SOLN 2b

Homework 2
CM4650
Spring 2020

Part A: Problems 1-4 are Due *Monday 3 February 2020, in class*
Part B: Problems 5-7 are Due *Wednesday 12 February 2020, in class*

Please do not write on the back side of any page of your solution. Please write legibly and large.

You may find this page helpful in this homework:
http://pages.mtu.edu/~fmorriso/cm4650/formula_sheet_for_exam1_2018.pdf

Please make note of the discussion about the stress sign convention and the difference between the two symbols $\underline{\underline{\Pi}} = \underline{\underline{\tau}} + p\underline{\underline{I}}$ (*Understanding Rheology*) and $\underline{\underline{\tilde{\Pi}}} = \underline{\underline{\tilde{\tau}}} - p\underline{\underline{I}}$ (*An Introduction to Fluid Mechanics*).

Note that the *Understanding Rheology* textbook has some typos:
www.chem.mtu.edu/~fmorriso/cm4650/URerrata.html

Part A

1. (20 points) For the vectors given below, what are the following quantities equal to? Show your work in Einstein notation before substituting the specific vectors $\underline{a}$ and $\underline{b}$ from below.
   a. the gradient of $\underline{a}$, $\nabla\underline{a}$
   b. the divergence of $\underline{b}$, $\nabla\cdot\underline{b}$
   c. the Laplacian of $\underline{b}$, $\nabla\cdot\nabla\underline{b} = \nabla^2\underline{b}$
   d. $\underline{a} - \underline{b}$

$$\underline{a} = \begin{pmatrix} 7x^2y \\ 11y^3 - 2 \\ 2y^2 + 5x^2 \end{pmatrix}_{xyz}$$

$$\underline{b} = (8 - 3xz)\hat{e}_x + 4x^2\hat{e}_y - 8xy\hat{e}_z$$

2. (10 points) What is the correct way to write the quantity $\beta v_i A_{ci}\hat{e}_c$ (currently written in Einstein notation) when we write it in Gibbs (vector-tensor) notation? What is $\frac{\partial v_j}{\partial x_m}\hat{e}_j\hat{e}_m$ in Gibbs notation?

3. (10 points) The flow rate through a finite surface S can be written as:

$$Q = \iint_S [\hat{n} \cdot \underline{v}]_{surface} \, dS$$

where $\hat{n}$ is the unit normal to the surface dS and $\underline{v}$ is the velocity at infinitesimal surface dS. The solution for the velocity field in Poiseuille flow in a tube is given in the text (see section 3.5.3). Starting with the equation above, calculate the flow rate through the slit cross-section. Show your work.

4. (10 points) The fluid force $\underline{F}$ on a finite surface S can be written as (stress convention of our book):

$$\underline{F} = \iint_S \left[\hat{n} \cdot \left(-\underline{\underline{\Pi}}\right)\right]_{surface} dS$$

where $\hat{n}$ is the unit normal to the infinitesimal surface dS and $\underline{\underline{\Pi}}$ is the total stress tensor with the sign convention of our text. The solution for the velocity field in Poiseuille flow in a slit is given in the text (see example 3.5.2). Starting with the equation above, calculate the total vector force (three components) on the upper wall. Show your work.

Part B

5. (10 points) Sketch (by hand is all I require; you can use MATLAB or something else if you want) the following vector velocity field (x, y, z in millimeters):

$$\underline{v}\left(\frac{mm}{s}\right) = \begin{pmatrix} -2x \\ -2y \\ 4z \end{pmatrix}_{xyz}$$

You may confine yourself to the first quadrant (x, y, z all positive) and the plane where $x = 0$.
Hint: You will need to choose some points, calculate $v = |\underline{v}|$ and the direction of $\underline{v}$ at those points, and then draw arrows of the appropriate lengths and directions at the points. Usually we center the vector at the points chosen.

6. (20 points) Text 3.17 (Drag flow in a tilted slit). Do not use tables for the momentum balance; use Einstein notation as in the text, begin from Gibbs notation (do not use tables).
7. (20 points) Text 3.18 (Tangential annular flow; you may use tables, http://pages.mtu.edu/~fmorriso/cm4650/Operations_with_Del_cyl_sph.pdf)

HW 26 # 5

						divide by 8		
x	y	z	vx	vy	vz	vx	vy	vz
0	0	0	0	0	0	0	0	0
0	1	0	0	-2	0	0	-0.25	0
0	2	0	0	-4	0	0	-0.5	0
0	3	0	0	-6	0	0	-0.75	0
0	4	0	0	-8	0	0	-1	0
0	0	1	0	0	4	0	0	0.5
0	1	1	0	-2	4	0	-0.25	0.5
0	2	1	0	-4	4	0	-0.5	0.5
0	3	1	0	-6	4	0	-0.75	0.5
0	4	1	0	-8	4	0	-1	0.5
0	0	2	0	0	8	0	0	1
0	1	2	0	-2	8	0	-0.25	1
0	2	2	0	-4	8	0	-0.5	1
0	3	2	0	-6	8	0	-0.75	1
0	4	2	0	-8	8	0	-1	1
0	0	3	0	0	12	0	0	1.5
0	1	3	0	-2	12	0	-0.25	1.5
0	2	3	0	-4	12	0	-0.5	1.5
0	3	3	0	-6	12	0	-0.75	1.5
0	4	3	0	-8	12	0	-1	1.5
0	0	4	0	0	16	0	0	2
0	1	4	0	-2	16	0	-0.25	2
0	2	4	0	-4	16	0	-0.5	2
0	3	4	0	-6	16	0	-0.75	2
0	4	4	0	-8	16	0	-1	2

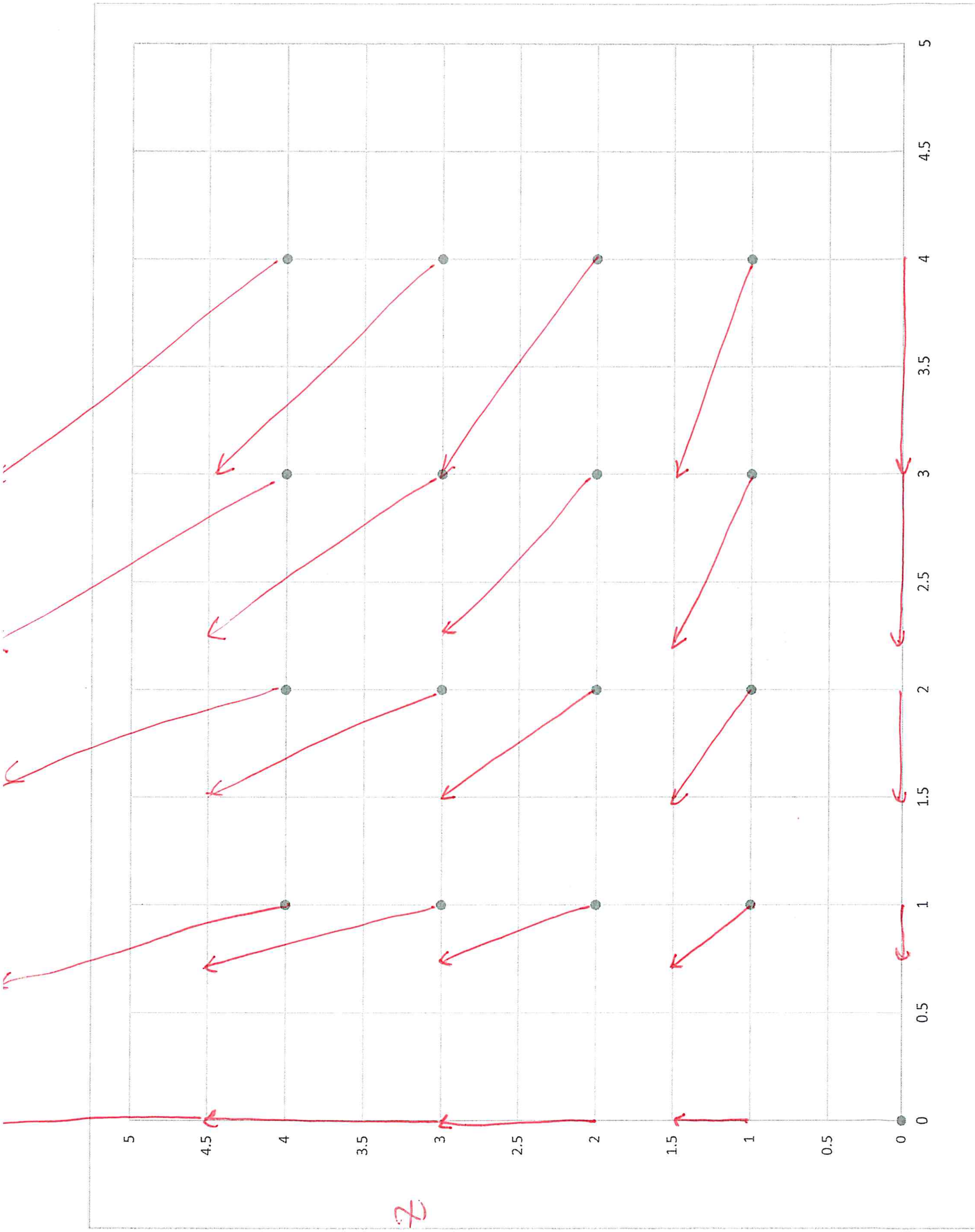
chosen points

calculated $\underline{v}$

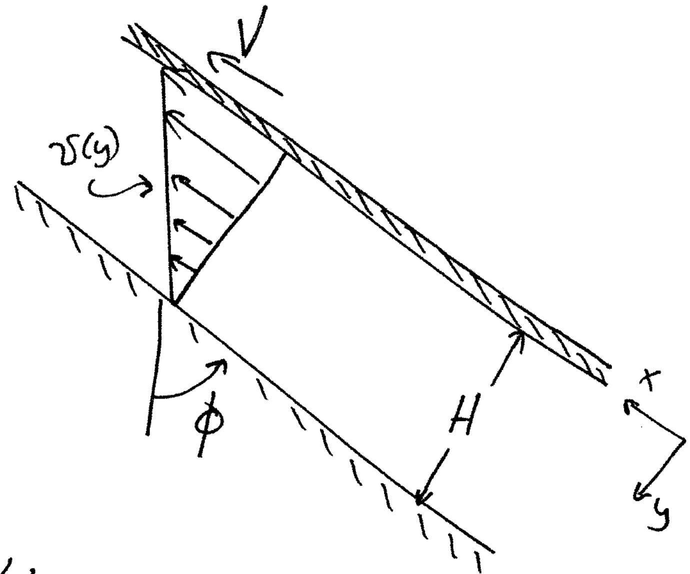
shrink the size for easier sketching

see plot →

HW2B #5



3.17 Drag flow in a tilted slit



$$\underline{v} = \begin{pmatrix} v_x \\ 0 \\ 0 \end{pmatrix}_{xyz}$$

incompressible
Newtonian
steady

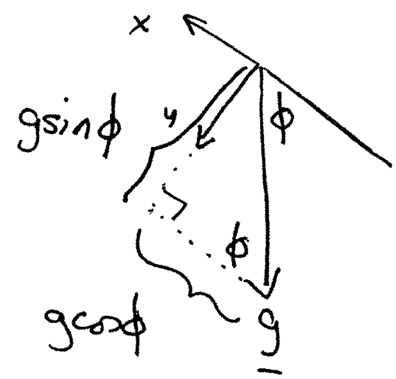
continuity : $\nabla \cdot \underline{v} = 0$ $\quad \dfrac{\partial v_x}{\partial x} + \underbrace{\cancel{\dfrac{\partial v_y}{\partial y}}}_{0} + \underbrace{\cancel{\dfrac{\partial v_z}{\partial z}}}_{0} = 0$

$$\boxed{\frac{\partial v_x}{\partial x} = 0}$$

EOM:

$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underbrace{\underline{v}\cdot\nabla\underline{v}}_{\text{unidirectional}}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$$

$$\underline{g} = \begin{pmatrix} -g\cos\phi \\ g\sin\phi \\ 0 \end{pmatrix}_{xyz}$$



$$\nabla^2 \underline{v} = \begin{pmatrix} \cancel{\frac{\partial^2 v_x}{\partial x^2}} + \frac{\partial^2 v_x}{\partial y^2} + \overset{0}{\frac{\partial^2 v_x}{\partial z^2}} \\ \cancel{\frac{\partial^2 v_y}{\partial x^2}} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}} \\ \cancel{\frac{\partial v_z}{\partial x^2}} + \cancel{\frac{\partial v_z}{\partial y^2}} + \cancel{\frac{\partial v_z}{\partial z^2}} \end{pmatrix}_{xyz}$$

assume wide slit
$\Rightarrow$ no z-variation

$$v_y = v_z = 0$$

$$\frac{\partial v_x}{\partial x} = 0$$

$$\nabla^2 \underline{v} = \begin{pmatrix} \frac{\partial^2 v_x}{\partial y^2} \\ 0 \\ 0 \end{pmatrix}_{xyz}$$

EOM:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{xyz} = \begin{pmatrix} -\frac{\partial P}{\partial x} \\ -\frac{\partial P}{\partial y} \\ -\frac{\partial P}{\partial z} \end{pmatrix}_{xyz} + \begin{pmatrix} \mu \frac{\partial^2 v_x}{\partial y^2} \\ 0 \\ 0 \end{pmatrix}_{xyz} + \begin{pmatrix} -\rho g\cos\phi \\ \rho g\sin\phi \\ 0 \end{pmatrix}_{xyz} \qquad (118)$$

z-component EOM

$$\boxed{\frac{\partial P}{\partial z} = 0}$$

y-component EOM

$$\boxed{\frac{\partial P}{\partial y} = \rho g \sin\phi}$$

x-component EOM

$$0 = -\underbrace{\cancel{\frac{\partial P}{\partial x}}}_{0} + \mu \frac{\partial^2 v_x}{\partial y^2} - \rho g \cos\phi$$

no imposed
pressure gradient
in x-direction

$$\underbrace{\frac{\rho g \cos\phi}{\mu}}_{\text{constant}} = \frac{\partial^2 v_x}{\partial y^2}$$

$$\frac{\partial v_x}{\partial y} = \frac{\rho g \cos\phi}{\mu} y + C_1$$

$$\boxed{v_x = \frac{\rho g \cos\phi}{\mu} \frac{y^2}{2} + C_1 y + C_2} \quad (119)$$

BOUNDARY CONDITIONS:

$$y = 0 \quad v_x = V \quad \Longrightarrow C_2 = V$$

$$y = H \quad v_x = 0$$

$$0 = \frac{\rho g \cos\phi}{\mu} \frac{H^2}{2} + C_1 H + V$$

$$\left(-V - \frac{\rho g \cos\phi\, H^2}{2\mu} \right) \frac{1}{H} = C_1$$

$$v_x = \frac{\rho g \cos\phi}{2\mu} y^2 - \left(\frac{V}{H} + \frac{\rho g \cos\phi\, H}{2\mu} \right) y + V$$

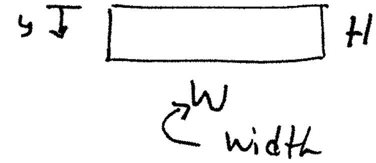
$$= \frac{\rho g \cos\phi}{2\mu} \left(y^2 - Hy \right) - V\left(\frac{y}{H}\right) + V$$

$$\boxed{v_x = \frac{\rho g \cos\phi\, H^2}{2\mu} \left(\left(\frac{y}{H}\right)^2 - \left(\frac{y}{H}\right) \right) + \left(1 - \frac{y}{H}\right) V}$$

Calculate flow rate, Q

$$Q = \int_{A} v_x \, dA$$

A ← cross-sectional area



$$dA = W\,dy$$

$$Q = \int_0^H v_x(y)\, W\, dy$$

$$\frac{Q}{W} = \int_0^H \alpha \left[\left(\frac{y}{H}\right)^2 - \left(\frac{y}{H}\right) \right] + V\left(1 - \frac{y}{H}\right) dy$$

$$\alpha = \frac{\rho g \cos\phi H^2}{2\mu}$$

$$\xi = \frac{y}{H} \qquad d\xi = \frac{1}{H}\, dy$$

$y = 0 \quad \xi = 0$

$y = H \quad \xi = 1$

$$\frac{Q}{W} = H\int_0^1 \alpha\left(\xi^2-\xi\right) + V\left(1-\xi\right)\, d\xi$$

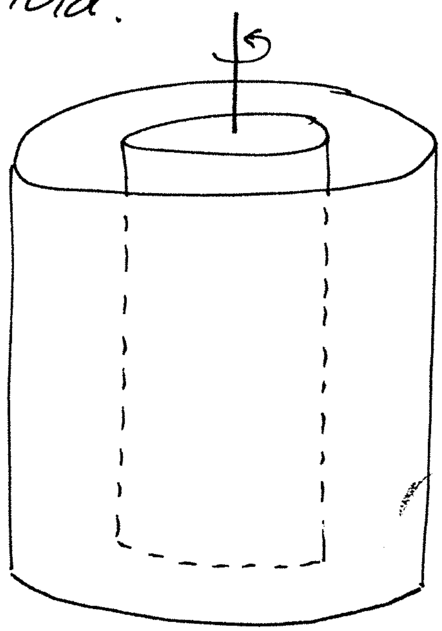
$$= H\left[\alpha\left(\frac{\xi^3}{3}-\frac{\xi^2}{2}\right) + V\left(\xi - \frac{\xi^2}{2}\right)\right]_0^1$$

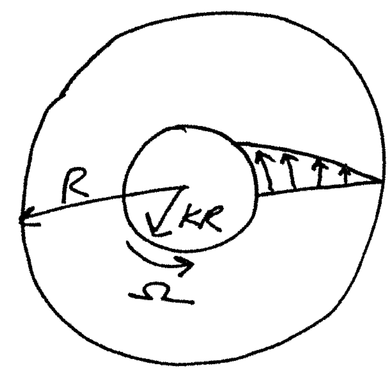
$$\frac{Q}{WH} = \alpha\left[\frac{1}{3}-\frac{1}{2}\right] + V\left(1-\frac{1}{2}\right) - 0$$

$$= \alpha\left(\frac{-1}{6}\right) + \frac{V}{2}$$

$$\boxed{Q = \frac{WH}{2}\Big[\underbrace{V}_{\substack{\text{forward}\\ \text{flow rate}\\ \text{due to}\\ \text{drag}\\ \text{flow}}} - \underbrace{\frac{\rho g \cos\phi H^2}{6\mu}}_{\substack{\text{backward}\\ \text{flow due}\\ \text{to gravity}}}\Big]}$$

3.18 Tangential, annular flow of an incompressible, Newtonian fluid.





$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}$$

<u>continuity</u>: $\overset{0}{\cancel{\frac{\partial v_z}{\partial z}}} + \frac{1}{r}\frac{\partial}{\partial r}(r \overset{0}{\cancel{v_r}}) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$

$\nabla \cdot \underline{v} = 0$

$$\Rightarrow \boxed{\frac{\partial v_\theta}{\partial \theta} = 0}$$

EQN OF MOTION:

$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady state}} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \underbrace{\cancel{\rho\underline{g}}}_{\text{neglect}}$$

$v_r = v_z = 0$

$$\underline{v}\cdot\nabla\underline{v} = \begin{pmatrix} v_r\left(\cancelto{0}{\frac{\partial v_r}{\partial r}}\right) + v_\theta\left(\frac{1}{r}\cancelto{0}{\frac{\partial v_r}{\partial \theta}} - \frac{v_\theta}{r}\right) + \cancelto{0}{v_z}\left(\frac{\partial v_r}{\partial z}\right) \\ \cancelto{0}{v_r}\left(\frac{\partial v_\theta}{\partial r}\right) + v_\theta\left(\frac{1}{r}\cancelto{\underline{\nabla}\cdot\underline{v}=0}{\frac{\partial v_\theta}{\partial \theta}} + \cancelto{0}{\frac{v_r}{r}}\right) + \cancelto{0}{v_z}\left(\frac{\partial v_\theta}{\partial z}\right) \\ \cancelto{0}{v_r}\left(\frac{\partial v_z}{\partial r}\right) + v_\theta\left(\frac{1}{r}\cancelto{0}{\frac{\partial v_z}{\partial \theta}}\right) + \cancelto{0}{v_z}\left(\frac{\partial v_z}{\partial z}\right) \end{pmatrix}$$

$$\underline{v}\cdot\nabla\underline{v} = \begin{pmatrix} -\dfrac{v_\theta^2}{r} \\ 0 \\ 0 \end{pmatrix}$$

$$v_r = v_z = 0$$

$$\nabla^2 \underline{v} = \begin{pmatrix} -\frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}}^{\;0 \;\; \nabla\cdot\underline{v}=0} \\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}}^{\;\nabla\cdot\underline{v}=0} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}^{\;\text{long tube, neglect ends}} \\ 0 \end{pmatrix}$$

$$\nabla^2 \underline{v} = \begin{pmatrix} 0 \\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) \\ 0 \end{pmatrix}$$

PUT TOGETHER:

$$\rho \begin{pmatrix} -\frac{v_\theta^2}{r} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\partial P}{\partial r} \\ -\frac{1}{r}\cancel{\frac{\partial P}{\partial \theta}}_{\;\text{symmetry}} \\ -\frac{\partial P}{\partial z} \end{pmatrix} + \mu \begin{pmatrix} 0 \\ \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) \\ 0 \end{pmatrix}$$

z-component: $\boxed{\frac{\partial P}{\partial z} = 0}$ (125)

r-component: $$\boxed{\rho \frac{v_\theta^2}{r} = \frac{\partial P}{\partial r}}$$

$P(r)$ can be solved for after v_θ

θ-component:

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) = 0$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) = C_1$$

$$\frac{\partial}{\partial r}(r v_\theta) = C_1 r$$

$$r v_\theta = C_1 \frac{r^2}{2} + C_2$$

$$v_\theta = C_1 \frac{r}{2} + \frac{C_2}{r}$$

BC: $r = R \quad v_\theta = 0$
$r = KR \quad v_\theta = KR\Omega$

$$0 = C_1 \frac{R}{2} + C_2 \frac{1}{R}$$

$$\kappa R \Omega = C_1 \frac{\kappa R}{2} + C_2 \frac{1}{\kappa R}$$

$$C_2 = - C_1 \frac{R^2}{2}$$

$$\kappa \cancel{R} \Omega = C_1 \frac{\kappa \cancel{R}}{2} + \frac{1}{\kappa \cancel{R}} (-C_1) \frac{\cancel{R^2}}{2}$$

$$2\kappa\Omega = C_1 \left(\kappa + \left(\frac{-1}{\kappa}\right) \right)$$

$$2\kappa\Omega = C_1 \left(\frac{\kappa^2 - 1}{\kappa} \right)$$

$$\boxed{C_1 = \frac{2\kappa^2\Omega}{\kappa^2 - 1}}$$

$$C_2 = -C_1 \frac{R^2}{2} = \frac{-\cancel{2}\kappa^2 R^2 \Omega}{\cancel{2}(\kappa^2-1)}$$

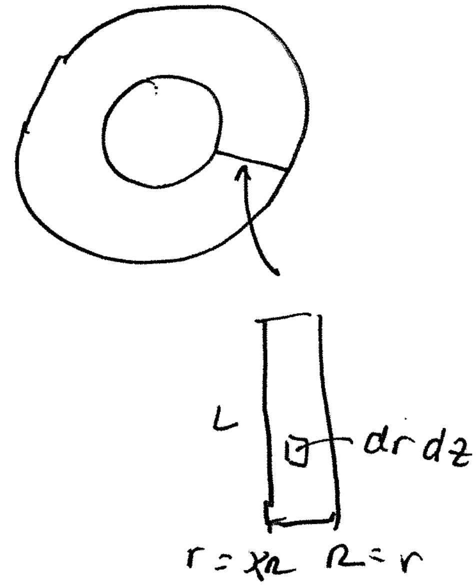
$$\boxed{C_2 = - \frac{\kappa^2 R^2 \Omega}{\kappa^2 - 1}}$$

$$v_\theta = \frac{\cancel{2}\kappa^2\Omega}{\kappa^2-1}\,\frac{r}{\cancel{2}} - \frac{\kappa^2 R^2 \Omega}{\kappa^2-1}\,\frac{1}{r}$$

$$= \frac{\kappa^2\Omega}{\kappa^2-1}\left(r - R^2\frac{1}{r}\right)$$

$$\boxed{v_\theta = \left(\frac{\kappa^2\Omega}{\kappa^2-1}\right) R \left(\left(\frac{r}{R}\right) - \left(\frac{R}{r}\right)\right)}$$

b) Calc Q



$$Q = \int_A v_\theta(r)\, dA$$

$$dA = L\, dr$$

$$\frac{Q}{L} = \int_{xR}^{R} v_\theta(r)\, dr$$

$$v_\theta = \underbrace{\left(\frac{x^2 \Omega R}{x^2-1}\right)}_{\zeta} \left(\xi - \frac{1}{\xi}\right) \qquad \xi = \frac{r}{R}$$

$$\frac{Q}{L} = \zeta R \int_x^1 \xi - \frac{1}{\xi}\, d\xi$$

$$\frac{Q}{L\zeta R} = \frac{\xi^2}{2} - \ln \xi \Big|_x^1$$

$$\frac{Q}{LR\zeta} = \frac{1}{2} - 0 - \frac{\kappa^2}{2} + \ln\kappa$$

$$= \frac{1}{2}\left(1 - \kappa^2 + 2\ln\kappa\right)$$

$$\boxed{Q = \frac{L\kappa^2 R^2 \Omega}{2(\kappa^2-1)}\left(1-\kappa^2+2\ln\kappa\right)}$$

c) calc $P(r)$

$$\frac{\partial P}{\partial r} = \rho\left(\frac{v_\theta^2}{r}\right)$$

$$v_\theta = \underbrace{\left(\frac{\kappa^2 R\,\Omega}{\kappa^2-1}\right)}_{\zeta}\left(\xi - \frac{1}{\xi}\right) \qquad \xi = \frac{r}{R}$$

$$v_\theta = \zeta\left(\xi - \frac{1}{\xi}\right)$$

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \xi}\frac{\partial \xi}{\partial r} = \frac{\partial P}{\partial \xi}\,\frac{1}{R}$$

$$\frac{\partial P}{\partial \xi} = \rho\left(\frac{R}{r}\right)v_\theta^2 = \rho\,\frac{1}{\xi}\,v_\theta^2$$

$$= \rho\frac{1}{\xi}\,\zeta^2\left(\xi - \frac{1}{\xi}\right)^2$$

$$= \rho\zeta^2\,\frac{1}{\xi}\left(\xi^2 - 2 + \frac{1}{\xi^2}\right)$$

$$\frac{\partial P}{\partial \xi} = \rho\,\zeta\left(\xi - \frac{2}{\xi} + \frac{1}{\xi^3}\right) \qquad (131)$$

$$P = \rho \zeta \left(\frac{\xi^2}{2} - 2\ln\xi + \frac{\xi^{-2}}{-2} \right) + C$$

BC: $\xi = \kappa \quad P = P_{atm} = P_0$

$$P_0 = \rho\zeta\left(\frac{\kappa^2}{2} - 2\ln\kappa - \frac{1}{2\kappa^2}\right) + C$$

$$\boxed{P - P_0 = \frac{\rho\zeta}{2}\left((\xi^2 - \kappa^2) - 4\ln\frac{\xi}{\kappa} - (\xi^{-2} - \kappa^{-2})\right) \qquad \xi = \frac{r}{R}}$$

note: since $\kappa \sim 1$
$P \sim P_0 \ \forall r$

d. Calculate the torque required to turn the cylinder.

See later for start from $\underline{R} \times \hat{r} \cdot \underline{\underline{\Pi}}$

$$\mathcal{T} = (\text{FORCE})(\text{LEVER ARM})$$

$$(\text{STRESS})(\text{AREA}) \qquad \kappa R$$

$$\left.\tau_{r\theta}\right|_{r=\kappa R} \qquad 2\pi \kappa R L$$

$$\tau_{r\theta} = -\mu \dot{\gamma}_{r\theta}$$

$$= -\mu \left[r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\left(\cancel{\frac{\partial v_r}{\partial \theta}}\right) \right]$$

$$\tau_{r\theta} = -\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)$$

$$\frac{v_\theta}{r} = \frac{\zeta}{R} \left(1 - \frac{R^2}{r^2}\right)$$

$$\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = \frac{\zeta}{R}\left(-R^2\right)(-2)\left(\frac{1}{r^3}\right)$$

$$-\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = \frac{-\zeta \mu R^{\cancel{2}} 2}{\cancel{R}} \frac{1}{r^2}$$

$$\tau_{r\theta}\Big|_{r=\kappa R} = -2\zeta\mu R \left.\frac{1}{r^2}\right|_{r=\kappa R}$$

$$= -\frac{2\zeta\mu \cancel{R}}{\kappa^2 R^{\cancel{2}}} = \frac{-2\zeta\mu}{\kappa^2 R}$$

$$J = \left(\frac{-2\zeta\mu}{\cancel{\kappa^2}\cancel{R}}\right) 2\pi \cancel{\kappa}\cancel{R} L \, \cancel{\kappa} R$$

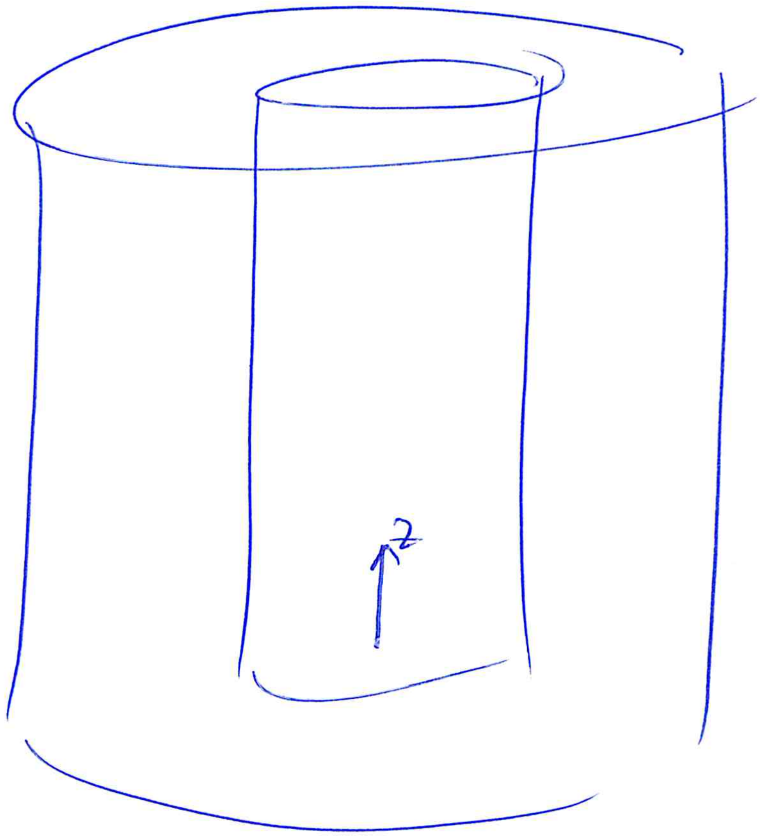
$$= -4\zeta\mu\pi R L$$

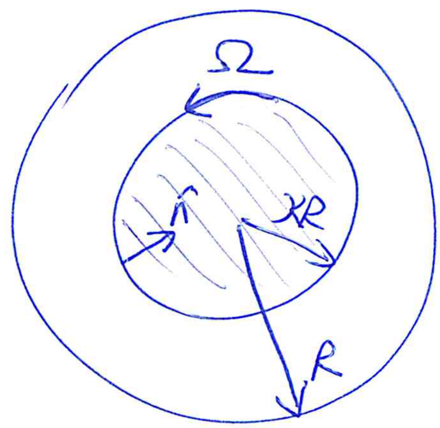
$$\boxed{J = -4\pi R^2 L \mu \kappa^2 \Omega \left(\frac{1}{\kappa^2 - 1}\right)}$$

19Feb2020

1

3.18 calculate the torque on inner cylinder





$$\underline{T}_{in} = \iint_S \underline{\tilde{R}} \times \left(\hat{n} \cdot -\underline{\underline{\Pi}}\right)_{\substack{\text{at} \\ \text{surface}}} dS$$

$$\hat{n} = -\hat{e}_r = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}_{r\theta z}$$

$$\underline{\tilde{R}} = \kappa R \hat{e}_r = \begin{pmatrix} \kappa R \\ 0 \\ 0 \end{pmatrix}_{r\theta z}$$

3

Exam 1 Formulas

Polymer Rheology Prof. Faith Morrison

Rate of deformation tensor: $\underline{\underline{\dot{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T$

Rate of deformation: $\dot{\gamma} = |\underline{\underline{\dot{\gamma}}}|$

Tensor magnitude: $A = |\underline{\underline{A}}| = +\sqrt{\frac{\underline{\underline{A}}:\underline{\underline{A}}}{2}}$

Total stress tensor: $\underline{\underline{\Pi}} = p\underline{\underline{I}} + \underline{\underline{\tau}}$
(Bird, UR sign convention on stress)

Navier-Stokes Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$

Cauchy Momentum Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$
(Bird, UR sign convention on stress)

Continuity Equation: $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v})$

Newtonian, incompressible constitutive equation: $\underline{\underline{\tau}} = -\mu(\nabla \underline{v} + (\nabla \underline{v})^T)$
(Bird sign convention on stress)

Fluid force $\underline{F}_{on}$ on a surface S:
(Bird, UR sign convention on stress)

$$\underline{F}_{on} = \iint_S [\hat{n} \cdot -\underline{\underline{\Pi}}]|_{surface} dS$$

TORQUE

Flow rate Q through a surface S:

$$Q = \iint_S [\hat{n} \cdot \underline{v}]_{surface} dS$$

Fluid torque $\underline{T}_{on}$ on a surface S: ($\underline{\tilde{R}}$ is the lever arm vector from the axis of rotation to the point of application of the force)
(Bird, UR sign convention on stress)

$$\underline{T}_{on} = \iint_S [\underline{\tilde{R}} \times (\hat{n} \cdot -\underline{\underline{\Pi}})]_{surface} dS$$

---

17 February 2020
formula_sheet_for_exam1_2020

$$\underline{\underline{\Pi}} = p\,\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$\underline{\underline{\tau}} = -\mu \underbrace{\left(\nabla \underline{v} + (\nabla \underline{v})^T\right)}_{\dot{\underline{\underline{\gamma}}}}$$

$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

see next pg:

$$\nabla \underline{v} = \begin{pmatrix} 0 & \frac{\partial v_\theta}{\partial r} & 0 \\ -\frac{v_\theta}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$\dot{\underline{\underline{\gamma}}} = \begin{pmatrix} 0 & \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & 0 \\ \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{r\theta z}$$

4

C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{w} = \frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r}\frac{\partial w_z}{\partial \theta} - \frac{\partial w_\theta}{\partial z} \\ \frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r w_\theta)}{\partial r} - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

$$\underline{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix}_{r\theta z}$$

$$\nabla \underline{v}$$

$$v_\theta = v_\theta(r)$$

$$\nabla \underline{w} = \begin{pmatrix} \cancel{\frac{\partial w_r}{\partial r}} & \frac{\partial w_\theta}{\partial r} & \cancel{\frac{\partial w_z}{\partial r}} \\ \cancel{\frac{1}{r}\frac{\partial w_r}{\partial \theta}} - \frac{w_\theta}{r} & \cancel{\frac{1}{r}\frac{\partial w_\theta}{\partial \theta}} + \cancel{\frac{w_r}{r}} & \cancel{\frac{1}{r}\frac{\partial w_z}{\partial \theta}} \\ \cancel{\frac{\partial w_r}{\partial z}} & \cancel{\frac{\partial w_\theta}{\partial z}} & \cancel{\frac{\partial w_z}{\partial z}} \end{pmatrix}_{r\theta z}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_r)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_r}{\partial \theta^2} + \frac{\partial^2 w_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial w_\theta}{\partial \theta} \\ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_\theta)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_\theta}{\partial \theta^2} + \frac{\partial^2 w_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial w_r}{\partial \theta} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial w_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_z}{\partial \theta^2} + \frac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z}$$

F. A. Morrison, Understanding Rheology (Oxford, 2001)

(5)

$$v_\theta = \underbrace{\left(\frac{\kappa^2 \Omega R}{\kappa^2 - 1}\right)}_{= \alpha} \left(\frac{r}{R} - \frac{R}{r}\right)$$

$$= \left(\frac{\alpha}{R}\right) r - \alpha R \left(\frac{1}{r}\right)$$

$$\frac{\partial v_\theta}{\partial r} = \frac{\alpha}{R} + \alpha R \frac{1}{r^2}$$

$$\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} = \cancel{\frac{\alpha}{R}} + \alpha R \left(\frac{1}{r^2}\right)$$

$$- \left[\cancel{\frac{\alpha}{R} r} - \alpha R \frac{1}{r}\right] \frac{1}{r}$$

$$= 2 \alpha R \frac{1}{r^2}$$

$$\tau_{r\theta} = \left(-2\mu \alpha R\right) \frac{1}{r^2}$$

$$P = P(r)$$

$$\underline{\underline{\tau}} = -\mu \underline{\underline{\dot{\gamma}}}$$

$$\underline{\underline{\Pi}} = \begin{pmatrix} P(r) & -\mu\frac{2\alpha R}{r^2} & 0 \\ -\mu\frac{2\alpha R}{r^2} & P(r & 0 \\ 0 & 0 & P(r) \end{pmatrix}_{r\theta z}$$

$$\hat{n} \cdot \underline{\underline{\Pi}} = \begin{pmatrix} -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} & \underline{\underline{\Pi}} & \end{pmatrix}_{r\theta z}$$

$$= \begin{pmatrix} -P(r) & \frac{2\mu\alpha R}{r^2} & 0 \end{pmatrix}_{r\theta z}$$

$$-\hat{n} \cdot \underline{\underline{\Pi}} = \begin{pmatrix} P(r) \\ -\frac{2\mu\alpha R}{r^2} \\ 0 \end{pmatrix}_{r\theta z}$$

$$\underline{\tilde{R}} \times \left(n \cdot -\underline{\underline{\Pi}} \right)$$

$$= \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ \chi R & 0 & 0 \\ p(r) & -\frac{2\mu\alpha R}{r^2} & 0 \end{vmatrix}$$ ← determinant

$$= \hat{e}_r (0) - \hat{e}_\theta (0) + \hat{e}_z \left(\frac{-2\mu\alpha \chi R^2}{r^2} \right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ \frac{-2\mu\alpha\chi R^2}{r^2} \end{pmatrix}_{r\theta z}$$

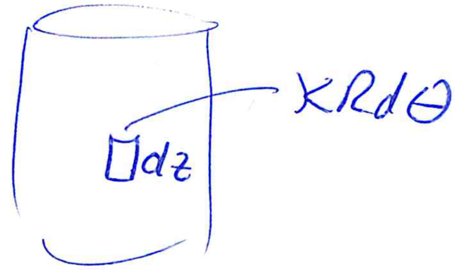
What is Torque?

$$\underline{T}_m = \iint_S \tilde{\underline{R}} \times \left(n \cdot -\underline{\underline{T}}\right)\Big|_{\text{surface}} \, dS$$

$r = \kappa R$

$$dS = \kappa R \, d\theta \, dz$$

nothing varies w/ θ or z



$0 \le \theta \le 2\pi$

$0 \le z \le L$

$$T_m = \begin{pmatrix} 0 \\ 0 \\ \dfrac{-2\mu\alpha\cancel{\kappa}\cancel{R^2}}{\cancel{\kappa^2}\cancel{R^2}} \end{pmatrix} \underbrace{\cancel{\kappa}R \, 2\pi L}_{\text{from the integration}}$$

$$T_z = -4\pi R\mu L \overbrace{\left(\frac{\kappa^2 \Omega R}{\kappa^2 - 1}\right)}^{\alpha}$$

$$\boxed{T_z = \frac{-4\pi R^2 \mu \Omega \kappa^2 L}{\kappa^2 - 1}}$$