Homework 4 Solution
CM4650
Spring 2020 Morrison

1. (30 points) Please answer in your own words (i.e. don't quote me directly in your answer; don't quote the internet).
    1. Give an example of a fluid exhibiting a memory effect. Explain why we use the word "memory" to describe this kind of behavior. A memory effect is when the current state of stress depends on the motion or deformation at some earlier time. The key concept is that the stress now depends on more than what is happening now. We must take into account what happened in the past. An example is slow relaxation, slow stress growth, non-zero step strain. In the video when the flow was stopped the stress did not immediately drop to zero; that's a memory effect.
    2. Many materials exhibit "rate-dependent effects" in their rheological response. Give an example of such an effect. A rate effect is when the stress response depends on how fast the deformation is. In the video he showed that for some fluids when you double the driving pressure you do not double the flow rate. This is a rate-dependent effect. Shear thinning and shear thickening are rate effects – the viscosity depends on the rate of deformation.
    3. Why are memory and rate-dependent effects interesting or important?A fluid's current stress may depend on its past deformation and it may also depend on the rate of the deformation. We discussed shear start-up experiments that showed slow start up (memory effect) but the exact trace of the stress depended on the rate that you start it up to (rate effect). This is why rheology is so complicated—there are both memory and rate effects occurring at the same time.

2.

a. What are the "*rate-based*" material functions in shear? In elongation?
b. What are the "*strain-based*" material functions in shear? In elongation?

Only the answers in black are required.

	Shear	**Elongation**
steady	rate-based	rate-based
startup	rate based	rate-based
cessation	rate-based	*not observed*
step strain	strain-based	*strain-based, but rarely used*
creep	strain-based	*strain-based, but rarely used*
small-amplitude oscillatory deformation	strain-based	*Strain-based, but the results are usually expressed with the SAOS*

3. (20 points)
   1. For general shear flow, derive the rate-of-deformation tensor $\underline{\underline{\dot{\gamma}}}$. What does the rate-of-deformation tensor become for steady shear flow?
   2. Derive the magnitude of the rate-of-deformation tensor for steady shear flow.

For steady shear $\dot{\zeta}(t) = \dot{\gamma}_0$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} 0 & \dot{\gamma}_0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$|\underline{\underline{\dot{\gamma}}}| = \sqrt{\frac{\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}}{2}}$$

δ_{ik}

$$\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} = \dot{\gamma}_{ij}\hat{e}_i\hat{e}_j : \dot{\gamma}_{pk}\hat{e}_p\hat{e}_k$$

δ_{jp}

$$\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}} = \dot{\gamma}_{kp}\dot{\gamma}_{pk}$$

$$= \sum_{k=1}^{3}\sum_{p=1}^{3} \dot{\gamma}_{pk}\dot{\gamma}_{kp}$$

$$= 2\left(\dot{\zeta}(t)\right)^2$$

$$|\underline{\underline{\dot{\gamma}}}| = \sqrt{\frac{\cancel{2}\dot{\zeta}^2(t)}{\cancel{2}}} = |\dot{\zeta}(t)|$$

for steady shear $|\underline{\underline{\dot{\gamma}}}| = |\dot{\gamma}_0|$

4. (10 points)

a) For *general* uniaxial elongational flow, derive the rate-of-deformation tensor $\underline{\underline{\dot{\gamma}}}$. What does the rate-of-deformation tensor become for steady uniaxial elongational flow?

b) Derive the magnitude of the rate-of-deformation tensor for steady uniaxial elongational flow.

steady elongation

$$\underline{v} = \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123}$$

$$\nabla \underline{v} = \frac{\partial v_i}{\partial x_p} \hat{e}_p \hat{e}_i$$

$$= \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{pmatrix}_{123}$$

$$= \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} & 0 & 0 \\ 0 & -\frac{\dot{\varepsilon}(t)}{2} & 0 \\ 0 & 0 & \dot{\varepsilon}(t) \end{pmatrix}_{123}$$

$$\underline{\underline{\dot{\gamma}}} = \begin{pmatrix} -\dot{\varepsilon}(t) & 0 & 0 \\ 0 & -\dot{\varepsilon}(t) & 0 \\ 0 & 0 & 2\dot{\varepsilon}(t) \end{pmatrix}_{123}$$

for steady elongation: $\dot{\varepsilon}(t) = \dot{\varepsilon}_0$

$$\dot{\underline{\underline{\gamma}}} = \begin{pmatrix} -\dot{\varepsilon}_0 & 0 & 0 \\ 0 & -\dot{\varepsilon}_0 & 0 \\ 0 & 0 & 2\dot{\varepsilon}_0 \end{pmatrix}_{123}$$

$$|\dot{\underline{\underline{\gamma}}}| = \sqrt{\frac{\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}}{2}}$$

$$\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}} = \sum_{k=1}^{3} \sum_{p=1}^{3} \dot{\gamma}_{pk} \dot{\gamma}_{kp}$$

$$\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}} = (\dot{\varepsilon}(t))^2 + (\dot{\varepsilon}(t))^2 + 4(\dot{\varepsilon}(t))^2$$

$$= 6(\dot{\varepsilon}(t))^2$$

$$|\dot{\underline{\underline{\gamma}}}| = \sqrt{\frac{6(\dot{\varepsilon}(t))^2}{2}} = \boxed{|\dot{\varepsilon}(t)|\sqrt{3}}$$

for steady elongation
$\dot{\varepsilon}(t) = \dot{\varepsilon}_0$ //

5. Calculate the shear strain for the flows associated with the following material functions:

   a. Steady shear, use 0 for reference time
   b. Start-up of steady shear, use $-\infty$ for reference time

a) steady shear strain: $\dot{\zeta}(t') = \dot{\gamma}_0$

$$\gamma(0,t) = \int_0^t \dot{\zeta}(t')\,dt'$$

$$= \int_0^t \dot{\gamma}_0 \, dt' = \dot{\gamma}_0 t' \Big|_0^t$$

$$= \boxed{\dot{\gamma}_0 t}$$

b) start-up $\dot{\zeta}(t') = \begin{cases} 0 & t' < 0 \\ \dot{\gamma}_0 & t' \geq 0 \end{cases}$

$$\gamma(-\infty, t) = \int_{-\infty}^{t} \dot{\zeta}(t')\,dt'$$

$$= \underbrace{\cancel{\int_{-\infty}^{0} 0 \, dt'}}_{0} + \int_0^t \dot{\gamma}_0 \, dt'$$

$$= \dot{\gamma}_0 t' \Big|_0^t = \boxed{\dot{\gamma}_0 t}$$