Final Exam
Soln
CM 4650
2018

1. $\underline{a} \cdot \nabla \underline{b}$

$$a_i \hat{e}_i \cdot \frac{\partial}{\partial x_p} \hat{e}_p \, b_m \hat{e}_m$$

δ_{ip} "i becomes p"

$$= a_p \frac{\partial b_m}{\partial x_p} \hat{e}_m \quad \underline{\text{vector}}$$

$$= \sum_{p=1}^{3} \sum_{m=1}^{3} a_p \frac{\partial b_m}{\partial x_p} \hat{e}_m$$

$$= \begin{pmatrix} \sum_{p=1}^{3} a_p \frac{\partial b_1}{\partial x_p} \\ \sum_{p=1}^{3} a_p \frac{\partial b_2}{\partial x_p} \\ \sum_{p=1}^{3} a_p \frac{\partial b_3}{\partial x_p} \end{pmatrix}_{123}$$

2.

a) False. Cox Merz is only true when it's true, i.e. it must be shown to hold before we use it.

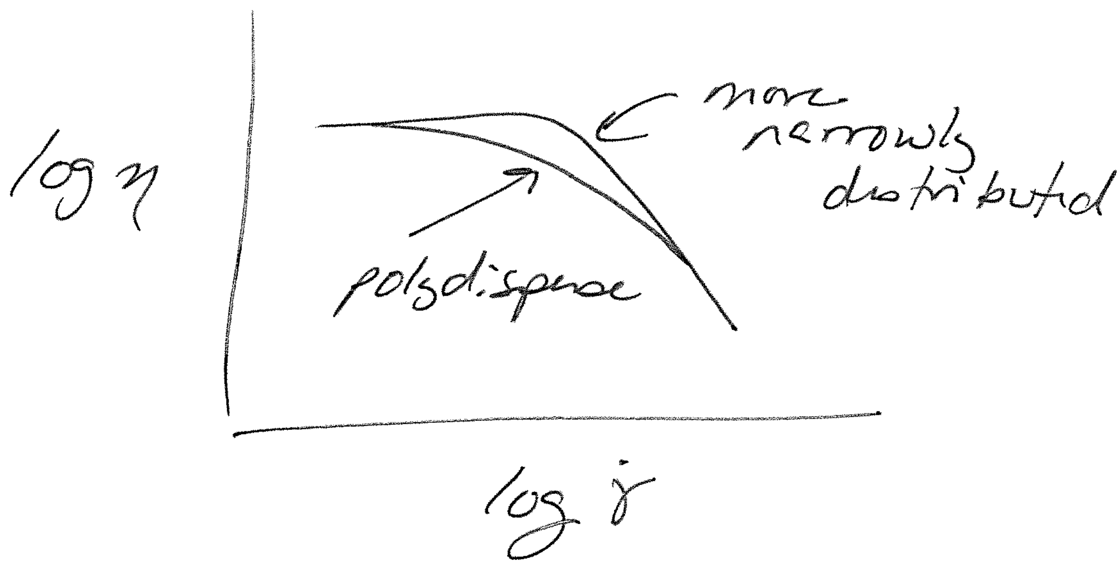
b) False. Trouton's rule generally does not hold at large $\dot{\gamma}$.

c) We can tell if a material is entangled

① from the "hockey stick graph" i.e. $\log \eta_0$ vs $\log M$, which shows a break at M_c, the critical MW for entanglement

② In SAOS $G'(\omega)$ has a plateau ~ $G' \sim 10^4 - 10^5$ when entangled.

d)



3. What is Ψ_1 for Lodge model?

<u>Lodge</u>

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} \underline{\underline{C}}^{-1}(t',t)\,dt'$$

For Ψ_1, $\dot{\varsigma}(t'') = \dot{\gamma}_0$

$$\Psi_1 = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^{\,2}}$$

<u>Table 9.3</u>

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We need γ; $\gamma = \int_{t'}^{t} \dot{\varsigma}(t'')\,dt''$

$$= \int_{t'}^{t} \dot{\gamma}_0\,dt''$$

$$= \dot{\gamma}_0 t'' \Big|_{t'}^{t} = \boxed{\dot{\gamma}_0(t-t')} = \gamma$$

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We need $\tau_{11} - \tau_{22}$

$$-\tau_{11} = \int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} (1+\gamma^2)\, dt'$$

$$-\tau_{22} = \int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} (1)\, dt'$$

$$-(\tau_{11} - \tau_{22}) = \int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} \underbrace{(\cancel{1}+\gamma^2 \cancel{-1})}_{\gamma^2 = \dot{\gamma}_0^2 (t-t')^2} dt'$$

$$\frac{-(\tau_{11}-\tau_{22})\lambda^2}{\eta_0 \dot{\gamma}_0^2} = \lambda^2 \lambda \int_{-\infty}^{t} \underbrace{e^{\frac{-(t-t')}{\lambda}}}_{\equiv e^{u}} \underbrace{\frac{(t-t')^2}{\lambda^2}}_{u^2} \underbrace{\frac{dt'}{\lambda}}_{du}$$

$$\Rightarrow u = \frac{-t}{\lambda} + \frac{t'}{\lambda} = -\frac{1}{\lambda}(t-t')$$

$$\frac{du}{dt'} = \frac{1}{\lambda} \qquad \Rightarrow \frac{(t-t')^2}{\lambda^2} = u^2$$

$$\rightarrow du = \frac{1}{\lambda} dt'$$

$t' = -\infty \quad u = -\infty$

$t' = t \quad u = 0$

$= \psi_1$

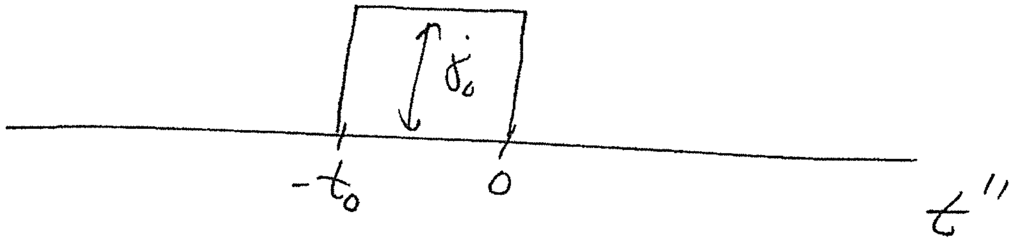
$$\frac{-(\tau_{11}-\tau_{22})}{\eta_0 \lambda \dot{\gamma}_0^2} = \int_{-\infty}^{0} e^{u} u^2 \, du$$

$$= e^{u}(u^2 - 2u + 2)\Big|_{-\infty}^{0}$$

$$= 2 - 0 = 2$$

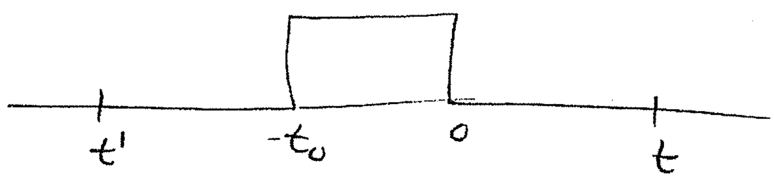
$$\psi_1 = \frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0^2} = \boxed{2\eta_0 \lambda}$$

7.



$$\gamma_{21}(t';t) = \int_{t'}^{t} \dot{\gamma}(t'')\,dt''$$

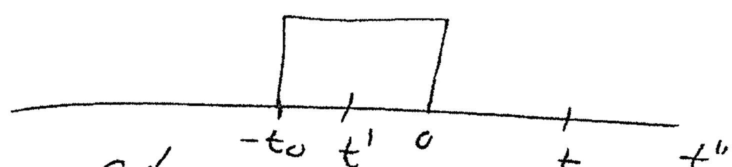
CASE I: $t' < -t_0$



$$\gamma = \int_{t'}^{-t_0} 0\,dt'' + \int_{-t_0}^{0} \dot{\gamma}_0\,dt'' + \int_{0}^{t} 0\,dt''$$

$$\boxed{\gamma = t_0 \dot{\gamma}_0}$$

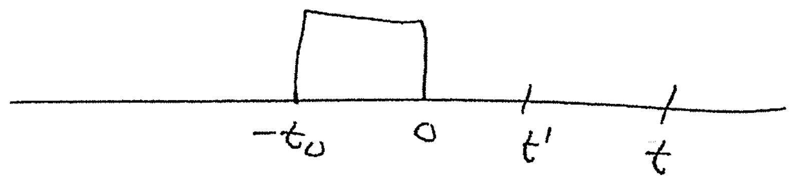
CASE II $-t_0 \le t' \le 0$



$$\gamma = \int_{t'}^{0} \dot{\gamma}_0\,dt'' + \int_{0}^{t} 0\,dt''$$

$$\boxed{\gamma = -\dot{\gamma}_0 t'}$$ (note $t'<0$ $\therefore$ γ positive)

CASE III $\quad t' > 0$



$$\gamma = \int_{t'}^{t} 0 \, dt'' = 0$$

$$\boxed{\gamma = \begin{cases} t_0 \dot{\gamma}_0 & t' < -t_0 \\ -\dot{\gamma}_0 t' & -t_0 \le t' \le 0 \\ 0 & t' > 0 \end{cases}}$$

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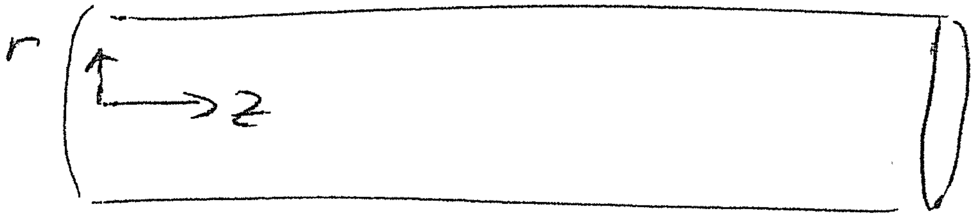
5. PL-GNF



$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v}\cdot\nabla\underline{v}}}_{\text{uni dir}}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \underbrace{\cancel{\rho \underline{g}}}_{\text{neglect}}$$

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}$$

$$\nabla\cdot\underline{V} = 0 \Rightarrow \boxed{\frac{\partial V_z}{\partial z} = 0}$$

See next pg

$$\underline{\underline{\tau}} = \eta(\dot\gamma)\,\underline{\underline{\dot\gamma}} = \eta(\dot\gamma)\left(\nabla\underline{v} + (\nabla \underline{v})^T\right)$$

(see next pg)

$$\eta = m\dot\gamma^{\,n-1}$$

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C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla a = \begin{pmatrix} \dfrac{\partial a}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial a}{\partial \theta} \\ \dfrac{\partial a}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

for $\underline{w} = \underline{v}$ $\longrightarrow$

$$\nabla \cdot \underline{w} = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(r w_r)} + \cancel{\frac{1}{r}\frac{\partial w_\theta}{\partial \theta}} + \frac{\partial w_z}{\partial z} = 0$$

$$\nabla \times \underline{w} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial w_z}{\partial \theta} - \dfrac{\partial w_\theta}{\partial z} \\ \dfrac{\partial w_r}{\partial z} - \dfrac{\partial w_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial (r w_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z}$$

$\Longrightarrow$ $\boxed{\dfrac{\partial v_z}{\partial z} = 0}$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

for $\underline{w} = \underline{v}$ $\longrightarrow$

$$\nabla \underline{w} = \begin{pmatrix} \dfrac{\partial w_r}{\partial r} & \dfrac{\partial w_\theta}{\partial r} & \dfrac{\partial w_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} - \dfrac{w_\theta}{r} & \dfrac{1}{r}\dfrac{\partial w_\theta}{\partial \theta} + \dfrac{w_r}{r} & \dfrac{1}{r}\dfrac{\partial w_z}{\partial \theta} \\ \dfrac{\partial w_r}{\partial z} & \dfrac{\partial w_\theta}{\partial z} & \dfrac{\partial w_z}{\partial z} \end{pmatrix}_{r\theta z}.$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial (r w_r)}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 w_r}{\partial \theta^2} + \dfrac{\partial^2 w_r}{\partial z^2} - \dfrac{2}{r^2}\dfrac{\partial w_\theta}{\partial \theta} \\ \dfrac{\partial}{\partial r}\left[\dfrac{1}{r}\dfrac{\partial (r w_\theta)}{\partial r}\right] + \dfrac{1}{r^2}\dfrac{\partial^2 w_\theta}{\partial \theta^2} + \dfrac{\partial^2 w_\theta}{\partial z^2} + \dfrac{2}{r^2}\dfrac{\partial w_r}{\partial \theta} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r \dfrac{\partial w_z}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2 w_z}{\partial \theta^2} + \dfrac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z}$$

F. A. Morrison, Understanding Rheology (Oxford, 2001)

Table C.3

$$\nabla \underline{v} = \begin{pmatrix} \cancel{\frac{\partial v_r}{\partial r}} & \cancel{\frac{\partial v_\theta}{\partial r}} & \frac{\partial v_z}{\partial r} \\ \frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}} - \cancel{\frac{v_\theta}{r}} & \frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{v_r}{r}} & \frac{1}{r}\cancel{\frac{\partial v_z}{\partial \theta}} \\ \cancel{\frac{\partial v_r}{\partial z}} & \cancel{\frac{\partial v_\theta}{\partial z}} & \cancel{\frac{\partial v_z}{\partial z}} \end{pmatrix}_{r\theta z}$$

θ sym

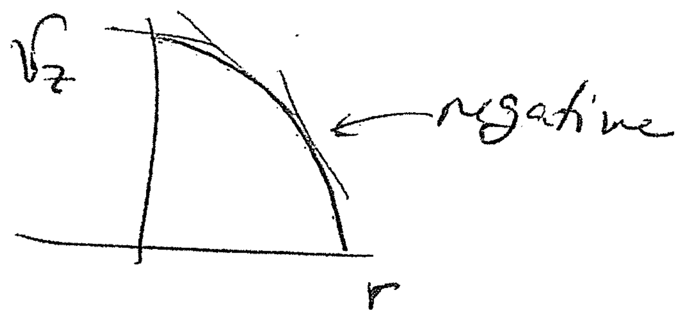
$\nabla \cdot \underline{v} = 0$

$$= \begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$\dot{\underline{\underline{\gamma}}} = \begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial v_z}{\partial r} & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$|\underline{\underline{\dot\gamma}}| = \sqrt{\frac{\underline{\underline{\dot\gamma}}:\underline{\underline{\dot\gamma}}}{2}} = \sqrt{\left(\frac{\partial v_z}{\partial r}\right)^2}$$

$$= \left|\frac{\partial v_z}{\partial r}\right| = \left(-\frac{\partial v_z}{\partial r}\right)$$



$$\underline{\underline{\tau}} = -\eta\,\underline{\underline{\dot\gamma}} = -m\,\dot\gamma^{\,n-1}\,\underline{\underline{\dot\gamma}}$$

$$= -m\left(-\frac{\partial v_z}{\partial r}\right)^{n-1}\begin{pmatrix} 0 & 0 & \frac{\partial v_z}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial v_z}{\partial r} & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$= \begin{pmatrix} \tau_{rr} & \tau_{r\theta} & \tau_{rz} \\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z} \\ \tau_{zr} & \tau_{z\theta} & \tau_{zz} \end{pmatrix}_{r\theta z}$$

Back to Cauchy Momentum Eqn

$$0 = -\nabla p - \underbrace{\nabla \cdot \underline{\underline{\tau}}}_{\text{need from Table C.3}}$$

(see next page)

v_z does not vary w/ z

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{r\theta z} = \begin{pmatrix} -\frac{\partial p}{\partial r} \\ -\frac{1}{r}\frac{\partial p}{\partial \theta} \\ -\frac{\partial p}{\partial z} \end{pmatrix}_{r\theta z} - \begin{pmatrix} \cancel{\frac{\partial}{\partial z}\left(\tau_{zr}\right)} \\ 0 \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right) \end{pmatrix}_{r\theta z}$$

Now we consider each component individually $\longrightarrow$

for $\underline{\underline{A}} = \underline{\underline{\tau}}$

$$\nabla \cdot \underline{\underline{A}} = \begin{pmatrix} \cancel{\frac{1}{r}\frac{\partial}{\partial r}(rA_{rr})} + \cancel{\frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta}} + \frac{\partial A_{zr}}{\partial z} - \cancel{\frac{A_{\theta\theta}}{r}} \\ \cancel{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_{r\theta})} + \cancel{\frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta}} + \cancel{\frac{\partial A_{z\theta}}{\partial z}} + \cancel{\frac{A_{\theta r} - A_{r\theta}}{r}} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_{rz}) + \cancel{\frac{1}{r}\frac{\partial A_{\theta z}}{\partial \theta}} + \cancel{\frac{\partial A_{zz}}{\partial z}} \end{pmatrix}_{r\theta z} \qquad \text{(C.3-10)}$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r \left(\frac{\partial w_r}{\partial r}\right) + u_\theta \left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + u_z \left(\frac{\partial w_r}{\partial z}\right) \\ u_r \left(\frac{\partial w_\theta}{\partial r}\right) + u_\theta \left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + u_z \left(\frac{\partial w_\theta}{\partial z}\right) \\ u_r \left(\frac{\partial w_z}{\partial r}\right) + u_\theta \left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + u_z \left(\frac{\partial w_z}{\partial z}\right) \end{pmatrix}_{r\theta z} \qquad \text{(C.3-11)}$$

TABLE C.4
Differential Operations in the Spherical Coordinate System r, θ, ϕ

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_\phi \end{pmatrix}_{r\theta\phi} \qquad \text{(C.4-1)}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi} \qquad \text{(C.4-2)}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{1}{r \sin\theta}\frac{\partial a}{\partial \phi} \end{pmatrix}_{r\theta\phi} \qquad \text{(C.4-3)}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial a}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial a}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 a}{\partial \phi^2} \qquad \text{(C.4-4)}$$

$$\nabla \cdot \underline{w} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(w_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial w_\phi}{\partial \phi} \qquad \text{(C.4-5)}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(w_\phi \sin\theta) - \frac{1}{r \sin\theta}\frac{\partial w_\theta}{\partial \phi} \\ \frac{1}{r \sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(r w_\phi) \\ \frac{1}{r}\frac{\partial}{\partial r}(r w_\theta) - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta\phi} \qquad \text{(C.4-6)}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi} \end{pmatrix}_{r\theta\phi} \qquad \text{(C.4-7)}$$

continued

r - component:

$$\frac{\partial P}{\partial r} = 0$$

θ - component:

$$\frac{1}{r}\frac{\partial P}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{\partial P}{\partial \theta} = 0$$

z - component:

$$\lambda = \underbrace{\frac{\partial P}{\partial z}}_{f(z)} = \underbrace{-\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\tau_{rz}\right)}_{f(r)}$$

$$\frac{dP}{dz} = \lambda$$

$$P = \lambda z + C_1 \qquad BC: \; z=0 \;\; P=P_0$$
$$z=L \;\; P=P_L$$

$$C_1 = P_0$$

$$\lambda = \frac{P_L - P_0}{L}$$

$$\boxed{P = \frac{P_L - P_0}{L} z + P_0}$$

$$-\frac{\Delta P}{L} = -\frac{1}{r}\frac{d}{dr}(r\tau_{rz})$$

$$\frac{\Delta P\, r}{L} = \frac{d}{dr}(r\tau_{rz})$$

$$r\tau_{rz} = \frac{\Delta P}{L}\frac{r^2}{2} + C_2$$

$$\boxed{\tau_{rz} = \frac{\Delta P}{2L} r + \frac{C_2}{r}}$$

BC: $r = 0$

$\tau_{rz} = \text{finite}$

$\Rightarrow C_2 = 0$

$$\boxed{\tau_{rz} = \frac{\Delta P\, r}{2L}}$$

From constitutive eqn:

$$-m\left(-\frac{\partial V_z}{\partial r}\right)^{n-1}\left(\frac{\partial V_z}{\partial r}\right) = \frac{\Delta P r}{2L}$$

$$\left(-\frac{\partial V_z}{\partial r}\right)^{n} = \frac{\Delta P r}{2Lm}$$

$$-\frac{dV_z}{dr} = \left(\frac{\Delta P}{2mL}\right)^{\frac{1}{n}} r^{\frac{1}{n}}$$

$$\boxed{V_z = -\left(\frac{\Delta P}{2mL}\right)^{\frac{1}{n}} \frac{r^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_3}$$

BC: $r = R$ $V_z = 0$

(already used $r = 0$ BC but could say $\frac{dV_z}{dr} = 0$ there) //