CM 4650
EXAM 1
SOLN
2020

1.) $\underline{v} = (\nabla \underline{w})^T \cdot \underline{u} = \left(\frac{\partial}{\partial x_p}\hat{e}_p w_n \hat{e}_n\right)^T \cdot u_f \hat{e}_f$

δ_{pf} "p becomes f"

$$= \frac{\partial w_n}{\partial x_p} \hat{e}_n \hat{e}_p \cdot u_f \hat{e}_f$$

$$= \frac{\partial w_n}{\partial x_f} u_f \hat{e}_n$$

$$\underline{v} = \begin{pmatrix} \sum_{f=1}^{3} \frac{\partial w_1}{\partial x_f} u_f \\ \sum_{f=1}^{3} \frac{\partial w_2}{\partial x_f} u_f \\ \sum_{f=1}^{3} \frac{\partial w_3}{\partial x_f} u_f \end{pmatrix}_{123}$$

a

b) $$V_2 = \sum_{f=1}^{3} \frac{\partial w_2}{\partial x_f} u_f$$

$$\boxed{V_2 = \frac{\partial w_2}{\partial x_1} u_1 + \frac{\partial w_2}{\partial x_2} u_2 + \frac{\partial w_2}{\partial x_3} u_3}$$

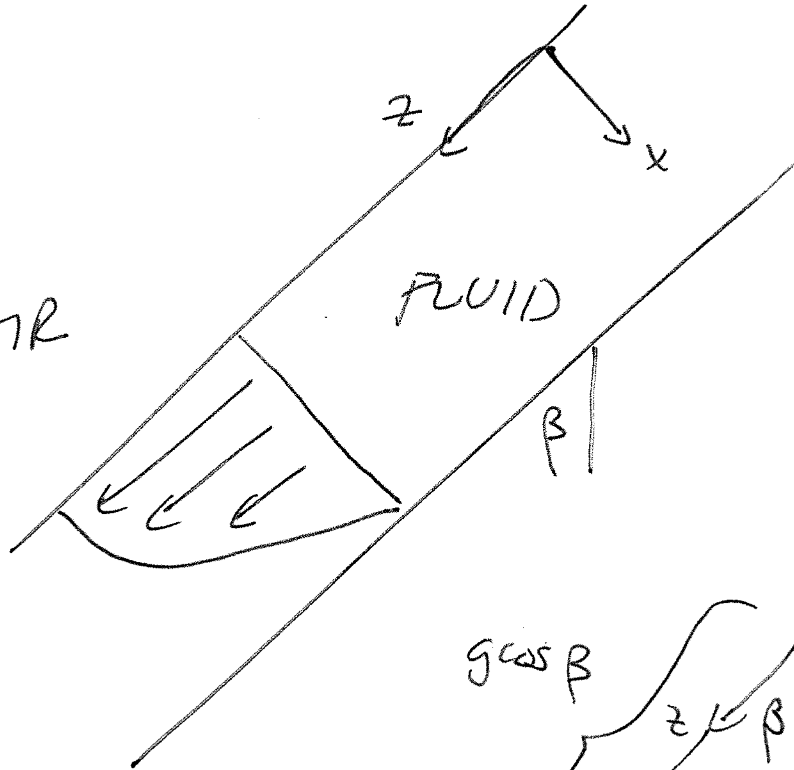
2) $$\underline{m} = \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_1}{\partial x_3^2} \\ \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_3^2} \\ \frac{\partial^2 w_3}{\partial x_1^2} + \frac{\partial^2 w_3}{\partial x_2^2} + \frac{\partial^2 w_3}{\partial x_3^2} \end{pmatrix}_{123}$$

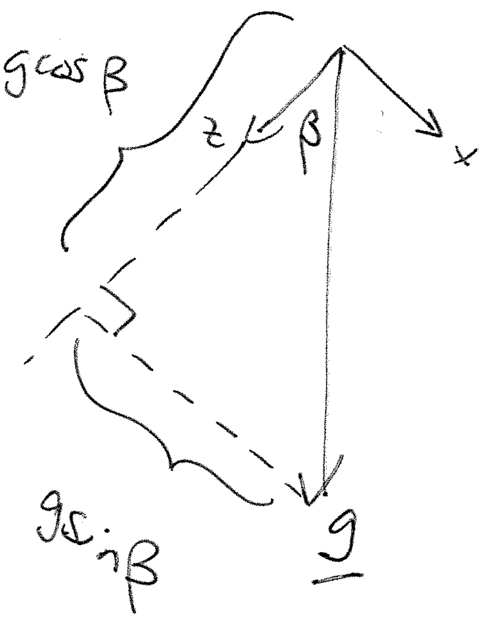
try $\nabla \cdot \nabla \underline{w}$: $$\frac{\partial}{\partial x_p} \overbrace{e_p \cdot \frac{\partial}{\partial x_n} \hat{e}_n}^{\delta_{pn} \text{ "p becomes n"}} w_k \hat{e}_k$$

$$= \frac{\partial}{\partial x_n} \frac{\partial w_k}{\partial x_n} \hat{e}_k$$

this matches //

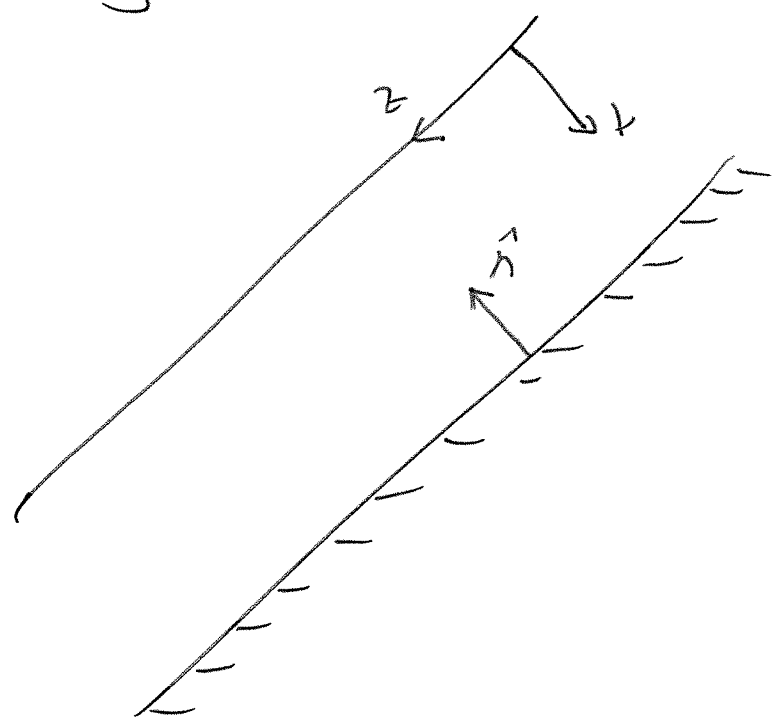
3)





a) $$\underline{g} = \begin{pmatrix} g\sin\beta \\ 0 \\ g\cos\beta \end{pmatrix}_{xyz}$$

b) $$\underline{F} = \iint_S \left(\hat{n} \cdot -\underline{\underline{\Pi}}\right)\Big|_{\text{surface}} dS$$



$$\hat{n} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{xyz} = -\hat{e}_x$$

"surface" is at $x = H$

$dS = dz\,dy$ (into wall)

$$\underline{\underline{F}} = \int_0^W \int_0^L \left((-\hat{e}_x) \cdot -\underline{\underline{\Pi}} \right) \Bigg|_{x=H} dz\, dy$$

//

4) Boundary conditions?

$x = H \qquad v_z = 0$ (no slip at the wall)

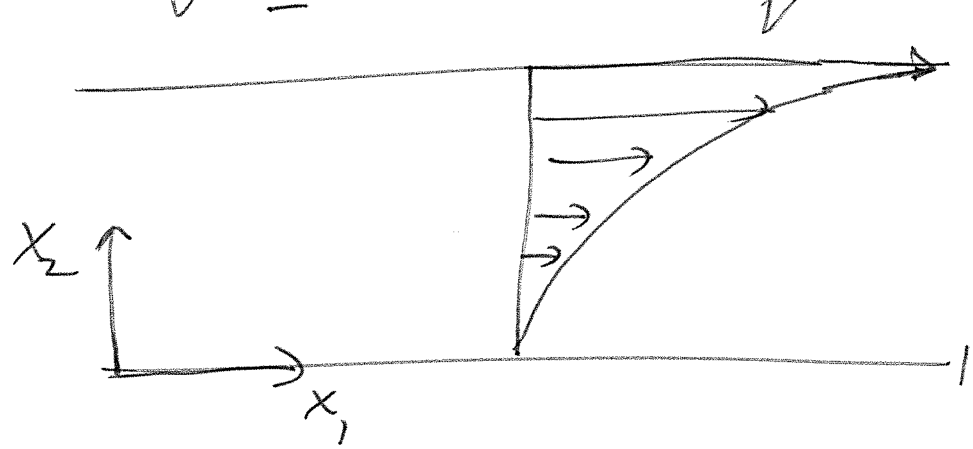
$x = 0 \qquad \dfrac{dv_z}{dx} = 0$ (maximum at surface)

This is a stress-matching BC. In the air $\tau_{xz_{air}} = -\mu_{AIR} \dfrac{dV_z}{dx} \approx 0$ due to small μ_{AIR}. At "free" surface $\tau_{xz_{AIR}} = \tau_{xz_{FLUID}}$ //

momentum + mass conservation:

5.) $$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{v}\cdot\nabla\underline{v}}}_{\text{unidirectional}}\right) = -\nabla p + \mu\nabla^2\underline{v} + \underbrace{\cancel{\rho\underline{g}}}_{0}$$

$$\nabla\cdot\underline{v} = 0$$



$X_1 = 0$
$P = P_0$

$X_1 = L$
$P = P_L$

steady
unidirectional
$\underline{g}$ neglected

$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{xyz}$$

δ_{pk} "p becomes k"

<u>MASS</u>

$$\nabla\cdot\underline{v} = 0 = \frac{\partial}{\partial x_p}\overbrace{\hat{e}_p \cdot v_k \hat{e}_k}$$

$$= \frac{\partial v_k}{\partial x_k} = \frac{\partial v_1}{\partial x_1} + \cancel{\frac{\partial v_2}{\partial x_2}} + \cancel{\frac{\partial v_3}{\partial x_3}}$$

$$\Rightarrow \boxed{\frac{\partial v_1}{\partial x_1} = 0}$$

δ_{fc} "f becomes c" (7)

$$\nabla^2 \underline{v} = \frac{\partial}{\partial x_f}\hat{e}_f \cdot \frac{\partial}{\partial x_c}\hat{e}_c\, v_j \hat{e}_j$$

$$= \frac{\partial}{\partial x_c}\frac{\partial v_j}{\partial x_c}\hat{e}_j = \frac{\partial^2 v_j}{\partial x_c^2}\hat{e}_j$$

$$= \begin{pmatrix} \sum_{c=1}^{3} \frac{\partial^2 v_1}{\partial x_c^2} \\ \sum_{c=1}^{3} \cancel{\frac{\partial^2 v_2}{\partial x_c^2}} \\ \sum_{c=1}^{3} \cancel{\frac{\partial^2 v_3}{\partial x_c^2}} \end{pmatrix}_{123} \quad \begin{matrix} \\ v_2 = 0 \\ v_3 = 0 \end{matrix}$$

$\nabla \cdot \underline{v} = 0$

$$= \begin{pmatrix} \cancel{\frac{\partial^2 v_1}{\partial x_1^2}} + \frac{\partial^2 v_1}{\partial x_2^2} + \cancel{\frac{\partial^2 v_1}{\partial x_3^2}}^{\,0} \\ 0 \\ 0 \end{pmatrix}_{123}$$

wide flow

$$\nabla^2 \underline{V} = \begin{pmatrix} \frac{\partial^2 V_1}{\partial x_2^2} \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$\nabla p = \frac{\partial}{\partial x_f} \hat{e}_f \, P$$

$$= \begin{pmatrix} \frac{\partial P}{\partial x_1} \\ \frac{\partial P}{\partial x_2} \\ \frac{\partial P}{\partial x_3} \end{pmatrix}_{123}$$

Assemble NS:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} -\frac{\partial P}{\partial x_1} \\ -\frac{\partial P}{\partial x_2} \\ -\frac{\partial P}{\partial x_3} \end{pmatrix}_{123} + \mu \begin{pmatrix} \frac{\partial^2 V_1}{\partial x_2^2} \\ 0 \\ 0 \end{pmatrix}_{123}$$

X_2 - component:

$$\boxed{-\frac{\partial P}{\partial X_2} = 0} \Longrightarrow P \text{ not a function of } x_2$$

X_3 - component:

$$\boxed{-\frac{\partial P}{\partial X_3} = 0} \Longrightarrow P \text{ not a function of } X_3$$

X_1 - component:

$$\boxed{0 = -\frac{\partial P}{\partial X_1} + \mu \frac{\partial^2 V_1}{\partial X_2^2}} \quad (a)$$

(b) BC:

$X_2 = 0$	$V_1 = 0$	no slip at bottom
$X_2 = H$	$V_1 = V$	no slip at top

Solve

$P = P(x_1)$ only
$V_1 = V_1(x_2)$ only

$$\underbrace{\frac{dP}{dx_1}}_{\text{function of } x_1} = \underbrace{\mu \frac{d^2 V_1}{dx_2^2}}_{\text{function of } x_2} = \lambda$$

$\Rightarrow$ must be equal to a constant. We'll call it λ.

Pressure

$$\frac{dP}{dx_1} = \lambda$$

$$\boxed{P = \lambda x_1 + C_1}$$

BC: $x_1 = 0 \quad P = P_0 \Rightarrow \boxed{C_1 = P_0}$

$x_1 = L \quad P = P_L \Rightarrow P_L = \lambda L + P_0$

$$\boxed{\lambda = \frac{P_L - P_0}{L}}$$

$$\boxed{P = \frac{P_2 - P_0}{L} x_1 + P_0}$$

check: $x_1 = 0 \quad P = P_0$ ✓

$x_1 = L \quad P = P_2$ ✓

VELOCITY

$$\frac{d^2 V_1}{dx_2^2} = \frac{\lambda}{\mu}$$

$$\frac{d}{dx_2}\underbrace{\left(\frac{dV_1}{dx_2}\right)}_{\equiv \Phi} = \frac{\lambda}{\mu}$$

$$\frac{d\Phi}{dx_2} = \frac{\lambda}{\mu}$$

integrate: $\Phi = \frac{\lambda}{\mu} x_2 + C_2$

SUBSTITUTE BACK: $\Phi = \frac{dV_1}{dx_2} = \frac{\lambda}{\mu} x_2 + C_2$

integrate:

$$\boxed{V_1 = \frac{\lambda}{\mu}\frac{X_2^2}{2} + C_2 X_2 + C_3} \quad \#$$

BC: $X_2 = 0 \quad V_1 = 0 \quad \Rightarrow \boxed{C_3 = 0}$

$X_2 = H \quad V_1 = V$

$$\Rightarrow V = \frac{\lambda}{2\mu} H^2 + C_2 H$$

$$\boxed{C_2 = \left(V - \frac{\lambda H^2}{2\mu}\right)\frac{1}{H}}$$

SUBSTITUTE BACK: #

$$V_1 = \frac{\lambda}{2\mu} X_2^2 + \left[V - \frac{\lambda H^2}{2\mu}\right]\frac{X_2}{H}$$

note: $\lambda = \dfrac{P_L - P_0}{L}$

$$V_1 = \left(\frac{P_L - P_0}{2L\mu}\right)\left[x_2^2 - \cancel{H^2}\left(\frac{x_2}{\cancel{H}}\right)\right] + \frac{V x_2}{H}$$

$$\boxed{V_1 = \left(\frac{P_L - P_0}{2\mu L}\right)\left[x_2^2 - H x_2\right] + \frac{V x_2}{H}}$$

check BC:

$x_2 = 0 \quad V_1 = 0$ ✓

$x_2 = H \quad V_1 = V$ ✓

check units:

$$\left(\frac{P_L - P_0}{2\mu L}\right) x_2^2 \;[=]\; \frac{\cancel{Pa}\; m^{\cancel{2}}}{\cancel{Pa}\, s\; \cancel{m}} = \frac{m}{s}$$

✓

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