SOLN 2020

Homework 3
CM4650
Spring 2020

Due: *Wednesday 26 February 2020, in class*

Please do not write on the backside of the pages. Please write legibly and large. Thank you.

1. (10 points) Text 5.1 What is a stress constitutive equation? What is a rheological material function? What is the difference and how are these two concepts/definitions related? Please put the differences in your own words (don't directly quote the book please).

2. (10 points) Text 2.19: Using Einstein notation, show that:

$$\left(\underline{\underline{A}} \cdot \underline{\underline{B}} \cdot \underline{\underline{C}}\right)^T = \underline{\underline{C}}^T \cdot \underline{\underline{B}}^T \cdot \underline{\underline{A}}^T$$

3. (10 points) Calculate the magnitude of the tensor $\underline{\underline{A}}$ given below:

$$\underline{\underline{A}} = 5\hat{e}_1\hat{e}_1 + 2\hat{e}_1\hat{e}_2 - \hat{e}_2\hat{e}_2 + 2\hat{e}_2\hat{e}_3 + \hat{e}_3\hat{e}_1 - 2\hat{e}_3\hat{e}_3$$

4. (10 points) Tensors (more precisely, second-order tensors) have three invariants, which are scalars that are independent of coordinate system. One set of three invariants, I, II, III, is defined in Chapter 2; another set of invariants I_1, I_2, I_3 is defined in Appendix B (page 453); the two sets are interrelated in equations C.81-C.83 (p 476). For the tensors given below, what are the values of the invariants? Calculate both sets from the definitions and verify that the interrelating equations on page 476 hold.

$$\underline{\underline{A}} = \begin{pmatrix} 5 & 8.2 & 0 \\ 8.2 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}_{123}$$

$$\underline{\underline{B}} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -16 \end{pmatrix}_{123}$$

5. (20 points) What is the start-up of steady shearing material function $\eta^+(t, \dot{\gamma}_0)$ predicted by the proposed constitutive equation below? Derive your answer from the starting definitions on the "recipe card". Please sketch your answer for various values of $\dot{\gamma}_0$.

$$\underline{\underline{\tau}}(t, \dot{\gamma}_0) = \left(\frac{a}{\sqrt{\dot{\gamma}_0}}\right)(\nabla \underline{v} + (\nabla \underline{v})^T)$$

where $\dot{\gamma}_0$ is the parameter in the definition of the start-up material function. What are the units on a?

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1. TEXT 5.1 See next page

2. TEXT 2.19 See subsequent pages

3. $|\underline{\underline{A}}| = ?$

$$\underline{\underline{A}} = \begin{array}{c}\begin{pmatrix} 5 & 2 & 0 \\ 0 & -1 & 2 \\ 1 & 0 & -2 \end{pmatrix} \\ \hspace{3em}{}_{123}\end{array}$$

δ_{km} "k becomes m"

$$\underline{\underline{A}} : \underline{\underline{A}} = A_{pk}\hat{e}_p\hat{e}_k : A_{ms}\hat{e}_m\hat{e}_s$$

δ_{ps} "p becomes s"

$$= A_{sm}A_{ms}$$

$$= A_{11}^2 + A_{12}A_{21} + A_{31}A_{13} + A_{21}A_{12} + A_{22}^2 + A_{23}A_{32} + A_{31}A_{13} + A_{32}A_{23} + A_{33}^2$$

(continues p5)

#1

5.1 What is a constitutive eqn? mat'l fn?

A constitutive equation is a tensor eqn which relates the deformation experienced by a fluid w/ the stress generated

$$\underline{\underline{\tau}} = \underline{\underline{\tau}}(\underline{v}, \nabla \underline{v}, \text{etc.})$$

With the eqns of conservation a constitutive eqn can be used to solve flow problems.

A material function is a scalar function defined in terms of a material's response to a particular set of flow kinematics. The kinematics are chosen, e.g. shear flow w/ $\dot{\gamma}(t) = \dot{\gamma}_0$, and the material function is defined in terms of how the material responds to the imposed kinematics, e.g. $\eta \equiv \frac{-\tau_{21}}{\dot{\gamma}_0}$ ← material response

Since constitutive eqns can predict how a mat'l will behave in ANY flow situation, mat'l functions can be predicted from constitutive eqns by substituting the kinematics into the constitutive eqn + solving for the material response needed in the mat'l function.

Mat'l functions can be measured while constitutive eqns cannot. Comparisons btwn measured mat'l functions + predicted mat'l functions help us to choose appropriate constitutive eqns.

#2

4

2.19 Show $(\underline{\underline{A}}\cdot\underline{\underline{B}}\cdot\underline{\underline{C}})^T = \underline{\underline{C}}^T\cdot\underline{\underline{B}}^T\cdot\underline{\underline{A}}^T$

$$\underline{\underline{A}}\cdot\underline{\underline{B}}\cdot\underline{\underline{C}} = A_{ij}\,\hat{e}_i\,\underbrace{\hat{e}_j\cdot B_{pk}\,\hat{e}_p}_{\delta_{jp}}\,\underbrace{\hat{e}_k\cdot C_{rs}\,\hat{e}}_{\delta_{kr}}$$

$$= A_{ij}\,B_{jk}\,C_{ks}\,\hat{e}_i\,\hat{e}_s$$

$$(\underline{\underline{A}}\cdot\underline{\underline{B}}\cdot\underline{\underline{C}})^T = C_{ks}\,B_{jk}\,A_{ij}\;\hat{e}_s\,\hat{e}_i$$

$$\underline{\underline{C}}^T\cdot\underline{\underline{B}}^T\cdot\underline{\underline{A}}^T = C_{ij}\,\hat{e}_j\,\underbrace{\hat{e}_i\cdot B_{mn}\,\hat{e}_n}_{\delta_{in}}\,\underbrace{\hat{e}_m\cdot A_{fg}\,\hat{e}_g}_{\delta_{mg}}$$

$$= C_{ij}\,B_{mi}\,A_{fm}\;\hat{e}_j\,\hat{e}_f$$

indices are in the same places $\therefore$ they are the same//

5

$$\underline{\underline{A}} : \underline{\underline{A}} = 5^2 + \cancel{(2)(0)} + \cancel{(0)(1)}$$
$$+ \cancel{(0)(2)} + (-1)^2 + \cancel{(2)(0)}$$
$$+ \cancel{(1)(0)} + \cancel{(0)(2)} + (-2)^2$$

$$= 25 + 1 + 4 = 30$$

$$|\underline{\underline{A}}| = \sqrt{\frac{\underline{\underline{A}} : \underline{\underline{A}}}{2}} = \sqrt{\frac{30}{2}} = \boxed{\sqrt{15}}$$

4. Tensor invariants

$$I_{\underline{\underline{A}}} = \text{trace}\ \underline{\underline{A}} \qquad \text{eqn } 2.186$$

$$II_{\underline{\underline{A}}} = \text{trace}\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) \qquad \text{eqn } 1.87$$

$$III_{\underline{\underline{A}}} = \text{trace}\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) \qquad \text{eqn } 1.88$$

$$I_{\underline{\underline{A}}} = 5 + 0 + -5 = \boxed{0 = I_{\underline{\underline{A}}}}$$

$$I_{\underline{\underline{B}}} = 8 + 8 + (-16) = \boxed{0 = I_{\underline{\underline{B}}}}$$

$$\underline{\underline{A}} \cdot \underline{\underline{A}} = \begin{pmatrix} 5 & 8.2 & 0 \\ 8.2 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 5 & 8.2 & 0 \\ 8.2 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$= \begin{pmatrix} 92.240 & 41 & 0 \\ 41 & 67.240 & 0 \\ 0 & 0 & 25 \end{pmatrix}$$

$$\boxed{II_{\underline{\underline{A}}} = 184.48}$$

$$\underline{\underline{B}} \cdot \underline{\underline{B}} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -16 \end{pmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 256 \end{pmatrix}$$

$$\boxed{II_{\underline{\underline{B}}} = 384}$$

$$\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}} = \begin{pmatrix} 92.240 & 41 & 0 \\ 41 & 67.240 & 0 \\ 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 5 & 8.2 & 0 \\ 8.2 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 797.4 & 756.368 & 0 \\ 756.368 & 336.20 & 0 \\ 0 & 0 & -125 \end{pmatrix}$$

$$\boxed{III_{A} = 1008.6}$$

$$\underline{\underline{B}} \cdot \underline{\underline{B}} \cdot \underline{\underline{B}} = \begin{pmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 252 \end{pmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} 512 & 0 & 0 \\ 0 & 512 & 0 \\ 0 & 0 & -4096 \end{pmatrix}$$

$$\boxed{III_B = 3072}$$

Second definitions p453

$\underline{A}$: $\Theta = I$ ✓

$$\Phi = \cancel{(0)(-5)} - \cancel{(0)(0)} + (-5)(5)$$

$$- \cancel{(0)(0)} + \cancel{(5)(0)} - (8.2)(8.2)$$

$$= -25 - 8.2^2$$

$$\boxed{\Phi = -92.240}$$

P476:

$$\Phi = \frac{1}{2}\left[(I)^2 - (II)\right]$$

$$= \left(\frac{1}{2}\right)\left[(0)^2 - 184.48\right]$$

$$\boxed{\Phi = 92.240}$$

$$\psi = \det |A|$$

$$= \det \begin{vmatrix} 5 & 8.2 & 0 \\ 8.2 & 0 & 0 \\ 0 & 0 & -5 \end{vmatrix}$$

$$= 5(0-0) - 8.2((8.2)(-5) - 0) + 0(0-0)$$

$$\psi = \boxed{336.2}$$

P476: $\frac{1}{6}\left(\cancel{I}^3 - 3\cancel{I}\, II + 2\, III\right)$

$$\psi = \frac{1}{3}(III) = \frac{1008.6}{3} = \boxed{336.2 = \psi}$$

<u>B</u> $\Theta = I$

$$\Phi = (8)(-16) - \cancel{(0)}(0) + (-16)(8) - \cancel{(0)}(0) + (8)(8) + \cancel{(0)}(0)$$

$$\boxed{\Phi = -192}$$

P47C:
$$\Phi = \frac{1}{2}\left(\cancel{I^2}_{0} - II\right)$$

$$= \frac{1}{2}(-384) = \boxed{-192 = \Phi}$$

$$\Psi = \det \begin{vmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -16 \end{vmatrix}$$

$$= 8(8)(-16) - 0\cancel{(0-0)} + (0)\cancel{(0-0)}$$

$$\boxed{\Psi = -1024}$$

P476:

$$\psi = \frac{1}{6}\left(\cancel{I}^3 - 3\cancel{I}\,II + 2\,III\right)$$

$$= \frac{1}{3}\left(III\right) = \frac{-3072}{3}$$

$$\boxed{\psi = -1024}$$ ✓

Conclusion – both sets of definitions work out as described in the text. //

Calc

S. $\eta^+(t, \dot{\gamma}_0)$ for:

$$\underline{\underline{\tau}}(t, \dot{\gamma}_0) = \frac{a}{\sqrt{\dot{\gamma}_0}} \left(\nabla \underline{v} + (\nabla \underline{v})^T\right)$$

$$\underline{\underline{\tau}} \; [=] \; Pa = \frac{a}{\sqrt{(s^{-1})}} \frac{\cancel{m}}{s} \frac{1}{\cancel{m}}$$

$$\boxed{a \; [=] \; Pa \; s^{\frac{1}{2}}}$$

start up of
steam flow (from recipe cond):

$$\underline{v} = \begin{pmatrix} \dot{\varsigma}(t)\, x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \qquad \dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

B

$$\nabla \underline{v} = \begin{pmatrix} 0 & 0 & 0 \\ \dot{\zeta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\dot{\underline{\underline{\gamma}}} = \nabla \underline{v} + (\nabla \underline{v})^T = \begin{pmatrix} 0 & \dot{\zeta}(t) & 0 \\ \dot{\zeta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\tau}} = \frac{a}{\sqrt{\dot{\gamma}_0}} \, \dot{\underline{\underline{\gamma}}}$$

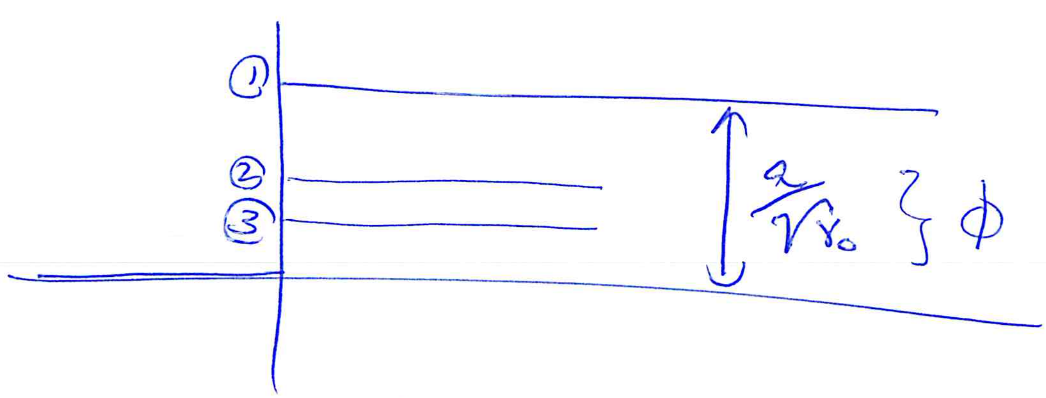
$$= \begin{pmatrix} 0 & \frac{a}{\sqrt{\dot{\gamma}_0}} \dot{\zeta}(t) & 0 \\ \frac{a}{\sqrt{\dot{\gamma}_0}} \dot{\zeta}(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Again, from the recipe card:

$$\eta^+ = \frac{-\tau_{21}}{\dot{\gamma}_0} = \frac{a}{\dot{\gamma}_0\sqrt{\dot{\gamma}_0}}\,\dot{\zeta}(t)$$

$$= \frac{a}{\dot{\gamma}_0^{3/2}}\begin{cases} 0 & t<0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$$

$$\eta^+ = \begin{cases} 0 & t<0 \\ \dfrac{a}{\sqrt{\dot{\gamma}_0}} & t \geq 0 \end{cases}$$



① $\dot{\gamma}_0 = b$ $\quad \phi = a/\sqrt{b}$

② $\dot{\gamma}_0 = 4b$ $\quad \phi = a/2\sqrt{b}$

③ $\dot{\gamma}_0 = 9b$ $\quad \phi = a/3\sqrt{b}$

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