Advanced Constitutive Modeling

Generalized Linear-Viscoelastic Model:

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} G(t-t')\,\underline{\underline{\dot{\gamma}}}(t')\,dt'$$

strain-rate tensor

Good only for small strains, small strain-rates

To develop constitutive equations for large strain, large strain-rate flows, the strain and strain history are important.

What is the strain measure that is used in the GLVE model?

1

What is the strain measure that is used in the GLVE model?

(see hand calculations)

2

Generalized Linear-Viscoelastic Model: (strain version)

infinitesimal strain tensor

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} M(t-t')\,\underline{\underline{\gamma}}(t,t')\,dt'$$

$$M(t-t') \equiv \frac{\partial G(t-t')}{\partial t'}$$

memory function

It is the use of the infinitesimal strain tensor as the strain measure that causes the frame-variance in the GLVE model.

3

We have seen the infinitesimal strain tensor before: when we first defined strain (when we discussed material functions).

Infinitesimal strain tensor $\underline{\underline{\gamma}} \equiv \nabla \underline{u} + \left(\nabla \underline{u}\right)^T$

Displacement function $\underline{u}(t_{ref}, t) \equiv \underline{r}(t) - \underline{r}(t_{ref})$

Particle tracking vector $\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123}$

4

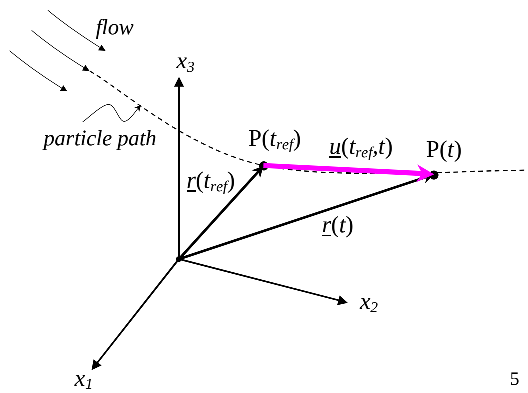
Strain in Shear Flow

$$\gamma_{21}(t_{ref},t) \equiv \frac{\partial u_1}{\partial x_2}$$ Shear strain

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref})$$ Displacement function

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123}$$



5

Deformation in shear flow (strain)

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123} = \begin{pmatrix} x_1(t_{ref}) + (t - t_{ref})\dot{\gamma}_0 x_2 \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t - t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

Displacement function in shear

6

Deformation in shear flow (strain)

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t-t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$

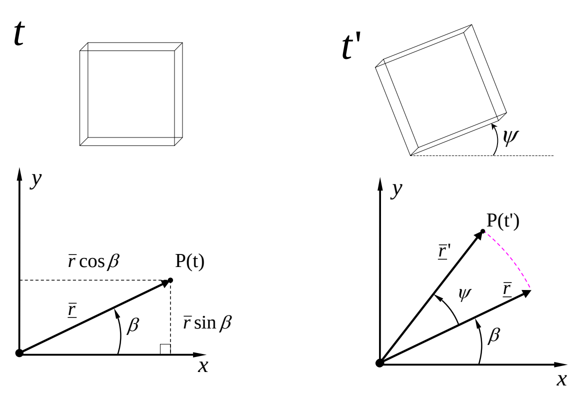
$$\nabla \underline{u} = \begin{pmatrix} 0 & 0 & 0 \\ (t-t_{ref})\dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

$$\underline{\underline{\gamma}} = \nabla \underline{u} + \left(\nabla \underline{u}\right)^T = \begin{pmatrix} 0 & (t-t_{ref})\dot{\gamma}_0 & 0 \\ (t-t_{ref})\dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Infinitesimal strain tensor in shear

7

No stress is generated when a fluid is rotated;
what does the GLVE predict?



- calculate the infinitesimal strain tensor for rigid body rotation
- use the strain-evident version of the GLVE

8

GLVE Prediction for Rigid-Body Rotation around the *z*-axis

(see hand calculations)

9

GLVE Prediction for Rigid-Body Rotation around the *z*-axis

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} M(t-t')\begin{pmatrix} 2(\cos\psi - 1) & 0 & 0 \\ 0 & 2(\cos\psi - 1) & 0 \\ 0 & 0 & 0 \end{pmatrix}_{xyz} dt'$$

Why does GLVE make this erroneous prediction?

$$\underline{\underline{\gamma}}(t,t') = \nabla \underline{u}(t,t') + \left[\nabla \underline{u}(t,t')\right]^T$$

$$\underline{u}(t,t') = \underline{r}(t') - \underline{r}(t)$$

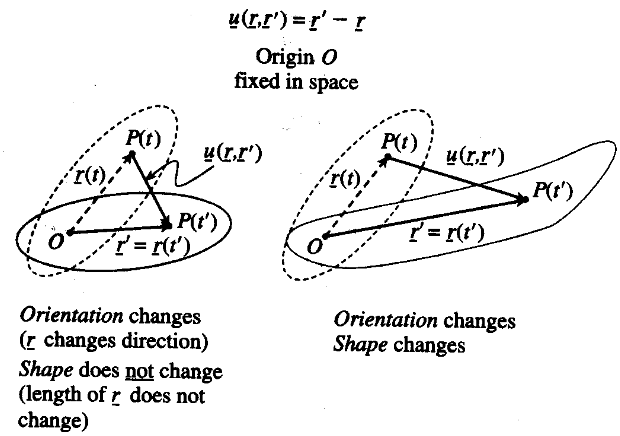
Because this vector, while accounting for deformation, ***also accounts for changes in orientation***.

10

$$\underline{\underline{\gamma}}(t,t') = \nabla \underline{u}(t,t') + \left[\nabla \underline{u}(t,t')\right]^T$$

$$\underline{u}(t,t') = \underline{r}(t') - \underline{r}(t)$$

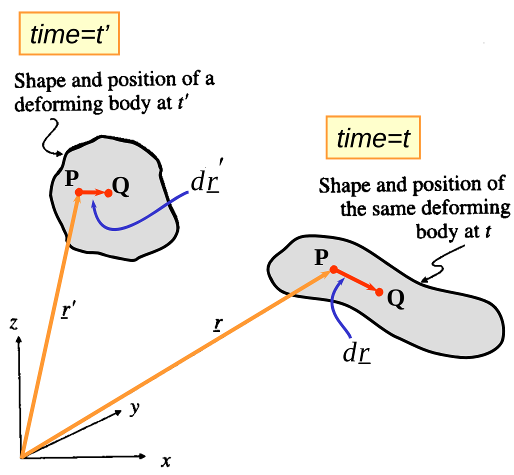
Accounts for changes in shape and orientation.



11

We desire a strain tensor that accurately captures large-strain deformation without being affected by rigid-body rotation.

Consider:



fixed coordinate system (xyz)

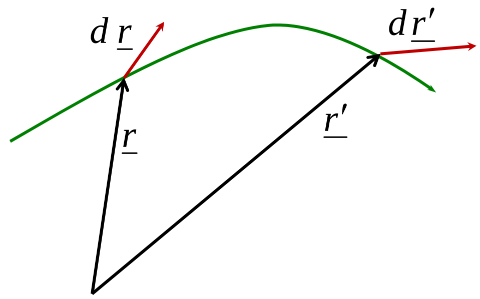
12

Define change-of-shape tensors that rely on relative location of two nearby particles

How does $d\underline{r}$ map to $d\underline{r}'$ along a particle path?

$\underline{r}$ particle label (reference time t)

$\underline{r}'$ location at time t' of the particle labeled $\underline{r}$



$$\underline{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}_{xyz} = f(\underline{r})$$

$$df = \begin{pmatrix} dx' \\ dy' \\ dz' \end{pmatrix}_{xyz} = ?$$

13

$$dx' = ?$$
$$dy' = ?$$
$$dz' = ?$$

(chain rule)

14

$$\begin{pmatrix} dx' & dy' & dz' \end{pmatrix}_{xyz} = \begin{pmatrix} dx & dy & dz \end{pmatrix}_{xyz} \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz}$$

$$d\underline{r}' = d\underline{r} \cdot \underline{\underline{F}}$$

Deformation-gradient tensor

$$\underline{\underline{F}}(t,t') \equiv \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz} = \frac{\partial \underline{r}'}{\partial \underline{r}} = \frac{\partial r_i'}{\partial r_p} \hat{e}_p \hat{e}_i$$

15

Define: $\underline{\underline{F}}^{-1} \cdot \underline{\underline{F}} = \underline{\underline{I}}$

Then use: $d\underline{r}' = d\underline{r} \cdot \underline{\underline{F}}$

$$\Rightarrow ?$$

16

Deformation-gradient tensor

$$d\underline{r}' = d\underline{r} \cdot \underline{\underline{F}}$$

$$\underline{\underline{F}}(t,t') \equiv \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{pmatrix}_{xyz} = \frac{\partial \underline{r}'}{\partial \underline{r}} = \frac{\partial r_i'}{\partial r_p} \hat{e}_p \hat{e}_i$$

Inverse deformation-gradient tensor

$$d\underline{r} = d\underline{r}' \cdot \underline{\underline{F}}^{-1}$$

$$\underline{\underline{F}}^{-1}(t',t) \equiv \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz} = \frac{\partial \underline{r}}{\partial \underline{r}'} = \frac{\partial r_m}{\partial r_j'} \hat{e}_j \hat{e}_m$$

17

We desire a strain tensor that accurately captures large-strain deformation without being affected by rigid-body rotation.

$$\left.\begin{matrix} \nabla \underline{u} \\ \underline{\underline{\gamma}} \\ \underline{\underline{F}} \\ \underline{\underline{F}}^{-1} \end{matrix}\right\}$$

All these strain measures include both deformation and orientation

We can separate the deformation and orientation information in $\underline{\underline{F}}$ and $\underline{\underline{F}}^{-1}$ using a technique called *polar decomposition*.

18

Polar Decomposition Theorem

Any tensor for which an inverse exists has two unique decompositions:

$$\boxed{\begin{aligned}\underline{\underline{A}} &= \underline{\underline{R}}\cdot\underline{\underline{U}}\\ &= \underline{\underline{V}}\cdot\underline{\underline{R}}\end{aligned}}$$

Pure rotation tensor

$$\underline{\underline{U}} = \left(\underline{\underline{A}}^T\cdot\underline{\underline{A}}\right)^{\frac{1}{2}}$$

$$\underline{\underline{V}} = \left(\underline{\underline{A}}\cdot\underline{\underline{A}}^T\right)^{\frac{1}{2}}$$

$$\underline{\underline{R}} = \underline{\underline{A}}\cdot\left(\underline{\underline{A}}^T\cdot\underline{\underline{A}}\right)^{-\frac{1}{2}} = \underline{\underline{A}}\cdot\underline{\underline{U}}^{-1}$$

$\underline{\underline{R}}^{-1} = \underline{\underline{R}}^T$

Orthogonal tensor

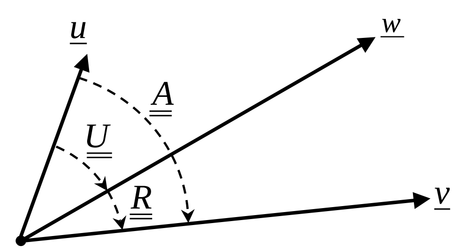
$\underline{\underline{U}}, \underline{\underline{V}}$

Symmetric, nonsingular tensors

19

EXAMPLE: Calculate the right stretch tensor and rotation tensor for a given tensor. Calculate the angle through which $\underline{\underline{R}}$ rotates the vector $\underline{u}$.

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 0 & 2\\ 0 & 3 & 2\\ 2 & 0 & 0\end{pmatrix}_{xyz} \qquad \underline{u} = \begin{pmatrix}1\\2\\1\end{pmatrix}_{xyz}$$



20

We have partially isolated the effect of rotation through polar decomposition.

rotation tensor

left stretch tensor

$$\underline{\underline{A}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

right stretch tensor

original (strain) tensor

We can further isolate stretch from rotation by considering the *eigenvectors* of $\underline{\underline{U}}$ and $\underline{\underline{V}}$.

21

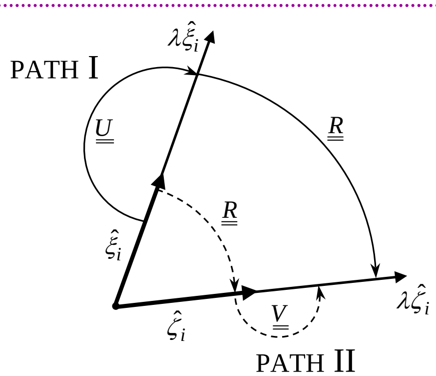
eigenvectors

$$\underline{\underline{U}} \cdot \hat{\xi}_k = \lambda_k \hat{\xi}_k$$

$$\underline{\underline{V}} \cdot \hat{\zeta}_j = \nu_j \hat{\zeta}_j$$

eigenvalues

Physical Interpretation



$$\underline{\underline{R}} \cdot \hat{\xi}_n = \hat{\zeta}_n$$

$$\lambda_n = \nu_n$$

22

Finite Strain Tensors

$\underline{\underline{A}}$	$\underline{\underline{V}}^2$	$\underline{\underline{U}}^2$
$\underline{\underline{F}}$	$\underline{\underline{F}}\cdot\underline{\underline{F}}^T$	$\underline{\underline{F}}^T\cdot\underline{\underline{F}}$
$\underline{\underline{F}}^T$	$\underline{\underline{F}}^T\cdot\underline{\underline{F}}$	$\underline{\underline{F}}\cdot\underline{\underline{F}}^T$
$\underline{\underline{F}}^{-1}$	$\underline{\underline{F}}^{-1}\cdot\left(\underline{\underline{F}}^{-1}\right)^T$	$\left(\underline{\underline{F}}^{-1}\right)^T\cdot\underline{\underline{F}}^{-1}$
$\left(\underline{\underline{F}}^{-1}\right)^T$	$\left(\underline{\underline{F}}^{-1}\right)^T\cdot\underline{\underline{F}}^{-1}$	$\underline{\underline{F}}^{-1}\cdot\left(\underline{\underline{F}}^{-1}\right)^T$

proposed deformation tensors; contain stretch and rotation

proposed deformation tensors; contain stretch of eigenvectors, **BUT NO ROTATION**

Cauchy tensor $\underline{\underline{C}}\equiv\underline{\underline{F}}\cdot\underline{\underline{F}}^T$

Finger tensor $\underline{\underline{C}}^{-1}\equiv\left(\underline{\underline{F}}^{-1}\right)^T\cdot\underline{\underline{F}}^{-1}$

23

EXAMPLE: Calculate stress predicted in rigid-body rotation by a finite-strain Hooke's law.

$$\underline{\underline{\tau}} = G\underline{\underline{C}}^{-1}(t,0)$$

EXAMPLE: Calculate stress predicted in shear by a finite-strain Hooke's law. Compare with experimental results.

$$\underline{\underline{\tau}} = G\underline{\underline{C}}^{-1}(t,0)$$

24

EXAMPLE: What is the inverse-deformation gradient tensor in steady shear flow? (deformation from t' to t) What is the Finger tensor?

$$\underline{v} = \begin{pmatrix} \dot{\gamma}_0 y \\ 0 \\ 0 \end{pmatrix}_{xyz} \qquad \underline{r}(t) = \begin{pmatrix} x' + (t - t')\dot{\gamma}_0 y' \\ y' \\ z' \end{pmatrix}_{xyz}$$

25

Table 9.3

tensor	shear in 1-direction with gradient in 2-direction	uniaxial elongation in 3-direction	ccw rotation around $\hat{e}_3$
$\underline{\underline{F}}(t,t')$	$\begin{pmatrix} 1 & 0 & 0 \\ -\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{-\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{F}}^{-1}(t',t)$	$\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\frac{\epsilon}{2}} & 0 & 0 \\ 0 & e^{-\frac{\epsilon}{2}} & 0 \\ 0 & 0 & e^{\epsilon} \end{pmatrix}_{123}$	$\begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$
$\underline{\underline{C}}(t,t')$	$\begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon} & 0 & 0 \\ 0 & e^{\epsilon} & 0 \\ 0 & 0 & e^{-2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{C}}^{-1}(t',t)$	$\begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon} & 0 & 0 \\ 0 & e^{-\epsilon} & 0 \\ 0 & 0 & e^{2\epsilon} \end{pmatrix}_{123}$	$\underline{\underline{I}}$
$\underline{\underline{\gamma}}^{[0]}(t,t')$	$\begin{pmatrix} 0 & -\gamma & 0 \\ -\gamma & \gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{\epsilon}-1 & 0 & 0 \\ 0 & e^{\epsilon}-1 & 0 \\ 0 & 0 & e^{-2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$
$\underline{\underline{\gamma}}_{[0]}(t,t')$	$\begin{pmatrix} -\gamma^2 & \gamma & 0 \\ \gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$	$\begin{pmatrix} e^{-\epsilon}-1 & 0 & 0 \\ 0 & e^{-\epsilon}-1 & 0 \\ 0 & 0 & e^{2\epsilon}-1 \end{pmatrix}_{123}$	$\underline{\underline{0}}$

$$\gamma = \gamma(t',t) = \int_{t'}^{t} \dot{\zeta}(t'')dt''$$

$$\epsilon = \epsilon(t',t) = \int_{t'}^{t} \dot{\epsilon}(t'')dt''$$

ψ is the angle from $\underline{r}'$ to $\underline{r}$ in ccw rotation around $\hat{e}_z$

26

TABLE D.1
Comparison of Nomenclature for Strain Tensors Used in the Literature

Name	This Text	Larson [138]	DPL [26]	Macosko [162]	Middleman [179]
Stress tensor	$\underline{\underline{\Pi}} = \underline{\underline{\tau}} + p\underline{\underline{I}}$	$-\underline{\underline{T}} = -\underline{\underline{\sigma}} + p\underline{\underline{I}}$	$\underline{\underline{\Pi}} = \underline{\underline{\tau}} + p\underline{\underline{I}}$	$-\underline{\underline{T}} = -\underline{\underline{\tau}} + p\underline{\underline{I}}$	$-\underline{\underline{T}} = -\underline{\underline{\tau}} + p\underline{\underline{I}}$
Gradient of a vector	$\nabla \underline{w} = \frac{\partial w_p}{\partial x_k}\hat{e}_k\hat{e}_p$	$\nabla \underline{w} = \frac{\partial w_p}{\partial x_k}\hat{e}_k\hat{e}_p$	$\nabla \underline{w} = \frac{\partial w_p}{\partial x_k}\hat{e}_k\hat{e}_p$	$\tilde{\nabla} \underline{w} = \frac{\partial w_p}{\partial x_k}\hat{e}_p\hat{e}_k$	$\tilde{\nabla} \underline{w} = \frac{\partial w_p}{\partial x_k}\hat{e}_p\hat{e}_k$
Deformation-gradient tensor	$\underline{\underline{F}}$	$\underline{\underline{F}}$	$\underline{\underline{\Delta}}^T$	$(\underline{\underline{F}}^{-1})^T$	—
Inverse deformation-gradient tensor	$\underline{\underline{F}}^{-1}$	$\underline{\underline{F}}^{-1}$	$\underline{\underline{E}}^T$	$\underline{\underline{F}}^T$	—
Cauchy tensor	$\underline{\underline{C}}$	$\underline{\underline{C}}$	$\underline{\underline{B}}^{-1}$	$\underline{\underline{B}}^{-1}$	—
Finger tensor	$\underline{\underline{C}}^{-1}$	$\underline{\underline{C}}^{-1}$	$\underline{\underline{B}}$	$\underline{\underline{B}}$	—
Finite strain based on Cauchy	$\underline{\underline{\gamma}}^{[0]}$	$\underline{\underline{C}} - \underline{\underline{I}}$	$\underline{\underline{\gamma}}^{[0]}$	$\underline{\underline{B}}^{-1} - \underline{\underline{I}}$	—
Finite strain based on Finger	$\underline{\underline{\gamma}}_{[0]}$	$\underline{\underline{I}} - \underline{\underline{C}}^{-1}$	$\underline{\underline{\gamma}}_{[0]}$	$-\underline{\underline{E}}$	—
Rate-of-strain tensor	$\underline{\underline{\dot{\gamma}}}$	$2\underline{\underline{D}}$	$\underline{\underline{\dot{\gamma}}}$	$2\underline{\underline{D}}$	$\underline{\underline{\Delta}}$
Green tensor	$\underline{\underline{F}}^{-1} \cdot (\underline{\underline{F}}^{-1})^T$	$\underline{\underline{F}}^{-1} \cdot (\underline{\underline{F}}^{-1})^T$	$\underline{\underline{E}}^T \cdot \underline{\underline{E}}$	$\underline{\underline{C}}$	—

27

Compare Finite-Strain Hooke's Law with Observations

NOTE: *for the first time we have predicted* *nonzero normal stresses* *in shear.*

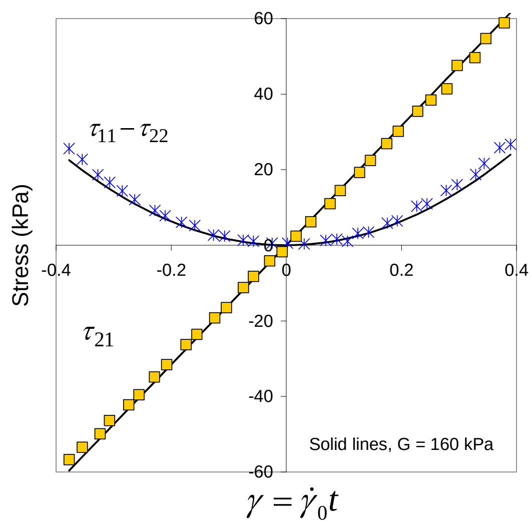


Figure 9.6, p. 325 DeGroot; solid rubber

28

Now, let's fix the Maxwell model.

Integral Maxwell model (rate version):

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda} e^{-\frac{(t-t')}{\lambda}} \underline{\underline{\dot{\gamma}}}(t')\, dt'$$

GLVE model (strain version):

$$\left\{ \begin{aligned} \underline{\underline{\tau}} &= +\int_{-\infty}^{t} M(t-t')\, \underline{\underline{\gamma}}(t,t')\, dt' \\ M(t-t') &\equiv \frac{\partial G(t-t')}{\partial t'} \end{aligned} \right.$$

Integral Maxwell model (strain version):

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{\gamma}}(t,t')\, dt'$$

29

Lodge model

Integral Maxwell model (strain version):

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{\gamma}}(t,t')\, dt'$$

substitute (-Finger tensor) for infinitesimal strain tensor $-\underline{\underline{C}}^{-1}(t',t)$

$$\boxed{\text{Lodge Model:}\quad \underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'}$$

what does it predict?

A finite-strain, viscoelastic constitutive equation

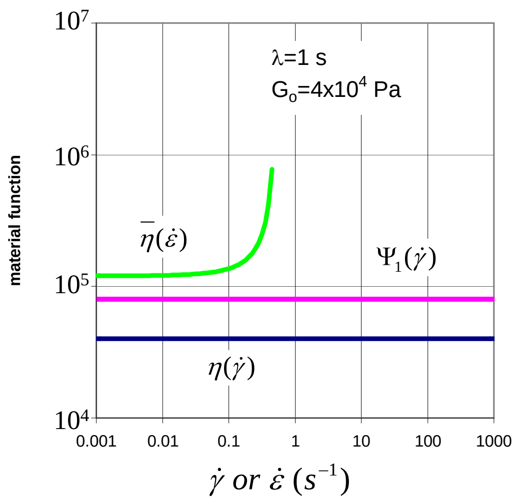
30

EXAMPLE: Calculate the material functions of steady shear flow for the Lodge model.

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

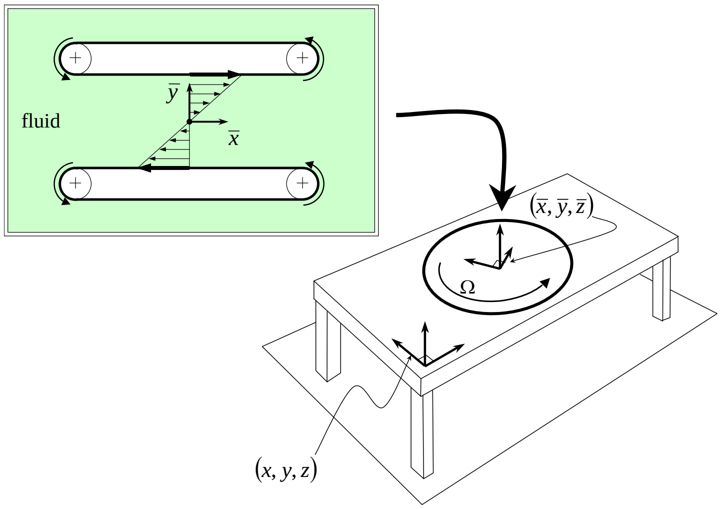
31

Lodge model



32

EXAMPLE: Does the Lodge model pass the test of objectivity posed by the turntable example? (remember, the GLVE failed this test)



33

Turntable Example

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

$$\underline{\underline{F}}^{-1}(t',t) \equiv \frac{\partial \underline{r}}{\partial \underline{r}'} = \frac{\partial r_m}{\partial r_j'} \hat{e}_j \hat{e}_m = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz}$$

$$\underline{r} = \begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix}_{xyz} = \begin{pmatrix} \bar{x}' + \dot{\gamma}_0 (t-t')\bar{y}' \\ \bar{y}' \\ \bar{z}' \end{pmatrix}_{xyz}$$

34

Deformation in shear flow (strain)

$$\underline{r}(t_{ref}) = \begin{pmatrix} x_1(t_{ref}) \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123} \qquad \gamma_{21}(t_{ref},t) \equiv \frac{\partial u_1}{\partial x_2}$$ **Shear strain**

$$\underline{r}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}_{123} = \begin{pmatrix} x_1(t_{ref}) + (t-t_{ref})\dot{\gamma}_0 x_2 \\ x_2(t_{ref}) \\ x_3(t_{ref}) \end{pmatrix}_{123}$$

$$\underline{u}(t_{ref},t) \equiv \underline{r}(t) - \underline{r}(t_{ref}) = \begin{pmatrix} (t-t_{ref})\dot{\gamma}_0 x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$$ **Displacement function**

35

Turntable Example

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

$$\underline{\underline{C}}^{-1} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

Lodge prediction: rotating frame

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz} dt'$$

36

Lodge turntable - from stationary frame

$$\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{xyz} = \begin{pmatrix} x_0 + (y' - y_0)\left[-SC' + CS' + CC'\gamma\right] + (x' - x_0)\left[SS' + CC' - CS'\gamma\right] \\ y_0 + (y' - y_0)\left[C'C + S'S + SC'\gamma\right] + (x' - x_0)\left[-CS' + SC' - SS'\gamma\right] \\ z' \end{pmatrix}_{xyz}$$

$$S = \sin \Omega t$$
$$S' = \sin \Omega t'$$
$$C = \cos \Omega t$$
$$C' = \cos \Omega t'$$
$$\gamma = \dot{\gamma}_0 (t - t')$$

Now, calculate $\underline{\underline{F}}^{-1}$ and $\underline{\underline{C}}^{-1}$.

$$\underline{\underline{F}}^{-1}(t',t) \equiv \frac{\partial \underline{r}}{\partial \underline{r}'} = \frac{\partial r_m}{\partial r_j'} \hat{e}_j \hat{e}_m = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial y}{\partial x'} & \frac{\partial z}{\partial x'} \\ \frac{\partial x}{\partial y'} & \frac{\partial y}{\partial y'} & \frac{\partial z}{\partial y'} \\ \frac{\partial x}{\partial z'} & \frac{\partial y}{\partial z'} & \frac{\partial z}{\partial z'} \end{pmatrix}_{xyz}$$

$$\underline{\underline{C}}^{-1} \equiv \left(\underline{\underline{F}}^{-1}\right)^T \cdot \underline{\underline{F}}^{-1}$$

Result:

$$\underline{\underline{C}}^{-1}(t',t) = \begin{pmatrix} 1 - 2CS\gamma + C^2\gamma^2 & (C^2 - S^2)\gamma + SC\gamma^2 & 0 \\ (C^2 - S^2)\gamma + SC\gamma^2 & 1 + 2CS\gamma + S^2\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz}$$

Lodge Model prediction in stationary frame:

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1 - 2CS\gamma + C^2\gamma^2 & (C^2 - S^2)\gamma + SC\gamma^2 & 0 \\ (C^2 - S^2)\gamma + SC\gamma^2 & 1 + 2CS\gamma + S^2\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz} dt'$$

$$S = \sin \Omega t \qquad C = \cos \Omega t$$
$$S' = \sin \Omega t' \qquad C' = \cos \Omega t'$$
$$\gamma = \dot{\gamma}_0 (t - t')$$

To compare to previous result, must consider shear coordinate system, e.g. *t=0*

Lodge prediction: stationary frame, t=0

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz} dt'$$

IDENTICAL

Lodge prediction: rotating frame

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{xyz} dt'$$

39

Lodge (Maxwell with Finger strain tensor) passes test of objectivity

What is the differential form of the Lodge model?

Lodge Model: $$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

$$\frac{d\underline{\underline{\tau}}}{dt} = ?$$

$$\frac{d\underline{\underline{C}}^{-1}}{dt} = ?$$ *(Homework)*

40

EXAMPLE: What is $\frac{\partial \underline{\underline{F}}^{-1}}{\partial t}$?

We can answer by writing the definition of the deformation gradient tensor in Einstein notation. We will also need the chain rule of differentiation.

(after this, use Table 9.1)

41

Aside: Why did we use $-\underline{\underline{C}}^{-1}(t',t)$ in the Lodge model?

EXAMPLE: What is $\underline{\underline{C}}^{-1}(t,t)$?

EXAMPLE: Define: $\underline{\underline{\gamma}}^{[0]}(t,t') \equiv \underline{\underline{C}} - \underline{\underline{I}}$ What is this strain tensor in the limit of small strains?

Hint: $\nabla \underline{u} = \frac{\partial(\underline{r}' - \underline{r})}{\partial r}$

EXAMPLE: Define: $\underline{\underline{\gamma}}_{[0]}(t,t') \equiv \underline{\underline{I}} - \underline{\underline{C}}^{-1}$ What is this strain tensor in the limit of small strains?

Hint: $\nabla' \underline{u} = \frac{\partial(\underline{r}' - \underline{r})}{\partial r'}$

42

(Differential Lodge Equation, continued)

$$\underline{\underline{\tau}} + \lambda \overset{\nabla}{\underline{\underline{\tau}}} = -\frac{\eta_0}{\lambda}\underline{\underline{I}}$$

$$\overset{\nabla}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} - \left(\nabla \underline{v}\right)^T \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{v}$$

upper-convected time derivative

$$\frac{D\underline{\underline{\tau}}}{Dt} \equiv \frac{\partial \underline{\underline{\tau}}}{\partial t} + \underline{v} \cdot \nabla \underline{\underline{\tau}}$$

If we define: $\underline{\underline{\tilde{\tau}}} = \underline{\underline{\tau}} + \frac{\eta_0}{\lambda}\underline{\underline{I}}$ (does not affect practical predictions since only normal stress differences can be measured)

43

Note on the total derivative/substantial derivative:

$$\frac{d\underline{\underline{\tau}}(t, x_1, x_2, x_3)}{dt} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + \frac{\partial \underline{\underline{\tau}}}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial \underline{\underline{\tau}}}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial \underline{\underline{\tau}}}{\partial x_3}\frac{\partial x_3}{\partial t}$$

$$\frac{d\underline{\underline{\tau}}}{dt} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + \sum_{m=1}^{3}\frac{\partial \underline{\underline{\tau}}}{\partial x_m}\frac{\partial x_m}{\partial t}$$

If the path along which we are taking the derivative is a particle path *(which we have already assumed when defining the Finger tensor)*, then

$$\frac{d\underline{\underline{\tau}}}{dt} = \frac{\partial \underline{\underline{\tau}}}{\partial t} + \sum_{m=1}^{3}\frac{\partial \underline{\underline{\tau}}}{\partial x_m}v_m = \frac{\partial \underline{\underline{\tau}}}{\partial t} + \underline{v} \cdot \nabla \underline{\underline{\tau}} = \frac{D\underline{\underline{\tau}}}{Dt}$$

44

Differential Lodge Equation (Upper Convected Maxwell Model)

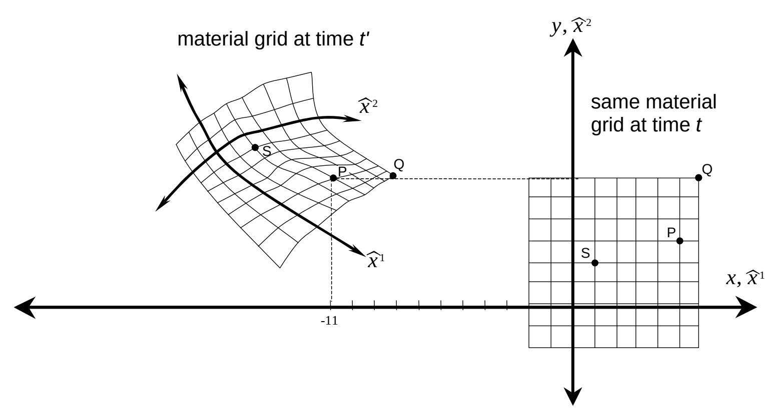
$$\underline{\underline{\tau}} + \lambda \overset{\nabla}{\underline{\underline{\tau}}} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

$$\overset{\nabla}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} - \left(\nabla \underline{v}\right)^T \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{v}$$

upper-convected time derivative

$$\frac{D\underline{\underline{\tau}}}{Dt} \equiv \frac{\partial \underline{\underline{\tau}}}{\partial t} + \underline{v} \cdot \nabla \underline{\underline{\tau}}$$

The *Upper-Convected time derivative* can be understood to be the time derivative calculated in a coordinate system that is translating and deforming with the fluid (see section 9.3).



Other Convected Derivatives

upper-convected time derivative

$$\overset{\triangledown}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} - (\nabla \underline{v})^T \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \nabla \underline{v}$$

lower-convected time derivative

$$\overset{\vartriangle}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} + \nabla \underline{v} \cdot \underline{\underline{\tau}} + \underline{\underline{\tau}} \cdot (\nabla \underline{v})^T$$

Corotational time derivative

$$\overset{\circ}{\underline{\underline{\tau}}} \equiv \frac{D\underline{\underline{\tau}}}{Dt} + \frac{1}{2}\left(\underline{\underline{\omega}} \cdot \underline{\underline{\tau}} - \underline{\underline{\tau}} \cdot \underline{\underline{\omega}}\right)$$

$$\underline{\underline{\omega}} \equiv \nabla \underline{v} - (\nabla \underline{v})^T$$

47

Lodge Model:
(upper-convected Maxwell)

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}^{-1}(t',t)\, dt'$$

Cauchy-Maxwell Model:
(lower-convected Maxwell)

$$\underline{\underline{\tau}} = +\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{C}}(t,t')\, dt'$$

Lodge Rubberlike Liquid Model:

$$\underline{\underline{\tau}} = -\int_{-\infty}^{t} M(t-t')\, \underline{\underline{C}}^{-1}(t',t)\, dt'$$

48

Lodge Equation (UCM)

TABLE D.2
Predictions of Lodge Equation or Upper Convected Maxwell Model in Shear and Extensional Flows

1. *Shear*		
Startup	$\eta^+(t,\dot\gamma)$	$\eta_0\left(1-e^{-\frac{t}{\lambda}}\right)$
	$\Psi_1^+(t,\dot\gamma)$	$2\eta_0\lambda\left[1-e^{\frac{-t}{\lambda}}\left(1+\frac{t}{\lambda}\right)\right]$
	$\Psi_2^+(t,\dot\gamma)$	0
Steady	$\eta(\dot\gamma)$	$\eta_0=G_0\lambda$
	$\Psi_1(\dot\gamma)$	$2G_0\lambda^2=2\eta_0\lambda$
	$\Psi_2(\dot\gamma)$	0
Cessation	$\eta^-(t,\dot\gamma)$	$\eta_0 e^{\frac{-t}{\lambda}}$
	$\Psi_1^-(t,\dot\gamma)$	$2\lambda\eta_0 e^{\frac{-t}{\lambda}}$
	$\Psi_2^-(t,\dot\gamma)$	0
Step shear strain	$G(t,\gamma_0)$	$G_0 e^{-\frac{t}{\lambda}}$
	$G_{\Psi_1}(t,\gamma_0)$	$G_0 e^{-\frac{t}{\lambda}}$
	$G_{\Psi_2}(t,\gamma_0)$	0
2. *Extension*		
Startup Uniaxial ($b=0,\dot\epsilon_0>0$) or biaxial ($b=0,\dot\epsilon_0<0$)	$\bar\eta^+(t,\dot\epsilon_0)$ or $\bar\eta_B^+(t,\dot\epsilon_0)$	$\frac{\eta_0}{\mathcal{A}\mathcal{B}}\left(3-2\mathcal{B}e^{-\frac{\mathcal{A}t}{\lambda}}-\mathcal{A}e^{-\frac{\mathcal{B}t}{\lambda}}\right)$ $\mathcal{A}=1-2\dot\epsilon_0\lambda$ $\mathcal{B}=1+\dot\epsilon_0\lambda$
Planar ($b=1,\dot\epsilon_0>0$)	$\bar\eta_{P_1}^+(t,\dot\epsilon_0)$	$\frac{2\eta_0}{\mathcal{A}C}\left(2-\mathcal{A}e^{-\frac{Ct}{\lambda}}-Ce^{-\frac{\mathcal{A}t}{\lambda}}\right)$ $\mathcal{A}=1-2\dot\epsilon_0\lambda$ $C=1+2\dot\epsilon_0\lambda$
	$\bar\eta_{P_2}^+(t,\dot\epsilon_0)$	$\frac{2\eta_0}{C}\left(1-e^{-\frac{Ct}{\lambda}}\right)$
Steady Uniaxial ($b=0,\dot\epsilon_0>0$) or biaxial ($b=0,\dot\epsilon_0<0$)	$\bar\eta(\dot\epsilon_0)$ or $\bar\eta_B(\dot\epsilon_0)$	$\frac{3\eta_0}{(1-2\lambda\dot\epsilon_0)(1+\lambda\dot\epsilon_0)}=\frac{3\eta_0}{\mathcal{A}\mathcal{B}}$
Planar ($b=1,\dot\epsilon_0>0$)	$\bar\eta_{P_1}(\dot\epsilon_0)$	$\frac{4\eta_0}{1-4\dot\epsilon_0^2\lambda^2}=\frac{4\eta_0}{\mathcal{A}C}$
	$\bar\eta_{P_2}(\dot\epsilon_0)$	$\frac{2\eta_0}{1+2\dot\epsilon_0\lambda}=\frac{2\eta_0}{C}$

49

Cauchy-Maxwell Equation (LCM)

TABLE D.3
Predictions of Cauchy–Maxwell Equation or Lower Convected Maxwell Model in Shear and Extensional Flows

1. *Shear*		
Startup	$\eta^+(t,\dot\gamma)$	$\eta_0\left(1-e^{-\frac{t}{\lambda}}\right)$
	$\Psi_1^+(t,\dot\gamma)$	$2\eta_0\lambda\left[1-e^{\frac{t}{\lambda}}\left(1+\frac{t}{\lambda}\right)\right]$
	$\Psi_2^+(t,\dot\gamma)$	$-\Psi_1^+$
Steady	$\eta(\dot\gamma)$	$\eta_0=G_0\lambda$
	$\Psi_1(\dot\gamma)$	$2G_0\lambda^2=2\eta_0\lambda$
	$\Psi_2(\dot\gamma)$	$-\Psi_1$
Cessation	$\eta^-(t,\dot\gamma)$	$\eta_0 e^{\frac{-t}{\lambda}}$
	$\Psi_1^-(t,\dot\gamma)$	$2\lambda\eta_0 e^{\frac{-t}{\lambda}}$
	$\Psi_2^-(t,\dot\gamma)$	$-\Psi_1^-$
Step shear strain	$G(t,\gamma_0)$	$G_0 e^{-\frac{t}{\lambda}}$
	$G_{\Psi_1}(t,\gamma_0)$	$G_0 e^{-\frac{t}{\lambda}}$
	$G_{\Psi_2}(t,\gamma_0)$	$-G_{\Psi_1}$
2. *Extension*		
Startup Uniaxial ($b=0,\dot\epsilon_0>0$) or biaxial ($b=0,\dot\epsilon_0<0$)	$\bar\eta^+(t,\dot\epsilon_0)$ or $\bar\eta_B^+(t,\dot\epsilon_0)$	$\frac{\eta_0}{C\mathcal{D}}\left(3-2\mathcal{D}e^{-\frac{Ct}{\lambda}}-Ce^{-\frac{\mathcal{D}t}{\lambda}}\right)$ $C=1+2\dot\epsilon_0\lambda$ $\mathcal{D}=1-\dot\epsilon_0\lambda$
Planar ($b=1,\dot\epsilon_0>0$)	$\bar\eta_{P_1}^+(t,\dot\epsilon_0)$	$\frac{-2\eta_0}{\mathcal{A}C}\left(2-\mathcal{A}e^{-\frac{Ct}{\lambda}}-Ce^{-\frac{\mathcal{A}t}{\lambda}}\right)$ $\mathcal{A}=1-2\dot\epsilon_0\lambda$
	$\bar\eta_{P_2}^+(t,\dot\epsilon_0)$	$\frac{-2\eta_0}{\mathcal{A}}\left(1-e^{-\frac{\mathcal{A}t}{\lambda}}\right)$
Steady Uniaxial ($b=0,\dot\epsilon_0>0$) or biaxial ($b=0,\dot\epsilon_0<0$)	$\bar\eta(\dot\epsilon_0)$ or $\bar\eta_B(\dot\epsilon_0)$	$\frac{3\eta_0}{(1+2\lambda\dot\epsilon_0)(1-\lambda\dot\epsilon_0)}=\frac{3\eta_0}{C\mathcal{D}}$
Planar ($b=1,\dot\epsilon_0>0$)	$\bar\eta_{P_1}(\dot\epsilon_0)$	$\frac{-4\eta_0}{1-4\dot\epsilon_0^2\lambda^2}=\frac{-4\eta_0}{\mathcal{A}C}$
	$\bar\eta_{P_2}(\dot\epsilon_0)$	$\frac{-2\eta_0}{1-2\dot\epsilon_0\lambda}=\frac{-2\eta_0}{\mathcal{A}}$

50

Approaches to finite-strain constitutive equations

replace with $\overset{\Delta}{\underline{\underline{\tau}}}, \overset{\nabla}{\underline{\underline{\tau}}}$ or other time derivatives

non-objective time derivative

equivalent

differential Maxwell model

$$\underline{\underline{\tau}} + \lambda \frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_o \dot{\underline{\underline{\gamma}}}$$

integral Maxwell model

$$\underline{\underline{\tau}} = \int_{-\infty}^{t} \frac{\eta_o}{\lambda^2} e^{\frac{-(t-t')}{\lambda}} \underline{\underline{\gamma}}(t,t')\, dt'$$

non-objective strain measure

replace with $-\underline{\underline{C}}^{-1}$, $\underline{\underline{C}}$ or other strain measures

51

Methods of Improving Constitutive Equations

Maxwell Model

We can improve with new time derivatives or new strain measures.

$$\underline{\underline{\tau}} + \lambda \frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \left[\frac{\eta_0}{\lambda^2} e^{-\frac{(t-t')}{\lambda}} \right] \underline{\underline{\gamma}}(t,t')\, dt'$$

We can also change the basic equation:

- *linear modifications*
- *non-linear modifications*

52

Other Constitutive Approaches

Simple Maxwell Model, shear

$$\tau_{21} + \lambda \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$$

Upper-Convected Maxwell Model, general

$$\underline{\underline{\tau}} + \lambda \overset{\nabla}{\underline{\underline{\tau}}} = -\eta_0 \underline{\underline{\dot{\gamma}}}$$

retardation time

Simple Jeffreys Model, shear

$$\tau_{21} + \lambda_1 \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \left(\dot{\gamma}_{21} + \lambda_2 \frac{\partial \dot{\gamma}_{21}}{\partial t} \right)$$

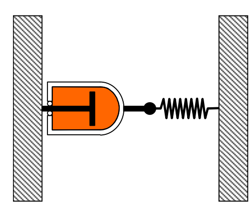
Upper-Convected Jeffreys Model, general (Oldroyd B Fluid)

$$\underline{\underline{\tau}} + \lambda_1 \overset{\nabla}{\underline{\underline{\tau}}} = -\eta_0 \left(\underline{\underline{\dot{\gamma}}} + \lambda_2 \overset{\nabla}{\underline{\underline{\dot{\gamma}}}} \right)$$

53

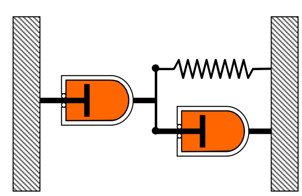
Maxwell Model - Mechanical Analog

$$\tau_{21} + \lambda \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \dot{\gamma}_{21}$$



Jeffreys Model - Mechanical Analog

$$\tau_{21} + \lambda_1 \frac{\partial \tau_{21}}{\partial t} = -\eta_0 \left(\dot{\gamma}_{21} + \lambda_2 \frac{\partial \dot{\gamma}_{21}}{\partial t} \right)$$



54

Unfortunately, this change only modifies G(t-t′);
the Jeffreys Model is a GLVE model

Simple Jeffreys Model (not frame-invariant) $\underline{\underline{\tau}} + \lambda_1 \frac{\partial \underline{\underline{\tau}}}{\partial t} = -\eta_0 \left(\underline{\underline{\dot{\gamma}}} + \lambda_2 \frac{\partial \underline{\underline{\dot{\gamma}}}}{\partial t} \right)$

Now, solving for τ_{21} explicitly we obtain,

$$\underline{\underline{\tau}}(t) = -\int_{-\infty}^{t} \underbrace{\left[\frac{\eta_0}{\lambda_1}\left(1 - \frac{\lambda_2}{\lambda_1}\right) e^{-\frac{t-t'}{\lambda_1}} + \frac{2\eta_0\lambda_2}{\lambda_1}\delta(t-t') \right]}_{G(t-t')} \underline{\underline{\dot{\gamma}}}(t')\,dt'$$

Other linear modifications of the Maxwell model motivated by springs and dashpots in series and parallel modify G(t-t') but do not otherwise introduce new behavior.

(Might as well use the Generalized Maxwell model)

55

Non-linear modifications of the Maxwell Model

White-Metzner Model

$$\underline{\underline{\tau}} + \frac{\eta(\dot{\gamma})}{G_0} \overset{\nabla}{\underline{\underline{\tau}}} = -\eta(\dot{\gamma}) \underline{\underline{\dot{\gamma}}}$$

Oldroyd 8-Constant Model

$$\underline{\underline{\tau}} + \lambda_1 \overset{\nabla}{\underline{\underline{\tau}}} + \frac{1}{2}(\lambda_1 - \mu_1)\left(\underline{\underline{\dot{\gamma}}} \cdot \underline{\underline{\tau}} + \underline{\underline{\tau}} \cdot \underline{\underline{\dot{\gamma}}}\right) + \frac{1}{2}\mu_0 \left(\mathrm{tr}\,\underline{\underline{\tau}}\right)\underline{\underline{\dot{\gamma}}} + \frac{1}{2}\nu_1 \left(\underline{\underline{\tau}} : \underline{\underline{\dot{\gamma}}}\right)\underline{\underline{I}}$$
$$= -\eta_0 \left(\underline{\underline{\dot{\gamma}}} + \lambda_2 \overset{\nabla}{\underline{\underline{\dot{\gamma}}}} + (\lambda_2 - \mu_2)\left(\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}\right) + \frac{1}{2}\nu_2 \left(\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}\right)\underline{\underline{I}} \right)$$

The Oldroyd 8-constant contains many other constitutive equations as special cases.

UCM

UCM + terms = UCJ

56

White-Metzner

TABLE D.5
Predictions of White–Metzner Equation in Shear and Extensional Flows [26]*

1. *Shear*		
Startup	$\eta^+(t,\dot\gamma)$	$\eta(\dot\gamma)\left(1-e^{-\frac{t}{\lambda(\dot\gamma)}}\right)$
	$\Psi_1^+(t,\dot\gamma)$	$2\eta(\dot\gamma)\lambda(\dot\gamma)\left[1-e^{-\frac{t}{\lambda(\dot\gamma)}}\left(1+\frac{t}{\lambda(\dot\gamma)}\right)\right]$
	$\Psi_2^+(t,\dot\gamma)$	0
Steady	$\eta(\dot\gamma)$	$\eta(\dot\gamma)$
	$\Psi_1(\dot\gamma)$	$2\eta(\dot\gamma)\lambda(\dot\gamma)$
	$\Psi_2(\dot\gamma)$	0
2. *Extension*		
Steady		
Uniaxial ($b=0, \dot\epsilon_0>0$) or biaxial ($b=0, \dot\epsilon_0<0$)	$\bar\eta(\dot\epsilon_0)$ or $\bar\eta_B(\dot\epsilon_0)$	$\frac{3\eta(\dot\gamma)}{\left[1-2\lambda(\dot\gamma)\dot\epsilon_0\right]\left[1+\lambda(\dot\gamma)\dot\epsilon_0\right]}=\frac{3\eta(\dot\gamma)}{\mathcal{A}(\dot\gamma)\mathcal{B}(\dot\gamma)}$ $\mathcal{A}(\dot\gamma)=1-2\dot\epsilon_0\lambda(\dot\gamma)$ $\mathcal{B}(\dot\gamma)=1+\dot\epsilon_0\lambda(\dot\gamma)$
Planar ($b=1, \dot\epsilon_0>0$)	$\bar\eta_{P_1}(\dot\epsilon_0)$	$\frac{4\eta(\dot\gamma)}{1-4\dot\epsilon_0^2\lambda(\dot\gamma)^2}=\frac{4\eta(\dot\gamma)}{\mathcal{A}(\dot\gamma)C(\dot\gamma)}$ $\mathcal{A}(\dot\gamma)=1-2\dot\epsilon_0\lambda(\dot\gamma)$ $C(\dot\gamma)=1+2\dot\epsilon_0\lambda(\dot\gamma)$
	$\bar\eta_{P_2}(\dot\epsilon_0)$	$\frac{2\eta(\dot\gamma)}{1+2\dot\epsilon_0\lambda(\dot\gamma)}=\frac{2\eta(\dot\gamma)}{C(\dot\gamma)}$

*$\lambda(\dot\gamma)=\eta(\dot\gamma)/G_0$ and $\dot\gamma=|\dot{\underline{\underline{\gamma}}}|$.

57

Oldroyd B (Convected Jeffreys)

TABLE D.4
Predictions of Oldroyd B or Convected Jeffreys Model in Shear and Extensional Flows [26]

1. *Shear*		
Startup	$\eta^+(t,\dot\gamma)$	$\eta_0\left[\frac{\lambda_2}{\lambda_1}+\left(1-\frac{\lambda_2}{\lambda_1}\right)\left(1-e^{-\frac{t}{\lambda_1}}\right)\right]$
	$\Psi_1^+(t,\dot\gamma)$	$2\eta_0(\lambda_1-\lambda_2)\left[1-e^{-\frac{t}{\lambda_1}}\left(1+\frac{t}{\lambda_1}\right)\right]$
	$\Psi_2^+(t,\dot\gamma)$	0
Steady	$\eta(\dot\gamma)$	η_0
	$\Psi_1(\dot\gamma)$	$2\eta_0(\lambda_1-\lambda_2)$
	$\Psi_2(\dot\gamma)$	0
Cessation	$\eta^-(t,\dot\gamma)$	$\eta_0\left(1-\frac{\lambda_2}{\lambda_1}\right)e^{-\frac{t}{\lambda_1}}$
	$\Psi_1^-(t,\dot\gamma)$	$2\eta_0(\lambda_1-\lambda_2)e^{-\frac{t}{\lambda_1}}$
	$\Psi_2^-(t,\dot\gamma)$	0
SAOS	$G'(\omega)$	$\eta_0\frac{(\lambda_1-\lambda_2)\omega^2}{1+\lambda_1^2\omega^2}$
	$G''(\omega)$	$\eta_0\omega\frac{1+\lambda_1\lambda_2\omega^2}{1+\lambda_1^2\omega^2}$
2. *Extension*		
Startup		
Uniaxial ($b=0, \dot\epsilon_0>0$) or biaxial ($b=0, \dot\epsilon_0<0$)	$\bar\eta^+(t,\dot\epsilon_0)$ or $\bar\eta_B^+(t,\dot\epsilon_0)$	$3\eta_0\frac{\lambda_2}{\lambda_1}+\frac{\eta_0}{\mathcal{A}\mathcal{B}}\left(1-\frac{\lambda_2}{\lambda_1}\right)\left(3-2\mathcal{B}e^{-\frac{\mathcal{A}t}{\lambda_1}}-\mathcal{A}e^{-\frac{\mathcal{B}t}{\lambda_1}}\right)$ $\mathcal{A}=1-2\dot\epsilon_0\lambda_1$ $\mathcal{B}=1+\dot\epsilon_0\lambda_1$
Planar ($b=1, \dot\epsilon_0>0$)	$\bar\eta_{P_1}^+(t,\dot\epsilon_0)$	$4\eta_0\frac{\lambda_2}{\lambda_1}+\frac{2\eta_0}{\mathcal{A}C}\left(1-\frac{\lambda_2}{\lambda_1}\right)\left(2-\mathcal{A}e^{-\frac{Ct}{\lambda_1}}-Ce^{-\frac{\mathcal{A}t}{\lambda_1}}\right)$ $\mathcal{A}=1-2\dot\epsilon_0\lambda_1$ $C=1+2\dot\epsilon_0\lambda_1$
	$\bar\eta_{P_2}^+(t,\dot\epsilon_0)$	$2\eta_0\frac{\lambda_2}{\lambda_1}+\frac{2\eta_0}{C}\left(1-\frac{\lambda_2}{\lambda_1}\right)\left(1-e^{-\frac{Ct}{\lambda_1}}\right)$
Steady		
Uniaxial ($b=0, \dot\epsilon_0>0$) or biaxial ($b=0, \dot\epsilon_0<0$)	$\bar\eta(\dot\epsilon_0)$ or $\bar\eta_B(\dot\epsilon_0)$	$3\eta_0\left(\frac{\lambda_2}{\lambda_1}+\frac{1-\frac{\lambda_2}{\lambda_1}}{\mathcal{A}\mathcal{B}}\right)$
Planar ($b=1, \dot\epsilon_0>0$)	$\bar\eta_{P_1}(\dot\epsilon_0)$	$4\eta_0\left(\frac{\lambda_2}{\lambda_1}+\frac{1-\frac{\lambda_2}{\lambda_1}}{\mathcal{A}C}\right)$
	$\bar\eta_{P_2}(\dot\epsilon_0)$	$2\eta_0\left(\frac{\lambda_2}{\lambda_1}+\frac{1-\frac{\lambda_2}{\lambda_1}}{C}\right)$

58

The Oldroyd 8-Constant model contains all terms *linear* in stress tensor and at most *quadratic* in rate-of-deformation tensor that are also consistent with frame invariance.

$$\underline{\underline{\tau}} + \lambda_1 \overset{\triangledown}{\underline{\underline{\tau}}} + \frac{1}{2}(\lambda_1 - \mu_1)\left(\dot{\underline{\underline{\gamma}}} \cdot \underline{\underline{\tau}} + \underline{\underline{\tau}} \cdot \dot{\underline{\underline{\gamma}}}\right) + \frac{1}{2}\mu_0 \left(\mathrm{tr}\,\underline{\underline{\tau}}\right)\dot{\underline{\underline{\gamma}}} + \frac{1}{2}\nu_1\left(\underline{\underline{\tau}} : \dot{\underline{\underline{\gamma}}}\right)\underline{\underline{I}}$$
$$= -\eta_0\left(\dot{\underline{\underline{\gamma}}} + \lambda_2 \overset{\triangledown}{\dot{\underline{\underline{\gamma}}}} + (\lambda_2 - \mu_2)\left(\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}\right) + \frac{1}{2}\nu_2\left(\dot{\underline{\underline{\gamma}}} : \dot{\underline{\underline{\gamma}}}\right)\underline{\underline{I}}\right)$$

Giesekus Model

$$\underline{\underline{\tau}} + \lambda \overset{\triangledown}{\underline{\underline{\tau}}} + \frac{\alpha\lambda}{\eta_0}\underbrace{\underline{\underline{\tau}} : \underline{\underline{\tau}}}_{\text{quadratic in stress}} = -\eta_0 \dot{\underline{\underline{\gamma}}}$$

The only way to choose among these nonlinear models is to compare predictions.

59

We can also **modify integral models** to add non-linearity and thus produce new constitutive equations.

Factorized Rivlin-Sawyers Model

$$\underline{\underline{\tau}}(t) = +\int_{-\infty}^{t} M(t-t')\left(\Phi_2(I_1, I_2)\underline{\underline{C}} - \Phi_1(I_1, I_2)\underline{\underline{C}}^{-1}\right)dt'$$

Factorized K-BKZ Model

$$\underline{\underline{\tau}}(t) = +\int_{-\infty}^{t} M(t-t')\left(2\frac{\partial U}{\partial I_2}\underline{\underline{C}} - 2\frac{\partial U}{\partial I_1}\underline{\underline{C}}^{-1}\right)dt'$$

I_1, I_2 are the invariants of the Finger or Cauchy strain tensors (these are related).

Again, the only way to choose among these nonlinear models is to compare predictions
(see R. G. Larson, Constitutive Equations for Polymer Melts).

60

Factorized Rivlin-Sawyers

TABLE D.6
Predictions of Factorized Rivlin–Sawyers Model in Shear and Extensional Flows [26]

1. *Shear*		
Steady	$\eta(\dot{\gamma})$	$\int_0^\infty M(s)s(\Phi_1+\Phi_2)\,ds$
	$\Psi_1(\dot{\gamma})$	$\int_0^\infty M(s)s^2(\Phi_1+\Phi_2)\,ds$
	$\Psi_2(\dot{\gamma})$	$-\int_0^\infty M(s)s^2\Phi_2\,ds$
SAOS	$G'(\omega)$	$\int_0^\infty M(s)(1-\cos\omega s)\,ds$
	$G''(\omega)$	$\int_0^\infty M(s)\sin\omega s\,ds$
2. *Extension*		
Steady		
Uniaxial ($b=0, \dot{\epsilon}_0>0$) or biaxial ($b=0, \dot{\epsilon}_0<0$)	$\bar{\eta}(\dot{\epsilon}_0)$ or $\bar{\eta}_B(\dot{\epsilon}_0)$	$\frac{1}{\dot{\epsilon}_0}\int_0^\infty M(s)\left[\Phi_1\left(e^{2\dot{\epsilon}_0 s}-e^{-\dot{\epsilon}_0 s}\right)+\Phi_2\left(e^{\dot{\epsilon}_0 s}-e^{-2\dot{\epsilon}_0 s}\right)\right]ds$
Planar ($b=1, \dot{\epsilon}_0>0$)	$\bar{\eta}_{P_1}(\dot{\epsilon}_0)$	$\frac{1}{\dot{\epsilon}_0}\int_0^\infty M(s)\left[\Phi_1\left(e^{2\dot{\epsilon}_0 s}-e^{-2\dot{\epsilon}_0 s}\right)+\Phi_2\left(e^{2\dot{\epsilon}_0 s}-e^{-2\dot{\epsilon}_0 s}\right)\right]ds$
	$\bar{\eta}_{P_2}(\dot{\epsilon}_0)$	$\frac{1}{\dot{\epsilon}_0}\int_0^\infty M(s)\left[\left(\Phi_1 e^{-\dot{\epsilon}_0 s}+\Phi_2 e^{\dot{\epsilon}_0 s}\right)\left(e^{\dot{\epsilon}_0 s}-e^{-\dot{\epsilon}_0 s}\right)\right]ds$

61

Choosing Constitutive Equations

We have fixed all the obvious flaws in our constitutive equations, and now we have too many choices!

We could make predictions and compare with experimental data, but some of the models (Rivlin Sawyer, K-BKZ) have undefined functions that must be specified.

How to proceed? ***We need some guidance.***

All along we have taken a *continuum-mechanics approach*. We have run that course all the way through. Now we must go back and seek some insight from molecular ideas of relaxation and polymer dynamics.

62

Some of what we have learned from Continuum Modeling

- We can model linear viscoelasticity. The GMM does a good job; there is no reason to play around with springs and dashpots to improve linear viscoelasticity
- We can model shear normal stresses. The kind of deformation described by the Finger tensor gives a first normal stress difference and zero second-normal stress; the kind of deformation described by the Cauchy tensor gives both stress differences, but too much second.
- We can model shear thinning. But only by brute force (GNF, White-Metzner)
- We can model elongational flows. But we predict singularities that do not appear to be present.
- Frame-Invariance is important. Calculations outside the linear viscoelastic regime are incorrect if the equations are not properly frame invariant.
- Thinking in terms of strain is an advantage. When we think only in terms of rate we can only model Newtonian fluids.
- Looking for contradictions when stretching a model to its limits is productive.
- Continuum models do not give molecular insight. We can fit continuum models and obtain material functions (viscosity, relaxation times) but we cannot predict these functions for new, related materials

63