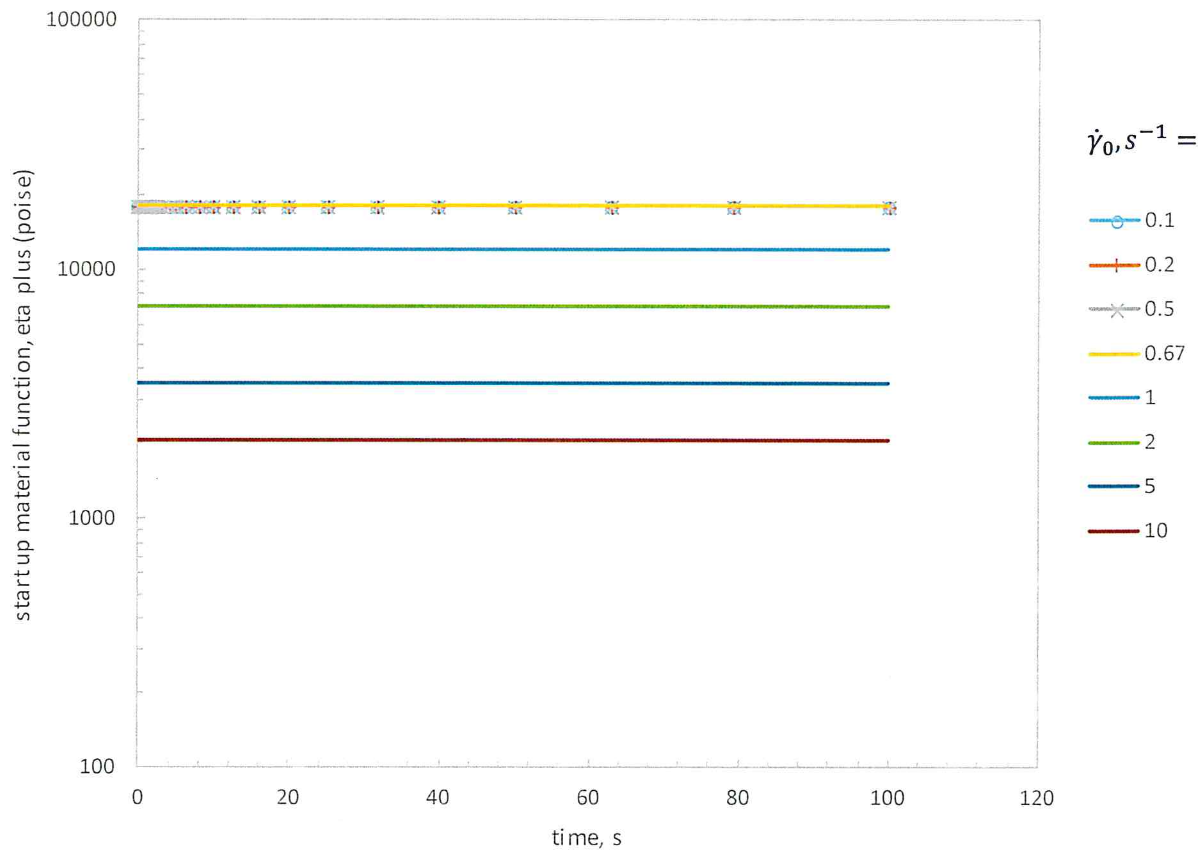
$$= \left(-\frac{1}{\dot{\gamma}_0}\right)\left(-M\,\dot{\zeta}(t)\right)$$

$$= \frac{M(\dot{\gamma}_0)\,\dot{\zeta}(t)}{\dot{\gamma}_0}$$

$$= \frac{\dot{\zeta}(t)}{\dot{\gamma}_0}\begin{cases} M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{\,n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}$$

$$\boxed{\eta^+(t) = \begin{cases} \dfrac{M_0\,\dot{\zeta}(t)}{\dot{\gamma}_0} & \dot{\gamma}_0 < \dot{\gamma}_c \\[2ex] \dfrac{m\dot{\gamma}_0^{\,n-1}}{\dot{\gamma}_0}\,\dot{\zeta}(t) & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases}}$$ see plot

(Problem 4): Note that for $\dot{\gamma} < \dot{\gamma}_c$, the step up is to 18,000 poise. For larger $\dot{\gamma}$, the start-up function rises to a lower and lower value (shear thinning).



$$\Psi_1^+ = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} = 0$$

$$\Psi_2^+ = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} = 0$$

5.) $$\underline{\underline{\tau}} = A(\nabla \underline{v} \cdot (\nabla \underline{v})^T) + B\nabla\underline{v} + C(\nabla\underline{v})^T$$

$$\nabla \underline{v} \cdot (\nabla \underline{v})^T = \frac{\partial}{\partial x_p} \hat{e}_p v_m \hat{e}_m \cdot \left(\frac{\partial}{\partial x_s} \hat{e}_s v_a \hat{e}_a\right)^T$$

$$= \frac{\partial v_m}{\partial x_p} \hat{e}_p \hat{e}_m \cdot \frac{\partial v_a}{\partial x_s} \hat{e}_a \hat{e}_s$$

δ_{ma} "m becomes a"

$$= \frac{\partial v_a}{\partial x_p} \frac{\partial v_a}{\partial x_s} \hat{e}_p \hat{e}_s$$

which is symmetric.

But, $\nabla \underline{v}$, $(\nabla \underline{v})^T$ are not symmetric.

For the overall expression to be symmetric $B(\nabla \underline{v}) + C(\nabla \underline{v})^T$ must be symmetric, i.e. $\boxed{B = C}$

(7) $\bar{\eta}^{+}(\dot{\varepsilon}) = \eta_e(\dot{\varepsilon})$ for Newtonian?

Newtonian:

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$$

Elongational flow:

$$\underline{v} = \begin{pmatrix} -\frac{1}{2}\dot{\varepsilon}(t)x_1 \\ -\frac{1}{2}\dot{\varepsilon}(t)x_2 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix}_{123}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} -\dot{\varepsilon}(t) & 0 & 0 \\ 0 & -\dot{\varepsilon}(t) & 0 \\ 0 & 0 & 2\dot{\varepsilon}(t) \end{pmatrix}_{123}$$

stress growth (start up): $\dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases}$

Newtonian:

$$\underline{\underline{\tau}} = -\mu \,\dot{\underline{\underline{\gamma}}}$$

$$= \begin{pmatrix} +\mu\dot{\varepsilon}(t) & 0 & 0 \\ 0 & +\mu\dot{\varepsilon}(t) & 0 \\ 0 & 0 & -2\mu\dot{\varepsilon}(t) \end{pmatrix}_{123}$$

$$\eta_e^+ = \frac{-(\tau_{33} - \tau_{11})}{\dot{\varepsilon}_0}$$

$$= \frac{-(-2\mu\dot{\varepsilon}(t) - \mu\dot{\varepsilon}(t))}{\dot{\varepsilon}_0}$$

$$\eta_e^+ = \frac{3\mu\,\dot{\varepsilon}(t)}{\dot{\varepsilon}_0} = \frac{3\mu}{\dot{\varepsilon}_0}\begin{cases} 0 & t<0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases}$$

$$\eta_e^+(t) = \begin{cases} 0 & t<0 \\ 3\mu & t \geq 0 \end{cases}$$