SOLN PART A

Homework 2
CM4650
Spring 2020

Part A: Problems 1-4 are Due *Monday 3 February 2020, in class*
Part B: Problems 5-7 are Due *Wednesday 12 February 2020, in class*

Please do not write on the back side of any page of your solution. Please write legibly and large.

You may find this page helpful in this homework:
http://pages.mtu.edu/~fmorriso/cm4650/formula_sheet_for_exam1_2018.pdf

Please make note of the discussion about the stress sign convention and the difference between the two symbols $\underline{\underline{\Pi}} = \underline{\underline{\tau}} + p\underline{\underline{I}}$ (*Understanding Rheology*) and $\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}}$ (*An Introduction to Fluid Mechanics*).

Note that the *Understanding Rheology* textbook has some typos:
www.chem.mtu.edu/~fmorriso/cm4650/URerrata.html

Part A

1. (20 points) For the vectors given below, what are the following quantities equal to? Show your work in Einstein notation before substituting the specific vectors $\underline{a}$ and $\underline{b}$ from below.
   a. the gradient of $\underline{a}$, $\nabla\underline{a}$
   b. the divergence of $\underline{b}$, $\nabla \cdot \underline{b}$
   c. the Laplacian of $\underline{b}$, $\nabla \cdot \nabla\underline{b} = \nabla^2\underline{b}$
   d. $\underline{a} - \underline{b}$

$$\underline{a} = \begin{pmatrix} 7x^2y \\ 11y^3 - 2 \\ 2y^2 + 5x^2 \end{pmatrix}_{xyz}$$

$$\underline{b} = (8 - 3xz)\hat{e}_x + 4x^2\hat{e}_y - 8xy\hat{e}_z$$

2. (10 points) What is the correct way to write the quantity $\beta v_i A_{ci}\hat{e}_c$ (currently written in Einstein notation) when we write it in Gibbs (vector-tensor) notation? What is $\frac{\partial v_j}{\partial x_m}\hat{e}_j\hat{e}_m$ in Gibbs notation?

3. (10 points) The flow rate through a finite surface S can be written as:

$$Q = \iint\limits_S [\hat{n} \cdot \underline{v}]_{surface}\, dS$$

where $\hat{n}$ is the unit normal to the surface dS and $\underline{v}$ is the velocity at infinitesimal surface dS. The solution for the velocity field in Poiseuille flow in a tube is given in the text (see section 3.5.3). Starting with the equation above, calculate the flow rate through the slit cross-section. Show your work.

4. (10 points) The fluid force $\underline{F}$ on a finite surface S can be written as (stress convention of our book):

$$\underline{F} = \iint\limits_S \left[\hat{n} \cdot \left(-\underline{\underline{\Pi}}\right)\right]_{surface} dS$$

where $\hat{n}$ is the unit normal to the infinitesimal surface dS and $\underline{\underline{\Pi}}$ is the total stress tensor with the sign convention of our text. The solution for the velocity field in Poiseuille flow in a slit is given in the text (see example 3.5.2). Starting with the equation above, calculate the total vector force (three components) on the upper wall. Show your work.

Part B

"A" only

5. (10 points) Sketch (by hand is all I require; you can use MATLAB or something else if you want) the following vector velocity field (x, y, z in millimeters):

$$\underline{v}\left(\frac{mm}{s}\right) = \begin{pmatrix} -2x \\ -2y \\ 4z \end{pmatrix}_{xyz}$$

You may confine yourself to the first quadrant (x, y, z all positive) and the plane where $x = 0$.
Hint: You will need to choose some points, calculate $v = |\underline{v}|$ and the direction of $\underline{v}$ at those points, and then draw arrows of the appropriate lengths and directions at the points. Usually we center the vector at the points chosen.

6. (20 points) Text 3.17 (Drag flow in a tilted slit). Do not use tables for the momentum balance; use Einstein notation as in the text, begin from Gibbs notation (do not use tables).
7. (20 points) Text 3.18 (Tangential annular flow; you may use tables, http://pages.mtu.edu/~fmorriso/cm4650/Operations_with_Del_cyl_sph.pdf)

CM4650
HW2
Spring 2020

1. $\nabla \underline{a} = \frac{\partial}{\partial x_p} \hat{e}_p \, a_n \hat{e}_n$

$= \frac{\partial a_n}{\partial x_p} \hat{e}_p \hat{e}_n$

$$= \begin{pmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_2}{\partial x_1} & \frac{\partial a_3}{\partial x_1} \\ \frac{\partial a_1}{\partial x_2} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_3}{\partial x_2} \\ \frac{\partial a_1}{\partial x_3} & \frac{\partial a_2}{\partial x_3} & \frac{\partial a_3}{\partial x_3} \end{pmatrix}_{123}$$

(2)

$$\nabla \underline{a} = \begin{pmatrix} 14xy & 0 & 10x \\ 7x^2 & 33y^2 & 4y \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

b) $\nabla \cdot \underline{b} = \frac{\partial}{\partial x_i} \overbrace{\hat{e}_i \cdot b_s \hat{e}_s}^{\delta_{is}}$ "i becomes s"

$$= \frac{\partial b_s}{\partial x_s}$$

$$= \frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3}$$

$$= (-3z) + 0 + 0$$

$$= \boxed{-3z}$$

δ_{fm} "f becomes m" 3

c) $$\nabla \cdot \nabla \underline{b} = \frac{\partial}{\partial x_f} \hat{e}_f \cdot \frac{\partial}{\partial x_m} \hat{e}_m \, b_i \hat{e}_i$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_m} b_i \hat{e}_i$$

$$= \begin{pmatrix} \frac{\partial^2 b_1}{\partial x_1^2} + \frac{\partial^2 b_1}{\partial x_2^2} + \frac{\partial^2 b_1}{\partial x_3^2} \\ \frac{\partial^2 b_2}{\partial x_1^2} + \frac{\partial^2 b_2}{\partial x_2^2} + \frac{\partial^2 b_2}{\partial x_3^2} \\ \frac{\partial^2 b_3}{\partial x_1^2} + \frac{\partial^2 b_3}{\partial x_2^2} + \frac{\partial^2 b_3}{\partial x_3^2} \end{pmatrix}_{123}$$

☆☆ VECTOR ☆☆

take second derivatives

$$= \begin{pmatrix} 0 + 0 + 0 \\ 8 + 0 + 0 \\ 0 + 0 + 0 \end{pmatrix}_{123} = \boxed{\begin{pmatrix} 0 \\ 8 \\ 0 \end{pmatrix}_{123}}$$

d) $\underline{a} - \underline{b} = a_p \hat{e}_{\underset{\downarrow\, j}{p}} - b_m \hat{e}_{\underset{\downarrow\, j}{m}}$

$$= \left(a_j \hat{e}_j - b_j \hat{e}_j\right)$$

$$= (a_j - b_j)\,\hat{e}_j$$

$$= \begin{pmatrix} 7x^2y - (8 - 3xz) \\ 11y^3 - 2 - 4x^2 \\ 2y^2 + 5x^2 + 8xy \end{pmatrix}_{xyz}$$

VECTOR

2. $\beta v_i A_{ci} \hat{e}_c$ This is a <u>vector</u>

try $\beta \underline{v} \cdot \underline{\underline{A}}$ and $\beta(\underline{\underline{A}} \cdot \underline{v})$

$$\underline{v} \cdot \underline{\underline{A}} = v_i \underbrace{\hat{e}_i A_{pk} \hat{e}_p}_{\delta_{ip}} \hat{e}_k$$

δ_{ip} "i becomes p"

$$= v_p \underbrace{A_{pk} \hat{e}_k}$$

not right

$$\underline{\underline{A}} \cdot \underline{v} = A_{mn} \hat{e}_m \underbrace{\hat{e}_n \cdot v_s}_{\delta_{ns}} \hat{e}_s$$

δ_{ns} "n becomes s"

$$= \underbrace{A_{ms} v_s \hat{e}_m}$$

yes

a) $\boxed{\beta \underline{\underline{A}} \cdot \underline{v}}$

b) $$\frac{\partial v_j}{\partial x_m}\hat{e}_j\hat{e}_m$$

try $\nabla \underline{v}$ and $(\nabla \underline{v})^T$

$$\nabla \underline{v} = \frac{\partial}{\partial x_f}\hat{e}_f\, v_m \hat{e}_m$$

$$= \frac{\partial v_m}{\partial x_f}\hat{e}_f\hat{e}_m$$

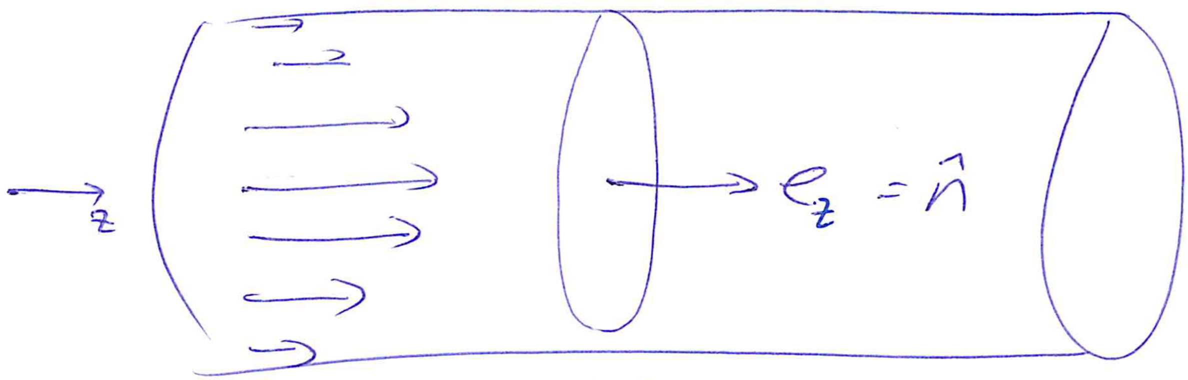
WRONG

CORRECT

$$(\nabla \underline{v})^T = \frac{\partial v_m}{\partial x_f} e_m e_f$$

$$\boxed{(\nabla \underline{v})^T}$$

3.



$$\underline{v} = \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$$

eqn 3.216

$$v_z = \overbrace{\frac{(P_0 - P_L)R^2}{4\mu L}}^{\equiv C}\left(1 - \left(\frac{r}{R}\right)^2\right)$$

$$Q = \iint_{S'} (\underbrace{\hat{n}}_{=\hat{e}_z} \cdot \underline{v})\Big|_{\underbrace{\text{surface}}_{z=0}} dS$$

(or L or anywhere but constant

$$\hat{n} \cdot \underline{v} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}_{r\theta z} \cdot \begin{pmatrix} 0 \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$$

$$= v_z$$

$ds = r\,dr\,d\theta$ (cross section)

$$\left(\hat{n}\cdot\underline{v}\right)\Big|_{z=\text{constant}} = v_z(r)$$

$$Q = \int_0^{2\pi}\int_0^{R} c\left(1-\left(\frac{r}{R}\right)^2\right) r\,dr\,d\theta$$

$$= 2\pi c \int_0^R \underbrace{\left(1-\left(\frac{r}{R}\right)^2\right) r}_{r-\frac{r^3}{R^2}}\,dr$$

$$= 2\pi c\ \underbrace{\left(\frac{r^2}{2}-\left(\frac{1}{R^2}\right)\frac{r^4}{4}\right)\Big|_0^R}_{\underbrace{\frac{R^2}{2}-\frac{R^2}{4}}_{R^2/4}}$$

$$= \cancel{2}\pi c\,\frac{R^2}{\cancel{4}2} = \frac{\pi R^2 c}{2}$$

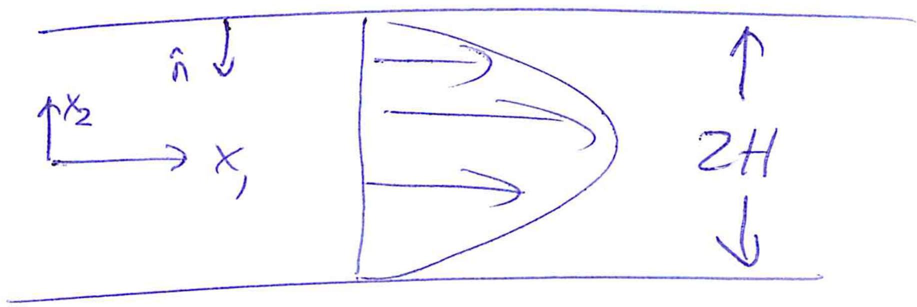
$$Q = \frac{(P_0 - P_L) R^2}{4 \mu L} \frac{\pi R^2}{2}$$

$$\boxed{Q = \frac{(P_0 - P_L) R^4 \pi}{8 \mu L}}$$

eqn 3.219

4) (p84)

$\hat{n} = -\hat{e}_2 = (0\ -1\ 0)_{123}$
@ $x_2 = H$



$$\underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

Eqn 3.177:

$$P(x_1) = -\left(\frac{P_0 - P_L}{L}\right) x_1 + P_0$$

eqn 3.178

$$v_1 = \underbrace{\frac{H^2 (P_0 - P_L)}{2 \mu L}}_{= C} \left(1 - \left(\frac{x_2}{H}\right)^2\right)$$

$$\underline{\underline{\Pi}} = P\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$= P\,\underline{\underline{I}} - \mu\left(\underbrace{\nabla\underline{v}}_{\text{start w/ } \nabla\underline{v}} + (\nabla\underline{v})^T\right)$$

$$\nabla\underline{v} = \frac{\partial}{\partial x_f}\hat{e}_f\, v_m \hat{e}_m \qquad \underline{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}$$

$$= \frac{\partial v_m}{\partial x_f}\hat{e}_f\,\hat{e}_m$$

$v_1(x_2)$ only

$$= \begin{pmatrix} \cancel{\frac{\partial v_1}{\partial x_1}} & \cancel{\frac{\partial v_2}{\partial x_1}} & \cancel{\frac{\partial v_3}{\partial x_1}} \\ \frac{\partial v_1}{\partial x_2} & \cancel{\frac{\partial v_2}{\partial x_2}} & \cancel{\frac{\partial v_3}{\partial x_2}} \\ \cancel{\frac{\partial v_1}{\partial x_3}} & \cancel{\frac{\partial v_2}{\partial x_3}} & \cancel{\frac{\partial v_3}{\partial x_3}} \end{pmatrix}_{123}$$

$v_2 = 0 \qquad v_3 = 0$

$$\nabla \underline{v} = \begin{pmatrix} 0 & 0 & 0 \\ C(-2)\frac{X_2}{H^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} 0 & \frac{-2CX_2}{H^2} & 0 \\ \frac{-2CX}{H^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\Pi}} = \begin{pmatrix} -P(x_1) & \frac{2\mu C X_2}{H^2} & 0 \\ \frac{2\mu C X_2}{H^2} & -P(x_1) & 0 \\ 0 & 0 & -P(x_1) \end{pmatrix}_{123}$$

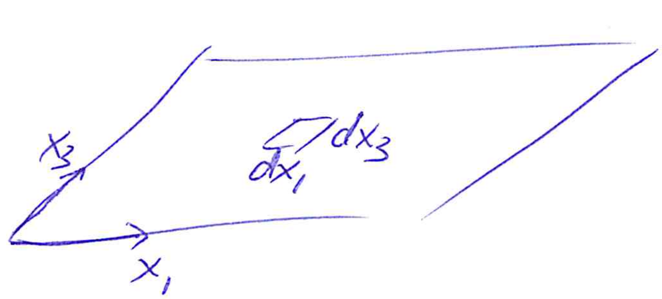
$$\hat{n}\cdot(-\underline{\underline{\Pi}})\Big|_{surf} = -\hat{e}_2 \cdot (-\underline{\underline{\Pi}})\Big|_{x_2=H}$$

$* \quad n = -\hat{e}_2$
top surface

$$\left(\hat{n}\cdot(-\underline{\underline{\Pi}})\right)\Big|_{x_2=H}$$

$$= \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}_{123} \begin{pmatrix} P(x_1) & -\frac{2\mu C}{H} & 0 \\ -\frac{2\mu C}{H} & P(x_1) & 0 \\ 0 & 0 & P(x_1) \end{pmatrix}_{123}$$

$$= \begin{pmatrix} \frac{2\mu C}{H} & -P(x_1) & 0 \end{pmatrix}_{123}$$

$dS =$ 

$$dS = dx_1\, dx_3$$

Put it all together:

$$\underline{F} = \iint_{S} \left(\hat{n} \cdot \left(-\underline{\underline{\Pi}} \right) \right)_{\substack{\text{at}\\ \text{surf}}} dS$$

$$= \int_0^W \int_0^L \begin{pmatrix} \frac{2\mu c}{H} \\ -P(x_1) \\ 0 \end{pmatrix}_{123} dx_1 \, dx_3$$

$$\frac{\underline{F}}{W} = \begin{pmatrix} \frac{2\mu c}{H} L \\ -\int_0^L P(x_1)\, dx_1 \\ 0 \end{pmatrix}_{123}$$

$$\int_0^L \left(-\left(\frac{P_0 - P_L}{L} \right) x_1 + P_0 \right) dx_1$$

$$\left(-\left(\frac{P_0 - P_L}{L} \right) \frac{x_1^2}{2} + P_0 x_1 \right) \Big|_0^L$$

$$= -\frac{(P_0 - P_L)}{\cancel{L}} \frac{L^{\cancel{2}}}{2} + P_0 L$$

$$= -\frac{(P_0 - P_L) L}{2} + P_0 L$$

$$= L \left(-\frac{P_0}{2} - \frac{P_L}{2} + P_0 \right)$$

$$= L \left(\frac{P_0 - P_L}{2} \right)$$

$$\frac{2\mu C L}{H} = \frac{\cancel{2\mu L}}{\cancel{H}} \frac{H^{\cancel{2}} (P_0 - P_L)}{\cancel{2\mu L}}$$

$$= H\,(P_0 - P_L)$$

$$\boxed{\underline{F} = \begin{pmatrix} HW(P_0 - P_L) \\ \frac{WL}{2}(P_0 - P_L) \\ 0 \end{pmatrix}_{123}}$$

X_1-component: <u>DRAG</u>

X_2-component: mean isotropic pressure normal to the surface

X_3-component no force in this direction