Chapter 1: Introduction

1. What is rheology, anyway?
2. Newtonian versus non-Newtonian
3. Key features of non-Newtonian behavior: Nonlinearity and Memory
What is rheology anyway?

To the layperson, rheology is:

- Mayonnaise does not flow even under stress for a long time; honey always flows
- Silly Putty bounces (is elastic) but also flows (is viscous)
- Dilute flour-water solutions are easy to work with but doughs can be quite temperamental
- Corn starch and water can display strange behavior – poke it slowly and it deforms easily around your finger; punch it rapidly and your fist bounces off of the surface

What is rheology anyway?

To the scientist, engineer, or technician, rheology is

- Yield stresses
- Viscoelastic effects
- Memory effects
- Shear thickening and shear thinning

For both the layperson and the technical person, rheology is a set of problems or observations related to how the stress in a material or force applied to a material is related to deformation (change of shape) of the material.
What is rheology anyway?

Rheology affects:

- Processing (design, costs, production rates)
- End use (food texture, product pour, motor-oil function)
- Product quality (surface distortions, anisotropy, strength, structure development)

Goal of the scientist, engineer, or technician:

- Understand the kinds of flow and deformation effects exhibited by complex systems
- Apply qualitative rheological knowledge to diagnostic, design, or optimization problems
- In diagnostic, design, or optimization problems, use or devise quantitative analytical tools that correctly capture rheological effects

How do we reach these goals?
How?

• **Understand** the kinds of flow and deformation effects exhibited by complex systems

• **Apply qualitative** rheological knowledge to diagnostic, design, or optimization problems

• In diagnostic, design, or optimization problems, use or devise quantitative analytical tools that correctly capture rheological effects

By observing the behavior of different systems

By making calculations with models in appropriate situations

By learning which quantitative models apply in what circumstances

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**Learning Rheology (bibliography)**

**Descriptive Rheology**


**Quantitative Rheology**


**Industrial Rheology**


**Polymer Behavior**


**Suspension Behavior**

Mewis, Jan and Norm Wagner, *Colloidal Suspension* (Cambridge, 2012)

The Physics Behind Rheology:

1. Conservation laws
   - mass
   - momentum
   - energy

2. Mathematics
   - differential equations
   - vectors
   - tensors

3. Constitutive law = law that relates stress to deformation for a particular fluid

Cauchy Momentum Equation
\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{T} + \rho \mathbf{g} \]

Polymer Rheology

Newtonian fluids: (fluid mechanics)
\[ \tau_{21} = -\mu \frac{dv_1}{dx_2} \]

Newton's Law of Viscosity
- This is an empirical law (measured or observed)
- May be derived theoretically for some systems

Non-Newtonian fluids: (rheology)

Need a new law or new laws
- These laws will also either be empirical or will be derived theoretically

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Polymer Rheology

**Newtonian fluids: (shear flow only)**

\[ \tau_{21} = -\frac{dv_1}{dx_2} \]

**Non-Newtonian fluids: (all flows)**

\[ \tau = f(\dot{\gamma}) \]

Stress tensor

Rate-of-deformation tensor

Non-linear function (in time and position)

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**Introduction to Non-Newtonian Behavior**

*Rheological Behavior of Fluids, National Committee on Fluid Mechanics Films, 1964*

<table>
<thead>
<tr>
<th>Type of fluid</th>
<th>Momentum balance</th>
<th>Stress–Deformation relationship (constitutive equation)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inviscid (zero viscosity, ( \mu = 0 ))</td>
<td>Euler equation (Navier-Stokes with zero viscosity)</td>
<td>Stress is isotropic</td>
</tr>
<tr>
<td>Newtonian (finite, constant viscosity, ( \mu ))</td>
<td>Navier-Stokes (Cauchy momentum equation with Newtonian constitutive equation)</td>
<td>Stress is a function of the instantaneous velocity gradient</td>
</tr>
<tr>
<td>Non-Newtonian (finite, variable viscosity ( \eta ) plus memory effects)</td>
<td>Cauchy momentum equation with memory constitutive equation</td>
<td>Stress is a function of the history of the velocity gradient</td>
</tr>
</tbody>
</table>
Rheological Behavior of Fluids - Newtonian

1. Strain response to imposed shear stress
   • shear rate is constant

\[ \frac{d\gamma}{dt} = \text{constant} \]

2. Pressure-driven flow in a tube (Poiseuille flow)
   • viscosity is constant

\[ Q = \frac{\pi \Delta P R^4}{8 \mu L} \]

3. Stress tensor in shear flow
   • only two components are nonzero

\[ \tau = \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Rheological Behavior of Fluids – non-Newtonian

1. Strain response to imposed shear stress
   • shear rate is variable

2. Pressure-driven flow in a tube (Poiseuille flow)
   • viscosity is variable

\[ Q = f(\Delta P) \]

3. Stress tensor in shear flow
   • all 9 components are nonzero

\[ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} \]
Rheological Behavior of Fluids – non-Newtonian

1. Strain response to imposed shear stress
   - shear rate is variable
   - viscosity is variable
   - all 9 components are nonzero

2. Pressure-driven flow in a tube (Poiseuille flow)
   - viscosity is variable
   - all 9 components are nonzero

Examples from the film of . . .

Dependence on the history of the deformation gradient
- Polymer fluid pours, but springs back
- Elastic ball bounces, but flows if given enough time
- Steel ball dropped in polymer solution “bounces”
- Polymer solution in concentric cylinders – has fading memory
- Quantitative measurements in concentric cylinders show memory and need a finite time to come to steady state

Non-linearity of the function $I = f \left( \frac{\dot{Y}}{\dot{Y}_0} \right)$
- Polymer solution draining from a tube is first slower, then faster than a Newtonian fluid
- Double the static head on a draining tube, and the flow rate does not necessarily double (as it does for Newtonian fluids); sometimes more than doubles, sometimes less
- Normal stresses in shear flow
- Die swell
Chapter 2: Mathematics Review

1. Vector review
2. Einstein notation
3. Tensors

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Motivation: We will be solving the momentum balance:

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}$$

Newtonian fluids:
- Linear
- Instantaneous
- $$\mathbf{\tau}(t) = -\mu \mathbf{\dot{v}}(t)$$

Non-Newtonian fluids:
- Non-linear
- Non-instantaneous
- $$\mathbf{\tau}(t) = ?$$

We’re going to be trying to identify the constitutive equation $$\mathbf{\tau}(t)$$ for non-Newtonian fluids.
Motivation: We will be solving the momentum balance:

$$\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{g} + \rho g$$

Newtonian fluids:
- Linear
- Instantaneous
  $$\mathcal{T}(t) = -\mu \dot{\mathbf{v}}(t)$$

Non-Newtonian fluids:
- Non-linear
- Non-instantaneous
  $$\mathcal{T}(t) = ?$$

We're going to need to calculate how different guesses affect the predicted behavior.

We need to understand and be able to manipulate this mathematical notation.
Chapter 2: Mathematics Review

1. Scalar – a mathematical entity that has magnitude only

   e.g.: temperature T
   speed \( v \)
   time \( t \)
   density \( r \)

   – scalars may be constant or may be variable

   **Laws of Algebra for Scalars:**
   - yes commutative \( ab = ba \)
   - yes associative \( a(bc) = (ab)c \)
   - yes distributive \( a(b+c) = ab + ac \)

2. Vector – a mathematical entity that has magnitude and direction

   e.g.: force on a surface \( f \)
   velocity \( \dot{v} \)

   – vectors may be constant or may be variable

   **Definitions**
   - magnitude of a vector – a scalar associated with a vector
     \[ |\mathbf{v}| = v \]
     \[ |\mathbf{f}| = f \]
   - unit vector – a vector of unit length
     \[ \frac{\mathbf{v}}{|\mathbf{v}|} = \hat{\mathbf{v}} \]
     a unit vector in the direction of \( \mathbf{v} \)
Laws of Algebra for Vectors:

1. Addition

\[ \mathbf{a} + \mathbf{b} \]

2. Subtraction

\[ \mathbf{a} + (-\mathbf{b}) \]

Laws of Algebra for Vectors (continued):

3. Multiplication by scalar \( \mathbf{v} \)

- commutative: \( \alpha \mathbf{v} = \mathbf{v} \alpha \)
- associative: \( \alpha (\beta \mathbf{v}) = (\alpha \beta) \mathbf{v} = \alpha \beta \mathbf{v} \)
- distributive: \( \alpha (\mathbf{v} + \mathbf{w}) = \alpha \mathbf{v} + \alpha \mathbf{w} \)

4. Multiplication of vector by vector

4a. scalar (dot) (inner) product

\[ \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \mathbf{w} \cos \theta \]

Note: we can find magnitude with dot product

\[ \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \mathbf{v} \cos 0 = \mathbf{v}^2 \]

\[ \mathbf{v} = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \]
Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con’t)

- yes commutative \( v \cdot w = w \cdot v \)
- NO associative \( v \cdot (w + z) \neq (v \cdot w) + (v \cdot z) \)
- yes distributive \( z \cdot (v + w) = z \cdot v + z \cdot w \)

4b. vector (cross) (outer) product

\( v \times w = v w \sin \theta \hat{e} \)

\( \hat{e} \) is a unit vector perpendicular to both \( v \) and \( w \) following the right-hand rule.

Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con’t)

- NO commutative \( v \times w \neq w \times v \)
- NO associative \( v \times w \times z \neq (v \times w) \times z \neq v \times (w \times z) \)
- yes distributive \( z \times (v + w) = (z \times v) + (z \times w) \)
Coordinate Systems

- Allow us to make actual calculations with vectors

Rule: any three vectors that are non-zero and linearly independent (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if $a$, $b$, and $g$ can be found such that:

$$\alpha a + \beta b + \gamma c = 0$$

for $\alpha, \beta, \gamma \neq 0$

If $\alpha$, $\beta$, and $\gamma$ are found to be zero, the vectors are linearly independent.

---

How can we do actual calculations with vectors?

Rule: any vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

$$\vec{a} = a_x \hat{\mathbf{e}}_x + a_y \hat{\mathbf{e}}_y + a_z \hat{\mathbf{e}}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{\text{xyz}}$$

$$= a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3$$

$$= \sum_{j=1}^{3} a_j \hat{\mathbf{e}}_j$$
Trial calculation: dot product of two vectors

\[ a \cdot b = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \]

\[ = a_1 \hat{e}_1 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) + \]

\[ a_2 \hat{e}_2 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) + \]

\[ a_3 \hat{e}_3 \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \]

\[ = a_1 \hat{e}_1 \cdot b_1 \hat{e}_1 + a_1 \hat{e}_1 \cdot b_2 \hat{e}_2 + a_1 \hat{e}_1 \cdot b_3 \hat{e}_3 + \]

\[ a_2 \hat{e}_2 \cdot b_1 \hat{e}_1 + a_2 \hat{e}_2 \cdot b_2 \hat{e}_2 + a_2 \hat{e}_2 \cdot b_3 \hat{e}_3 + \]

\[ a_3 \hat{e}_3 \cdot b_1 \hat{e}_1 + a_3 \hat{e}_3 \cdot b_2 \hat{e}_2 + a_3 \hat{e}_3 \cdot b_3 \hat{e}_3 \]

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

If we choose the basis to be orthonormal - mutually perpendicular and of unit length, then we can simplify.

\[ \hat{e}_1 \cdot \hat{e}_1 = 1 \]
\[ \hat{e}_1 \cdot \hat{e}_2 = 0 \]
\[ \hat{e}_1 \cdot \hat{e}_3 = 0 \]
\[ \ldots \]

\[ a \cdot b = a_1 \hat{e}_1 \cdot b_1 \hat{e}_1 + a_1 \hat{e}_1 \cdot b_2 \hat{e}_2 + a_1 \hat{e}_1 \cdot b_3 \hat{e}_3 + \]
\[ a_2 \hat{e}_2 \cdot b_1 \hat{e}_1 + a_2 \hat{e}_2 \cdot b_2 \hat{e}_2 + a_2 \hat{e}_2 \cdot b_3 \hat{e}_3 + \]
\[ a_3 \hat{e}_3 \cdot b_1 \hat{e}_1 + a_3 \hat{e}_3 \cdot b_2 \hat{e}_2 + a_3 \hat{e}_3 \cdot b_3 \hat{e}_3 \]

\[ = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

We can generalize this operation with a technique called Einstein notation.
Einstein Notation

A system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

\[ \mathbf{a} = a_1 \mathbf{\hat{e}}_1 + a_2 \mathbf{\hat{e}}_2 + a_3 \mathbf{\hat{e}}_3 \]

\[ = \sum_{j=1}^{3} a_j \mathbf{\hat{e}}_j \]

\[ = a_j \mathbf{\hat{e}}_j = a_m \mathbf{\hat{e}}_m \]

- The initial choice of subscript letter is arbitrary.
- The presence of a pair of like subscripts implies a missing summation sign.

Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function.

\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_1 = 1 \]
\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_2 = 0 \]
\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_3 = 0 \]
\[ \cdots \]

\[ \mathbf{\hat{e}}_i \cdot \mathbf{\hat{e}}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \]

Kronecker delta
Einstein Notation (con’t)

To carry out a dot product of two arbitrary vectors . . .

**Detailed Notation**

\[ \mathbf{a} \cdot \mathbf{b} = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \]

\[ = a_1 b_1 \hat{e}_1 \cdot \hat{e}_1 + a_1 b_2 \hat{e}_1 \cdot \hat{e}_2 + a_1 b_3 \hat{e}_1 \cdot \hat{e}_3 + \]

\[ + a_2 b_1 \hat{e}_2 \cdot \hat{e}_1 + a_2 b_2 \hat{e}_2 \cdot \hat{e}_2 + a_2 b_3 \hat{e}_2 \cdot \hat{e}_3 + \]

\[ + a_3 b_1 \hat{e}_3 \cdot \hat{e}_1 + a_3 b_2 \hat{e}_3 \cdot \hat{e}_2 + a_3 b_3 \hat{e}_3 \cdot \hat{e}_3 \]

\[ = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

**Einstein Notation**

\[ \mathbf{a} \cdot \mathbf{b} = a_j \hat{e}_j \cdot b_m \hat{e}_m \]

\[ \Rightarrow a \mathbf{b} = a_j \delta_{jm} b_m \]

\[ = a \mathbf{b}_j \]

---

3. Tensor – the indeterminate vector product of two (or more) vectors

**e.g.:** stress \( \mathbf{T} \)  
velocity gradient \( \mathbf{\dot{g}} \)

- tensors may be **constant** or may be **variable**

**Definitions**

- dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

\[ \mathbf{a} \cdot \mathbf{d} \quad \text{dyad} \]

\[ \mathbf{A} \quad \text{general representation of a tensor} \]
Laws of Algebra for Indeterminate Product of Vectors:

- **NO** commutative
  \[ a \cdot v \neq v \cdot a \]
- **yes** associative
  \[ b \cdot (a \cdot v) = (b \cdot a) \cdot v = b \cdot a \cdot v \]
- **yes** distributive
  \[ a \cdot (v + w) = a \cdot v + a \cdot w \]

How can we represent tensors with respect to a chosen coordinate system?  

**Just follow the rules of tensor algebra**

\[
a \cdot m = (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3)(m_1 \hat{e}_1 + m_2 \hat{e}_2 + m_3 \hat{e}_3)
= a_1 \hat{e}_1 m_1 \hat{e}_1 + a_1 \hat{e}_1 m_2 \hat{e}_2 + a_1 \hat{e}_1 m_3 \hat{e}_3 +
\]

\[
a_2 \hat{e}_2 m_1 \hat{e}_1 + a_2 \hat{e}_2 m_2 \hat{e}_2 + a_2 \hat{e}_2 m_3 \hat{e}_3 +
\]

\[
a_3 \hat{e}_3 m_1 \hat{e}_1 + a_3 \hat{e}_3 m_2 \hat{e}_2 + a_3 \hat{e}_3 m_3 \hat{e}_3
= \sum_{k=1}^{3} \sum_{w=1}^{3} a_k \hat{e}_k m_w \hat{e}_w
\]

**Any tensor may be written as the sum of 9 dyadic products of basis vectors**
What about $\mathbf{A}$?  

$$\mathbf{A} = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \hat{e}_i \hat{e}_j$$

Einstein notation for tensors: *drop the summation sign; every double index implies a summation sign has been dropped.*

$$\mathbf{A} = A_{ij} \hat{e}_i \hat{e}_j = A_{pk} \hat{e}_p \hat{e}_k$$

*Reminder: the initial choice of subscript letters is arbitrary*

---

How can we use Einstein Notation to calculate dot products between vectors and tensors?

It's the same as between vectors.

$$\mathbf{a} \cdot \mathbf{b} =$$

$$\mathbf{a} \cdot \mathbf{u} \mathbf{v} =$$

$$\mathbf{b} \cdot \mathbf{A} =$$
Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index
2. Drop the summation signs
3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:
• In Einstein notation, the presence of repeated indices implies a missing summation sign
• The choice of initial index (i, m, p, etc.) is arbitrary - it merely indicates which indices change together

3. Tensor – (continued)

Definitions

Scalar product of two tensors

\[ A \cdot M = A_{ip} \hat{e}_i \cdot \hat{e}_p : M_{km} \hat{e}_k \cdot \hat{e}_m \]

\[ = A_{ip} M_{km} \delta_{ik} \delta_{pm} \]

\[ = A_{ip} M_{km} \delta_{ik} \delta_{jm} \]

\[ = A_{mk} M_{km} \]

“p” becomes “k”
“i” becomes “m”
But, what is a tensor really?

A tensor is a handy representation of a *Linear Vector Function*

**Scalar function:** \( y = f(x) = x^2 + 2x + 3 \)
- a mapping of values of \( x \) onto values of \( y \)

**Vector function:** \( \mathbf{w} = f(\mathbf{v}) \)
- a mapping of vectors of \( \mathbf{v} \) into vectors \( \mathbf{w} \)

How do we express a vector function?

What is a linear function?

*Linear, in this usage, has a precise, mathematical definition.*

Linear functions (scalar and vector) have the following two properties:

\[
\begin{align*}
    f(\lambda x) &= \lambda f(x) \\
    f(x + \mathbf{w}) &= f(x) + f(\mathbf{w})
\end{align*}
\]

It turns out . . .
Tensors are **Linear Vector Functions**

Let \( f(\mathbf{a}) = \mathbf{b} \) be a linear vector function.

We can write \( \mathbf{a} \) in Cartesian coordinates.

\[
\mathbf{a} = a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3
\]

\[
f(\mathbf{a}) = f(a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3) = \mathbf{b}
\]

Using the linear properties of \( f \), we can distribute the function action:

\[
f(\mathbf{a}) = a_1 f(\hat{\mathbf{e}}_1) + a_2 f(\hat{\mathbf{e}}_2) + a_3 f(\hat{\mathbf{e}}_3) = \mathbf{b}
\]

These results are just vectors, we will name them \( \mathbf{v} \), \( \mathbf{w} \), and \( \mathbf{m} \).

---

Tensors are **Linear Vector Functions** (continued)

\[
f(\mathbf{a}) = a_1 f(\hat{\mathbf{e}}_1) + a_2 f(\hat{\mathbf{e}}_2) + a_3 f(\hat{\mathbf{e}}_3) = \mathbf{b}
\]

\[
\mathbf{v} \quad \mathbf{w} \quad \mathbf{m}
\]

\[
f(\mathbf{a}) = a_1 \mathbf{v} + a_2 \mathbf{w} + a_3 \mathbf{m} = \mathbf{b}
\]

Now we note that the coefficients \( a_i \) may be written as,

\[
a_1 = \mathbf{a} \cdot \hat{\mathbf{e}}_1 \quad a_2 = \mathbf{a} \cdot \hat{\mathbf{e}}_2 \quad a_3 = \mathbf{a} \cdot \hat{\mathbf{e}}_3
\]

Substituting,

\[
f(\mathbf{a}) = \mathbf{a} \cdot \hat{\mathbf{e}}_1 \mathbf{v} + \mathbf{a} \cdot \hat{\mathbf{e}}_2 \mathbf{w} + \mathbf{a} \cdot \hat{\mathbf{e}}_3 \mathbf{m} = \mathbf{b}
\]

The indeterminate vector product has appeared!
Using the distributive law, we can factor out the dot product with \( \mathbf{a} \):

\[
f(\mathbf{a}) = \mathbf{a} \cdot (\hat{e}_1 \mathbf{v} + \hat{e}_2 \mathbf{w} + \hat{e}_3 \mathbf{m}) = \mathbf{b}
\]

This is just a tensor (the sum of dyadic products of vectors):

\[
(\hat{e}_1 \mathbf{v} + \hat{e}_2 \mathbf{w} + \hat{e}_3 \mathbf{m}) = \mathbf{M}
\]

\[
f(\mathbf{a}) = \mathbf{a} \cdot \mathbf{M} = \mathbf{b}
\]

**CONCLUSION:** Tensor operations are convenient to use to express linear vector functions.

---

3. Tensor – (continued)

**More Definitions**

**Identity Tensor**

\[
\mathbf{I} = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\mathbf{A} \cdot \mathbf{I} = A_{\mathbf{p} \mathbf{k}} \hat{e}_\mathbf{p} \hat{e}_k
\]

\[
= A_{\mathbf{p} \mathbf{k}} \delta_{\mathbf{p} \mathbf{k}} \hat{e}_\mathbf{k}
\]

\[
= A_{\mathbf{k} \mathbf{k}} \hat{e}_\mathbf{k}
\]

\[
= \mathbf{A}
\]
### Mathematics Review

#### 3. Tensor – (continued)

#### More Definitions

<table>
<thead>
<tr>
<th>Zero Tensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \mathbf{0} = \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \end{pmatrix} ]</td>
</tr>
</tbody>
</table>

**Magnitude of a Tensor**

\[ |\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A} / 2} \]

Where \( \mathbf{A} : \mathbf{A} = A_{ip} \hat{e}_p \cdot A_{km} \hat{e}_m \)

\[ = A_{ip} A_{km} (\hat{e}_p \cdot \hat{e}_i) (\hat{e}_m \cdot \hat{e}_n) \]

\[ = A_{mk} A_{km} \]

Note that the book has a typo on this equation: the "2" is under the square root.

#### Tensor Transpose

\[ \mathbf{M}^T = (M_{ik} \hat{\mathbf{e}}_k) = M_{ik} \hat{\mathbf{e}}_k \]

*Exchange the coefficients across the diagonal*

**CAUTION:**

\[ (\mathbf{A} \cdot \mathbf{C})^T = (A_{ik} \hat{\mathbf{e}}_k \cdot C_{pj} \hat{\mathbf{e}}_j)^T = (A_{ik} C_{pj} \hat{\mathbf{e}}_i \delta_{kp})^T \]

\[ = (A_{ip} C_{pj} \hat{\mathbf{e}}_j)^T \]

\[ = A_{ip} C_{pj} \hat{\mathbf{e}}_j \]

It is **not** equal to: \( \mathbf{A} \cdot \mathbf{C} \neq A_{ip} C_{pj} \hat{\mathbf{e}}_j \)

I recommend you always interchange the indices on the basis vectors rather than on the coefficients.
3. Tensor – (continued) 

More Definitions

Symmetric Tensor

\[ \begin{align*}
M &= M^T \\
M_{ik} &= M_{ki}
\end{align*} \]

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{pmatrix}_{123}
\]

Antisymmetric Tensor

\[ \begin{align*}
M &= -M^T \\
M_{ik} &= -M_{ki}
\end{align*} \]

\[
\begin{pmatrix}
0 & -2 & -3 \\
2 & 0 & -5 \\
3 & 5 & 0
\end{pmatrix}_{123}
\]

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

\[
\begin{array}{ccc}
\text{order} & \text{0th-order tensors} & \text{3^0} \\
\text{vectors} & \text{1st-order tensors} & \text{3^1} \\
\text{tensors} & \text{2nd-order tensors} & \text{3^2} \\
\text{higher-order tensors} & \text{3rd-order tensors} & \text{3^3}
\end{array}
\]

Number of coefficients needed to express the tensor in 3D space
3. Tensor – (continued)

More Definitions

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

Vectors: \[ \|v\| = v \]

The magnitude of a vector is a scalar associated with the vector.

It is independent of coordinate system, i.e., it is an invariant.

Tensors: \[ A \]

There are three invariants associated with a second-order tensor.

For the tensor written in Cartesian coordinates:

\[ I_4 \equiv \text{trace} A = tr A \]

\[ II_4 \equiv \text{trace}(A \cdot A) = \dot{A} : A = A_{pk} A_{kp} \]

\[ III_4 \equiv \text{trace}(A \cdot A \cdot A) = A_{pj} A_{jh} A_{hp} \]

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.
4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

• To carry out the differentiation with respect to a single variable, differentiate each coefficient individually.
• There is no change in order (vectors remain vectors, scalars remain scalars, etc.

\[
\frac{\partial \alpha}{\partial t} = \left( \frac{\partial W_1}{\partial t} \right)_{123} \quad \frac{\partial W}{\partial t} = \left( \frac{\partial W_1}{\partial t} \right)_{123} \quad \frac{\partial B}{\partial t} = \left( \frac{\partial B_{11}}{\partial t} \right)_{123}
\]

• To carry out the differentiation with respect to 3D spatial variation, use the del (nabla) operator.
• This is a vector operator
• Del may be applied in three different ways
• Del may operate on scalars, vectors, or tensors

\[
\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \left( \frac{\partial}{\partial x_i} \right)_{123}
\]

This is written in Cartesian coordinates

\[
\nabla = \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \hat{e}_p \frac{\partial}{\partial x_p}
\]

Einstein notation for del
4. Differential Operations with Vectors, Tensors (continued)

A. Scalars - gradient

\[ \nabla \beta = \hat{e}_1 \frac{\partial \beta}{\partial x_1} + \hat{e}_2 \frac{\partial \beta}{\partial x_2} + \hat{e}_3 \frac{\partial \beta}{\partial x_3} = \begin{pmatrix} \frac{\partial \beta}{\partial x_1} \\ \frac{\partial \beta}{\partial x_2} \\ \frac{\partial \beta}{\partial x_3} \end{pmatrix} \]

This is written in Cartesian coordinates.

The gradient of a scalar field is a vector.

*Gradient operation increases the order of the entity operated upon.

B. Vectors - gradient

\[ \nabla \mathbf{w} = \hat{e}_1 \frac{\partial \mathbf{w}}{\partial x_1} + \hat{e}_2 \frac{\partial \mathbf{w}}{\partial x_2} + \hat{e}_3 \frac{\partial \mathbf{w}}{\partial x_3} = \begin{pmatrix} \frac{\partial w_1}{\partial x_1} \\ \frac{\partial w_2}{\partial x_2} \\ \frac{\partial w_3}{\partial x_3} \end{pmatrix} \]

This is all written in Cartesian coordinates (basis vectors are constant).

The basis vectors can move out of the derivatives because they are constant (do not change with position).
## 4. Differential Operations with Vectors, Tensors (continued)

### B. Vectors - gradient (continued)

The gradient of a vector field is a tensor.

\[
\nabla \mathbf{w} = \sum_{j=1}^{3} \sum_{k=1}^{3} \hat{e}_j \frac{\partial w_k}{\partial x_j} = \hat{e}_j \frac{\partial w_k}{\partial x_j}
\]

Einstein notation for gradient of a vector.

### C. Vectors - divergence

The divergence of a vector field is a scalar.

\[
\nabla \cdot \mathbf{w} = \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) \cdot w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3
\]

\[
= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}
\]

\[
= \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i}
\]

Einstein notation for divergence of a vector.
4. Differential Operations with Vectors, Tensors (continued)

C. Vectors - divergence (continued)

Using Einstein notation

\[ \nabla \cdot \mathbf{w} = \sum_{m} \frac{\partial}{\partial x_{m}} w_{j} \hat{e}_{j} = \frac{\partial w_{j}}{\partial x_{m}} \hat{e}_{j} = \frac{\partial w_{j}}{\partial x_{m}} \delta_{mj} \]

This is all written in Cartesian coordinates (basis vectors are constant)

• divergence operation decreases the order of the entity operated upon

D. Vectors - Laplacian

Using Einstein notation:

\[ \nabla \cdot \nabla \mathbf{w} \equiv \sum_{m} \frac{\partial}{\partial x_{m}} \left( \sum_{p} \frac{\partial}{\partial x_{p}} w_{j} \hat{e}_{j} \right) = \sum_{m} \frac{\partial}{\partial x_{m}} \left( \sum_{p} \frac{\partial}{\partial x_{p}} w_{j} \right) \hat{e}_{j} \]

\[ = \sum_{m} \frac{\partial}{\partial x_{m}} \left( \sum_{p} \frac{\partial}{\partial x_{p}} w_{j} \right) \hat{e}_{j} \]

The Laplacian of a vector field is a vector

\[ = \left( \frac{\partial^2 w_{1}}{\partial x_1^2} + \frac{\partial^2 w_{1}}{\partial x_2^2} + \frac{\partial^2 w_{1}}{\partial x_3^2} \right) \]

\[ \frac{\partial^2 w_{2}}{\partial x_1^2} + \frac{\partial^2 w_{2}}{\partial x_2^2} + \frac{\partial^2 w_{2}}{\partial x_3^2} \]

\[ \frac{\partial^2 w_{3}}{\partial x_1^2} + \frac{\partial^2 w_{3}}{\partial x_2^2} + \frac{\partial^2 w_{3}}{\partial x_3^2} \]

• Laplacian operation does not change the order of the entity operated upon

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4. Differential Operations with Vectors, Tensors (continued)

E. Scalar - divergence
\[ \nabla \cdot \alpha \]

F. Scalar - Laplacian
\[ \nabla \cdot \nabla \alpha \]

G. Tensor - gradient
\[ \nabla A \]

H. Tensor - divergence
\[ \nabla \cdot A \]

I. Tensor - Laplacian
\[ \nabla \cdot \nabla A \]

(Impossible; cannot decrease order of a scalar)

5. Curvilinear Coordinates

Cylindrical
\[ r, \theta, z \]
\[ \hat{e}_r, \hat{e}_\theta, \hat{e}_z \]

Spherical
\[ r, \theta, \phi \]
\[ \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \]

See figures 2.11 and 2.12

These coordinate systems are ortho-normal, but they are not constant (they vary with position).

This causes some non-intuitive effects when derivatives are taken.
First, we need to write this in cylindrical coordinates. Solve for Cartesian basis vectors and substitute above using chain rule (see next slide for details).

\[ \hat{e}_r = \cos \theta \hat{e}_x + \sin \theta \hat{e}_y \]
\[ \hat{e}_\theta = -\sin \theta \hat{e}_x + \cos \theta \hat{e}_y \]
\[ \hat{e}_z = \hat{e}_z \]

\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
\[ z = z \]

\[ \nabla \cdot \mathbf{v} = \nabla \cdot \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) \]
\[ = \left( \frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \right) \cdot \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) \]
Result: 
\[ \nabla = \left( \frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \right) \]
\[ \nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \]

Now, proceed:
\[ \nabla \cdot \mathbf{v} = \left( \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \]
\[ = \hat{e}_r \frac{\partial}{\partial r} \cdot (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) + \]
\[ \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \cdot (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) + \]
\[ \hat{e}_z \frac{\partial}{\partial z} \cdot (v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z) \]

(We cannot use Einstein notation because these are not Cartesian coordinates)
5. Curvilinear Coordinates (continued)

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position.

Final result for divergence of a vector in cylindrical coordinates:

\[
\nabla \cdot \mathbf{v} = \hat{\mathbf{e}}_r \frac{1}{r} \frac{\partial}{\partial r} \left( v_r \hat{\mathbf{e}}_r + v_{\theta} \hat{\mathbf{e}}_{\theta} + v_z \hat{\mathbf{e}}_z \right) + \\
\hat{\mathbf{e}}_{\theta} \frac{\partial}{\partial \theta} \left( v_r \hat{\mathbf{e}}_r + v_{\theta} \hat{\mathbf{e}}_{\theta} + v_z \hat{\mathbf{e}}_z \right) + \\
\hat{\mathbf{e}}_z \frac{\partial}{\partial z} \left( v_r \hat{\mathbf{e}}_r + v_{\theta} \hat{\mathbf{e}}_{\theta} + v_z \hat{\mathbf{e}}_z \right)
\]

\[
\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{1}{r} v_r + \frac{\partial v_z}{\partial z}
\]
5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

• The basis vectors are ortho-normal
• The basis vectors are non-constant (vary with position)
• These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
• *We cannot use Einstein notation* – must use Tables in Appendix C2 (pp464-468).

6. Vector and Tensor Theorems and definitions

In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

**Gauss Divergence Theorem**

\[ \iiint_V \nabla \cdot \mathbf{b} \, dV = \iiint_S \hat{n} \cdot \mathbf{b} \, dS \]

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.
6. Vector and Tensor Theorems (continued)

**Leibnitz Rule** for differentiating integrals

\[ I = \int_{\alpha}^{\beta} f(x,t) \, dx \]

\[ \frac{dI}{dt} = \frac{d}{dt} \int_{\alpha}^{\beta} f(x,t) \, dx = \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t} \, dx \]
6. Vector and Tensor Theorems (continued)

Leibnitz Rule for differentiating integrals

\[
J = \int_{\alpha(t)}^{\beta(t)} f(x, t) \, dx
\]

\[
\frac{dJ}{dt} = \frac{d}{dt} \int_{\alpha(t)}^{\beta(t)} f(x, t) \, dx
\]

\[
= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x, t)}{\partial t} \, dx + \frac{d\beta}{dt} f(\beta, t) - \frac{d\alpha}{dt} f(\alpha, t)
\]

for differentiating integrals with variable limits.

For one dimension, variable limits:

For three dimensions, variable limits:

Velocity of the surface element \(dS\).
Substantial Derivative

Consider a function \( f(x, y, z, t) \)

true for any path:

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial t}
\]

choose special path:

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} + \frac{\partial f}{\partial t}
\]

time rate of change of \( f \) along a chosen path

x-component of velocity along that path

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

\[
\frac{df}{dt} = \frac{\partial f}{\partial x} \mathbf{v}_x + \frac{\partial f}{\partial y} \mathbf{v}_y + \frac{\partial f}{\partial z} \mathbf{v}_z + \frac{\partial f}{\partial t}
\]

\[
\left( \frac{df}{dt} \right)_{\text{along path}} = \mathbf{v} \cdot \nabla f
\]

\[
\frac{df}{dt} = \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f
\]
Done with math background.

Chapter 3: Newtonian Fluids

Navier-Stokes Equation

\[ \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \right) = -\nabla p + \mu \nabla^2 \vec{v} + \rho \vec{g} \]
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

• Derive governing equations (mass and momentum balances)
• Solve governing equations for velocity and stress fields

QUICK START

First, before we get deep into derivation, let's do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.

EXAMPLE: Drag flow between infinite parallel plates

• Newtonian
• Steady state
• Incompressible fluid
• Very wide, long
• Uniform pressure
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

• Derive governing equations (mass and momentum balances)
• Solve governing equations for velocity and stress fields

Mass Balance

Consider an arbitrary control volume $V$ enclosed by a surface $S$

\[
\left( \text{rate of increase of mass in } CV \right) = \left( \text{net flux of mass into } CV \right)
\]
Consider an arbitrary volume $V$ enclosed by a surface $S$.

\[ \left( \text{rate of increase of mass in } V \right) = \frac{d}{dt} \left( \iiint_V \rho \, dV \right) \]

\[ \left( \text{net flux of mass into } V \text{ through surface } S \right) = -\oint_S \rho \hat{n} \cdot \mathbf{v} \, dS \]

The Leibnitz rule:

\[ \frac{d}{dt} \left( \iiint_V \rho \, dV \right) = -\oint_S \rho \hat{n} \cdot \mathbf{v} \, dS \]

\[ \iiint_V \frac{\partial \rho}{\partial t} \, dV = -\oint_S \hat{n} \cdot (\rho \mathbf{v}) \, dS \]

\[ = -\iiint_V \nabla \cdot (\rho \mathbf{v}) \, dV \]

\[ \iiint_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \, dV = 0 \]
Chapter 3: Newtonian Fluid Mechanics

Mass Balance (continued)

Since $V$ is arbitrary,

\[
\iiint_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \, dV = 0
\]

Continuity equation: microscopic mass balance

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Chapter 3: Newtonian Fluid Mechanics

Mass Balance (continued)

Continuity equation (general fluids)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0
\]

\[
\frac{D\rho}{Dt} + \rho (\nabla \cdot \mathbf{v}) = 0
\]

For $\rho=$constant (incompressible fluids):

\[
\nabla \cdot \mathbf{v} = 0
\]
Momentum Balance

Momentum is conserved.

\[
\left( \frac{\text{rate of increase}}{\text{of momentum in } CV} \right) = \left( \text{net flux of momentum into } CV \right) + \left( \text{sum of forces on } CV \right)
\]

- Rate term resembles the rate term in the mass balance.
- Flux term resembles the flux term in the mass balance.
- Forces:
  - Body (gravity)
  - Molecular forces

\[ \text{Consider an arbitrary control volume } V \text{ enclosed by a surface } S \]
Momentum Balance (continued)

\[ \left( \text{rate of increase of momentum in } V \right) = \frac{d}{dt} \left( \iiint_V \rho v \, dV \right) \]
\[ = \iiint_V \frac{\partial}{\partial t} (\rho v) \, dV \]

\[ \text{(net flux of momentum into } V) = -\oiint_S \hat{n} \cdot (\rho v v) \, dS \]
\[ = -\iiint_V \nabla \cdot (\rho v v) \, dV \]

Forces on \( V \)

Body Forces (non-contact)

\[ \left( \text{force on } V \right) \text{ due to } g = \iiint_V \rho g \, dV \]
Molecular Forces (contact) – this is the tough one

\[ f = \frac{\text{stress at } P}{\text{on } dS} \]

We need an expression for the state of stress at an arbitrary point \( P \) in a flow.

Molecular Forces (continued)

Think back to the molecular picture from chemistry:

The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.
Molecular Forces (continued)

- We will concentrate on expressing the molecular forces mathematically;
- We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:
- arbitrary shape
- small

\[
\text{stress at } P \quad \text{on } dS \\
\Rightarrow dS = \frac{f}{\hat{n}}
\]

What is \( f \)?

Consider the forces on three mutually perpendicular surfaces through point \( P \):

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Molecular Forces (continued)

\[ a = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3 \]
\[ \pi_{11}\hat{e}_1 + \pi_{12}\hat{e}_2 + \pi_{13}\hat{e}_3 \]

\[ b = b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3 \]
\[ \pi_{21}\hat{e}_1 + \pi_{22}\hat{e}_2 + \pi_{23}\hat{e}_3 \]

\[ c = c_1\hat{e}_1 + c_2\hat{e}_2 + c_3\hat{e}_3 \]
\[ \pi_{31}\hat{e}_1 + \pi_{32}\hat{e}_2 + \pi_{33}\hat{e}_3 \]

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.

\[ \pi_{pk} \]

Stress on a “p” surface in the \( k \)-direction
How can we write \( f \) (the force on an arbitrary surface \( dS \)) in terms of the \( \Pi_{pk} \)?

\[
f = f_1 \mathbf{\hat{e}_1} + f_2 \mathbf{\hat{e}_2} + f_3 \mathbf{\hat{e}_3}
\]

\( f_1 \) is force on \( dS \) in 1-direction

\( f_2 \) is force on \( dS \) in 2-direction

\( f_3 \) is force on \( dS \) in 3-direction

There are three \( \Pi_{pk} \) that relate to forces in the 1-direction:

\( \Pi_{11}, \Pi_{21}, \Pi_{31} \)

---

How can we write \( f \) (the force on an arbitrary surface \( dS \)) in terms of the quantities \( \Pi_{pk} \)?

\( f_1 \), the force on \( dS \) in 1-direction, can be broken into three parts associated with the three stress components:

\( \Pi_{11}, \Pi_{21}, \Pi_{31} \)

first part:

\[
\left( \Pi_{11} \right) \left( \text{projection of force onto the surface} \right) = \Pi_{11} \mathbf{\hat{n}} \cdot \mathbf{\hat{e}_1} dS
\]

\[
\left( \frac{\text{force}}{\text{area}} \right) \cdot (\text{area})
\]
Molecular Forces (continued)

\( f_1 \), the force on \( dS \) in 1-direction, is composed of THREE parts:

- **first part:**
  \[
  \left( \Pi_{11} \right)
  \left( \text{projection of } \Pi \text{ onto the } \text{1-surface} \right)
  = \Pi_{11} \hat{n} \cdot \hat{e}_1 \, dS
  
  \]

- **second part:**
  \[
  \left( \Pi_{21} \right)
  \left( \text{projection of } \Pi \text{ onto the } \text{2-surface} \right)
  = \Pi_{21} \hat{n} \cdot \hat{e}_2 \, dS
  
  \]

- **third part:**
  \[
  \left( \Pi_{31} \right)
  \left( \text{projection of } \Pi \text{ onto the } \text{3-surface} \right)
  = \Pi_{31} \hat{n} \cdot \hat{e}_3 \, dS
  
  \]

The sum of these three = \( f_1 \)

\( \hat{n} \) is the normal to the curved surface.
Molecular Forces (continued)

\( f_1 \), the force in the 1-direction on an arbitrary surface \( dS \) is composed of THREE parts.

\[
f_1 = \Pi_1 \hat{n} \cdot \hat{e}_1 \ dS + \Pi_2 \hat{n} \cdot \hat{e}_2 \ dS + \Pi_3 \hat{n} \cdot \hat{e}_3 \ dS
\]

Using the distributive law:

\[
f_1 = \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \ dS
\]

The same logic applies in the 2-direction and the 3-direction

\[
\begin{align*}
f_2 &= \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \ dS \\
f_3 &= \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \ dS
\end{align*}
\]

Assembling the force vector:

\[
\mathbf{f} = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3 \\
= dS \ \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \hat{e}_1 \\
+ dS \ \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \hat{e}_2 \\
+ dS \ \hat{n} \cdot (\Pi_1 \hat{e}_1 + \Pi_2 \hat{e}_2 + \Pi_3 \hat{e}_3) \hat{e}_3
\]
Molecular Forces  (continued)

Assembling the force vector:

\[ \mathbf{f} = f_1 \mathbf{\hat{e}_1} + f_2 \mathbf{\hat{e}_2} + f_3 \mathbf{\hat{e}_3} \]

\[ = dS \: \hat{n} \cdot \left( \Pi_{11} \mathbf{\hat{e}_1} + \Pi_{21} \mathbf{\hat{e}_2} + \Pi_{31} \mathbf{\hat{e}_3} \right) \hat{e}_1 \]
\[ + dS \: \hat{n} \cdot \left( \Pi_{12} \mathbf{\hat{e}_1} + \Pi_{22} \mathbf{\hat{e}_2} + \Pi_{32} \mathbf{\hat{e}_3} \right) \hat{e}_2 \]
\[ + dS \: \hat{n} \cdot \left( \Pi_{13} \mathbf{\hat{e}_1} + \Pi_{23} \mathbf{\hat{e}_2} + \Pi_{33} \mathbf{\hat{e}_3} \right) \hat{e}_3 \]

\[ = dS \: \hat{n} \cdot \left[ \Pi_{11} \mathbf{\hat{e}_1} \mathbf{\hat{e}_1} + \Pi_{21} \mathbf{\hat{e}_2} \mathbf{\hat{e}_1} + \Pi_{31} \mathbf{\hat{e}_3} \mathbf{\hat{e}_1} \right.\]
\[ + \left. \Pi_{12} \mathbf{\hat{e}_1} \mathbf{\hat{e}_2} + \Pi_{22} \mathbf{\hat{e}_2} \mathbf{\hat{e}_2} + \Pi_{32} \mathbf{\hat{e}_3} \mathbf{\hat{e}_2} \right.\]
\[ + \left. \Pi_{13} \mathbf{\hat{e}_1} \mathbf{\hat{e}_3} + \Pi_{23} \mathbf{\hat{e}_2} \mathbf{\hat{e}_3} + \Pi_{33} \mathbf{\hat{e}_3} \mathbf{\hat{e}_3} \right] \]

linear combination of dyadic products = tensor

Total stress tensor (molecular stresses)
Momentum Balance (continued)

\[
\left( \frac{\text{rate of increase}}{\text{of momentum in } V} \right) = \left( \text{net flux of momentum into } V \right) + \left( \text{sum of forces on } V \right)
\]

\[
\iiint_V \frac{\partial}{\partial t} (\rho \vec{v}) dV = -\iiint_V \nabla \cdot (\rho \vec{v}) dV + \iiint_V \rho g dV + \text{molecular forces}
\]

\[
\text{molecular forces} = \oiint_S (\text{molecular forces on } dS)
\]

\[
= \oiint_S n \cdot (-\Pi) dS
\]

\[
= \iiint_V \nabla \cdot (-\Pi) dV
\]
Momentum Balance (continued)

\[ F_{\text{on surface}} = \int_S \hat{n} \cdot (-\Pi) \, dS = \int_S \hat{n} \cdot \left( \Pi_{yx} \right) \, dS \]

- UR/Bird choice: fluid at lesser \( y \) exerts force on fluid at greater \( y \)
- (IFM/Mechanics choice: (opposite))

Final Assembly:

\[
\begin{align*}
\int_V \frac{\partial}{\partial t} (\rho \mathbf{v}) \, dV &= -\iiint_V \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \, dV + \iiint_V \rho \mathbf{g} \cdot \mathbf{n} \, dV - \iiint_V \nabla \cdot \mathbf{P} \, dV \\
\iiint_V \left[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \rho \mathbf{g} \cdot \mathbf{n} + \nabla \cdot \mathbf{P} \right] \, dV &= 0
\end{align*}
\]

Because \( V \) is arbitrary, we may conclude:

\[
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \rho \mathbf{g} \cdot \mathbf{n} + \nabla \cdot \mathbf{P} = 0
\]

Microscopic momentum balance

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Momentum Balance (continued)

Microscopic momentum balance

\[ \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho vv) - \rho g + \nabla \cdot \Pi = 0 \]

After some rearrangement:

\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla \cdot \Pi + \rho g \]

\[ \frac{Dv}{Dt} = -\nabla \cdot \Pi + \rho g \]

Now, what to do with \( \Pi \)?

Pressure

*definition:* An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

\[ \text{pressure} = p \mathbb{I} = p \varepsilon_1 + p \varepsilon_2 + p \varepsilon_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{23} \]

Test: what is the force on a surface with unit normal \( \hat{n} \)?
Momentum Balance (continued)

extra molecular stresses

Definition: The extra stresses are the molecular stresses that are not isotropic

\[ \tau \equiv \Pi - p I \]

Extra stress tensor, i.e. everything complicated in molecular deformation

Now, what to do with \( \tau \)?

This becomes the central question of rheological study.

Stress sign convention affects any expressions with \( \Pi, \tilde{\Pi} \) or \( \tau, \tilde{\tau} \)

\[ \Pi \equiv \tau + p I \]

UR/Bird choice: fluid at lesser \( y \) exerts force on fluid at greater \( y \)

\[ \tilde{\Pi} \equiv \tilde{\tau} - p I \]

(IFM/Mechanics choice: (opposite))
Constitutive equations for Stress

- are tensor equations
- relate the velocity field to the stresses generated by molecular forces
- are based on observations (empirical) or are based on molecular models (theoretical)
- are typically found by trial-and-error
- are justified by how well they work for a system of interest
- are observed to be symmetric

\[ \tau = f(\nabla \psi, \text{material properties}) \]

Observation: the stress tensor is symmetric

---

Microscopic momentum balance

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \cdot \mathbf{\Pi} + \rho \mathbf{g} \]

Equation of Motion

In terms of the extra stress tensor:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \tau + \rho \mathbf{g} \]

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):

---

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Momentum Balance (continued)

Newtonian Constitutive equation

\[ \tau = -\mu \left( \nabla v + (\nabla v)^T \right) \]

- for incompressible fluids (see text for compressible fluids)
- is empirical
- may be justified for some systems with molecular modeling calculations

Note: \( \tilde{\tau} = +\mu \left( \nabla v + (\nabla v)^T \right) \)

How is the Newtonian Constitutive equation related to Newton’s Law of Viscosity?

\[ \tau = -\mu \left( \nabla v + (\nabla v)^T \right) \]
\[ \tau_{21} = -\mu \frac{\partial v_1}{\partial x_2} \]

- incompressible fluids
- incompressible fluids
- rectilinear flow (straight lines)
- no variation in \( x_3 \)-direction
Back to the momentum balance . . .

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g} \]

**Equation of Motion**

\[ \mathbf{\tau} = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids.

**Navier-Stokes Equation**

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

- incompressible fluids
- Newtonian fluids

**Note:** The Navier-Stokes is unaffected by the stress sign convention.
Navier-Stokes Equation

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g} \]

Newtonian Problem Solving

EXAMPLE: Drag flow between infinite parallel plates

- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \]
EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Infinitely wide, long

\[ x_1 = 0 \]
\[ p = P_o \]
\[ x_1 = L \]
\[ p = P_L \]

EXAMPLE: Poiseuille flow in a tube

- Newtonian
- Steady state
- Incompressible fluid
- Long tube
EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- $\nu = z f(r)$

Chapter 4: Standard Flows

Newtonian fluids:

$$\tau = -\mu \dot{\gamma}$$

VS.

non-Newtonian fluids:

$$\tau \neq -\mu \dot{\gamma}$$

How can we investigate non-Newtonian behavior?
Chapter 4: Standard Flows for Rheology

We now know how to model Newtonian fluid motion, \( \mathbf{v}(\mathbf{x}, t), p(\mathbf{x}, t) \):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Continuity equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

Cauchy momentum equation

\[
\mathbf{\tau} = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right)
\]

Newtonian constitutive equation
Rheological Behavior of Fluids – Non-Newtonian

How do we model the motion of Non-Newtonian fluid fluids?

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Continuity equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

Cauchy Momentum Equation

\[\mathbf{\tau} = f(\mathbf{x}, t)\]

Non-Newtonian constitutive equation

This is the missing piece
Chapter 4: Standard Flows for Rheology

Chapter 4: Standard flows
Chapter 5: Material Functions
Chapter 6: Experimental Data

Chapter 7: GNF
Chapter 8: GLVE
Chapter 9: Advanced

To get to constitutive equations, we must first quantify how non-Newtonian fluids behave

Constitutive equations

What do we observe?

Rheological Behavior of Fluids – Newtonian

1. Strain response to imposed shear stress
   • shear rate is constant
   \[
   \dot{\gamma} = \text{constant}
   \]

2. Pressure-driven flow in a tube (Poiseuille flow)
   • viscosity is constant
   \[
   \frac{Q}{8\mu L} = \text{constant}
   \]

3. Stress tensor in shear flow
   • only two components are nonzero
   \[
   \tau = \begin{pmatrix}
   0 & \tau_{12} & 0 \\
   \tau_{21} & 0 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \]

What do we observe?

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1. Strain response to imposed shear stress
   - shear rate is variable

2. Pressure-driven flow in a tube (Poiseuille flow)
   - viscosity is variable

3. Stress tensor in shear flow
   - all 9 components are nonzero

Non-Newtonian Constitutive Equations

- We have observations that some materials are not like Newtonian fluids.
- How can we be systematic about developing new, unknown models for these materials?

Need measurements

For Newtonian fluids, measurements were easy:
- shear flow
- one stress, $\tau_{21}$
- one material constant, $\mu$ (viscosity)

$$\tau = -\mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$$
Non-Newtonian Constitutive Equations

Need measurements

For non-Newtonian fluids, measurements are **not easy**:

- shear flow (not the only choice)
- Four stresses in shear, \( \tau_{21}, \tau_{11}, \tau_{22}, \tau_{33} \)
- **Unknown** number of material constants in \( \mathbf{T}(\mathbf{v}) \)
- **Unknown** number of material functions in \( \mathbf{T}(\mathbf{v}) \)

\[ \mathbf{T} = ??? \]
Non-Newtonian Constitutive Equations

Need measurements

For non-Newtonian fluids, measurements are not easy:

- shear flow (not the only choice)
- Four stresses in shear, $\tau_{21}, \tau_{11}, \tau_{22}, \tau_{33}$
- Unknown number of material constants in $\mathbf{I}(\mathbf{u})$
- Unknown number of material functions in $\mathbf{I}(\mathbf{u})$

$\mathbf{I} = ???$

We know we need to make measurements to know more,

But, because we do not know the functional form of $\mathbf{I}(\mathbf{u})$, we don’t know what we need to measure to know more!

Non-Newtonian Constitutive Equations

What should we do?
Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows  Chapter 4: Standard flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab

2. Make calculations
3. Make measurements  Chapter 5: Material Functions
   Chapter 6: Experimental Data
Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab

2. Make calculations
3. Make measurements
4. Try to deduce $\tau(\dot{\gamma})$

Chapter 4: Standard flows

Chapter 5: Material Functions
Chapter 6: Experimental Data
Chapter 7: GNF
Chapter 8: GLVE
Chapter 9: Advanced

Tactic: Divide the Problem in half

Modeling Calculations

- Dream up models
- Calculate model predictions for stresses in standard flows
- Calculate material functions from model stresses
- Pass judgment on models
- Collect models and their report cards for future use

Experiments

- Build experimental apparatuses that allow measurements in standard flows
- Determine material functions from measured stresses
- Compare
- Standard Flows
Standard flows – choose a velocity field (not an apparatus or a procedure)

- For model predictions, calculations are straightforward
- For experiments, design can be optimized for accuracy and fluid variety

Material functions – choose a common vocabulary of stress and kinematics to report results

- Make it easier to compare model/experiment
- Record an “inventory” of fluid behavior (expertise)

Newtonian fluids:
\[ \tau = -\mu \dot{\gamma} \]

VS.

non-Newtonian fluids:
\[ \tau \neq -\mu \dot{\gamma} \]

How can we investigate non-Newtonian behavior?
Simple Shear Flow

velocity field

\[ v_1(H) = V = \dot{\gamma}H \]
\[ \dot{\gamma}_0 = \text{constant} \]

path lines

\[ \vec{V} \equiv \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix} \]

Near solid surfaces, the flow is shear flow.
Experimental Shear Geometries

Standard Nomenclature for Shear Flow
Why is shear a standard flow?

- simple velocity field
- represents all sliding flows
- simple stress tensor

How do particles move apart in shear flow?

Consider two particles in the same $x_1$-$x_2$ plane, initially along the $x_2$ axis.
How do particles move apart in shear flow?

Consider two particles in the same $x_1$-$x_2$ plane, initially along the $x_2$ axis ($x_1=0$).

Each particle has a different velocity depending on its $x_2$ position:

$\mathbf{V} = \begin{bmatrix} \dot{x}_0 x_2 \\ 0 \\ 0 \end{bmatrix}$

$\mathbf{v}_1 = \gamma_0 x_2$

$P_1: v_1 = \gamma_0 l_1$

$P_2: v_1 = \gamma_0 l_2$

The initial $x_1$ position of each particle is $x_1=0$. After $t$ seconds, the two particles are at the following positions:

$P_1(t): x_1 = \gamma_0 l_1 t$

$P_2(t): x_1 = \gamma_0 l_2 t$

The separation of the particles after time $t$ is

$l^2 = l_0^2 + \left[\gamma_0 t (l_2 - l_1)\right]^2$

$= l_0^2 + \gamma_0^2 t^2 l_0^2$

$= l_0^2 \left(1 + \gamma_0^2 t^2\right)$

$l = l_0 \sqrt{1 + \gamma_0^2 t^2} \approx l_0 \gamma_0 t$

In shear the distance between points is directly proportional to time.

Negligible as $t \rightarrow \infty$
Uniaxial Elongational Flow

\[
v = \begin{pmatrix}
-\dot{\varepsilon}(t) x_1 \\
\frac{1}{2} \dot{\varepsilon}(t) x_2 \\
\frac{1}{2} \dot{\varepsilon}(t) x_3
\end{pmatrix}_{123} \quad \dot{\varepsilon}(t) > 0
\]

velocity field

Uniaxial Elongational Flow

\[
v = \begin{pmatrix}
-\dot{\varepsilon}(t) x_1 \\
\frac{1}{2} \dot{\varepsilon}(t) x_2 \\
\frac{1}{2} \dot{\varepsilon}(t) x_3
\end{pmatrix}_{123} \quad \dot{\varepsilon}(t) > 0
\]

path lines

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Elongational flow occurs when there is stretching - die exit, flow through contractions

Experimental Elongational Geometries
**Sentmanat Extension Rheometer (2005)**

- Originally developed for rubbers, good for melts
- Measures elongational viscosity, startup, other material functions
- Two counter-rotating drums
- Easy to load; reproducible

[Image of Sentmanat Extension Rheometer]

http://www.xpansioninstruments.com/rheo-optics.htm

**Why is elongation a standard flow?**

- Simple velocity field
- Represents all stretching flows
- Simple stress tensor

[Diagram of elongation flow]
How do particles move apart in elongational flow?

Consider two particles in the same $x_1$-$x_3$ plane, initially along the $x_3$ axis.

$\begin{align*}
x_1 &= 0 \\
x_2 &= 0 \\
x_3 &\text{ varies}
\end{align*}$

$v_3 = \frac{dx_3}{dt} = \dot{\epsilon}_0 x_3$

$v_3 = \dot{\epsilon}_0 dt$

$\ln x_3 = \dot{\epsilon}_0 t + C_1$

$x_3 = x_3(0)e^{\dot{\epsilon}_0 t}$

$l = l_0 e^{\dot{\epsilon}_0 t}$

Particles move apart exponentially fast.
A second type of shear-free flow: **Biaxial Stretching**

\[
\nu = \begin{pmatrix}
\frac{\dot{\epsilon}(t)}{2} x_1 \\
-\frac{\dot{\epsilon}(t)}{2} x_2 \\
\frac{\dot{\epsilon}(t)}{2} x_3
\end{pmatrix}_{123}
\]

\[\dot{\epsilon}(t) < 0\]

How do uniaxial and biaxial deformations differ?

Consider a **uniaxial** flow in which a particle is doubled in length in the flow direction.
Consider a biaxial flow in which a particle is doubled in length in the flow direction.

A third type of shear-free flow: Planar Elongational Flow

\[ \mathbf{v} = \begin{pmatrix} -\dot{\varepsilon}(t)x_1 \\ 0 \\ \dot{\varepsilon}(t)x_3 \end{pmatrix} \]

where \( \dot{\varepsilon}(t) > 0 \)
All three shear-free flows can be written together as:

\[ v = \begin{pmatrix} -\frac{1}{2} \dot{\epsilon}(t)(1 + b)x_1 \\ -\frac{1}{2} \dot{\epsilon}(t)(1 - b)x_2 \\ \dot{\epsilon}(t)x_3 \end{pmatrix} \]

Elongational flow: \( b=0, \dot{\epsilon}(t) > 0 \)
Biaxial stretching: \( b=0, \dot{\epsilon}(t) < 0 \)
Planar elongation: \( b=1, \dot{\epsilon}(t) > 0 \)

Why have we chosen these flows?

**ANSWER:** Because these simple flows have symmetry.

And symmetry allows us to draw conclusions about the stress tensor that is associated with these flows for any fluid subjected to that flow.
In general:

\[
\tau = \begin{pmatrix}
\tau_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \tau_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \tau_{33}
\end{pmatrix}
\]

But the stress tensor is symmetric – leaving 6 independent stress components.

Can we choose a flow to use in which there are fewer than 6 independent stress components?

Yes we can – symmetric flows

How does the stress tensor simplify for shear (and later, elongational) flow?
What would the velocity function be for a Newtonian fluid in this coordinate system?

\[ \mathbf{V} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} \]

What would the velocity function be for a Newtonian fluid in this coordinate system?

\[ \mathbf{V} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} \]
Vectors are independent of coordinate system, but in general the coefficients will be different when the same vector is written in two different coordinate systems:

\[
\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} = \begin{pmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \end{pmatrix}_{123}
\]

For shear flow and the two particular coordinate systems we have just examined, however:

\[
\mathbf{v} = \begin{pmatrix} \frac{V}{2H} x_2 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} \frac{\bar{V}}{2H} \bar{x}_2 \\ 0 \\ 0 \end{pmatrix}_{123}
\]

If we plug in the same number in for \( x_2 \) and \( \bar{x}_2 \), we will NOT be asking about the same point in space, but we WILL get the same exact velocity vector.

Since stress is calculated from the velocity field, we will get the same exact stress components when we calculate them from either vector representation.
What do we learn if we formally transform $\mathbf{V}$ from one coordinate system to the other?

\[ \hat{e}_1 = -\bar{e}_1 \]
\[ \hat{e}_2 = -\bar{e}_2 \]
\[ \hat{e}_3 = \bar{e}_3 \]

What do we learn if we formally transform $\mathbf{T}$ from one coordinate system to the other?
What do we learn if we formally transform \( V \) from one coordinate system to the other?

\[
\mathbf{\tau} = \tau_{ms} \hat{e}_m \hat{e}_s = \tilde{\tau}_{ms} \tilde{e}_m \tilde{e}_s
\]

(now, substitute from previous slide and simplify)

**You try.**

Conclusion:
Because of symmetry, there are only 5 nonzero components of the extra stress tensor in shear flow.

**SHEAR:**

\[
\mathbf{\tau} = \begin{pmatrix}
\tau_{11} & \tau_{12} & 0 \\
\tau_{21} & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]

This greatly simplifies the experimentalists tasks as only four stress components must be measured instead of 6 (recall \( \tau_{12} = \tau_{21} \)).
Summary:

We have found a coordinate system (the shear coordinate system) in which there are only 5 non-zero coefficients of the stress tensor. In addition, $\tau_{21} = \tau_{12}$.

This leaves only four stress components to be measured for this flow, expressed in this coordinate system.

How does the stress tensor simplify for elongational flow?

There is $180^\circ$ of symmetry around all three coordinate axes.
Because of symmetry, there are only 3 nonzero components of the extra stress tensor in elongational flows.

**ELONGATION:**

\[
\tau = \begin{pmatrix}
\tau_{11} & 0 & 0 \\
0 & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]

This greatly simplifies the experimentalist's tasks as only three stress components must be measured instead of 6.

**Standard Flows Summary**

Choose velocity field:

\[
v = \begin{pmatrix}
\xi(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]

Symmetry alone implies:

(no constitutive equation needed yet)

\[
tau = \begin{pmatrix}
\tau_{11} & \tau_{12} & 0 \\
\tau_{21} & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]

\[
v = \begin{pmatrix}
-\frac{1}{2}\dot{\xi}(t)(1+b)x_1 \\
0 \\
0
\end{pmatrix}_{123}
\]

\[
tau = \begin{pmatrix}
\tau_{11} & 0 & 0 \\
0 & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]

By choosing these symmetric flows, we have reduced the number of stress components that we need to measure.
**Tactic: Divide the Problem in half**

<table>
<thead>
<tr>
<th>Modeling Calculations</th>
<th>Experiments</th>
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<tbody>
<tr>
<td>Dream up models</td>
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<td></td>
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<tr>
<td>Collect models and their report cards for future use</td>
<td></td>
</tr>
</tbody>
</table>

**Choose velocity field:**

\[
\mathbf{v} = \begin{pmatrix}
\dot{\zeta}(t)x_2 \\
0 \\
0
\end{pmatrix}_{123}
\]

\[
\mathbf{v} = \begin{pmatrix}
\frac{1}{2} \dot{\varepsilon}(t)(1+b)x_3 \\
0 \\
\frac{1}{2} \dot{\varepsilon}(t)(1-b)x_2
\end{pmatrix}_{10}
\]

**Symmetry alone implies:** (no constitutive equation needed yet)

\[
\mathbf{\tau} = \begin{pmatrix}
\tau_{11} & \tau_{12} & 0 \\
\tau_{21} & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]

**Measure and predict this**

\[
\mathbf{\tau} = \begin{pmatrix}
\tau_{11} & 0 & 0 \\
0 & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}_{123}
\]
One final comment on measuring stresses...

What is measured is the total stress, $\Pi$:

$$\Pi = \begin{pmatrix} p + \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & p + \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & p + \tau_{33} \end{pmatrix}_{123}$$

For the normal stresses we are faced with the difficulty of separating $p$ from $\tau_{ii}$.

Compressible fluids:

Get $p$ from measurements of $T$ and $V$.

Incompressible fluids:

$\rho = \frac{nRT}{V}$

Density does not vary (much) with pressure for polymeric fluids.
For incompressible fluids it is not possible to separate \( p \) from \( \tau_{ii} \).

Luckily, this is not a problem since we only need \( \nabla \cdot \Pi = \nabla p + \nabla \cdot \tau \).

**Equation of motion**

\[
\frac{\partial v}{\partial t} + v \cdot \nabla v = -\nabla \Pi + \rho g \\
= -\nabla P - \nabla \cdot \tau + \rho g
\]

We do not need \( \tau_{ii} \) directly to solve for velocities.

Solution? *Normal stress differences*

**Normal Stress Differences**

First normal stress difference

\[ N_1 = \Pi_{11} - \Pi_{22} = \tau_{11} - \tau_{22} \]

Second normal stress difference

\[ N_2 = \Pi_{22} - \Pi_{33} = \tau_{22} - \tau_{33} \]

In shear flow, three stress quantities are measured \( \tau_{21}, N_1, N_2 \)

In elongational flow, two stress quantities are measured \( \tau_{33} - \tau_{11}, \tau_{22} - \tau_{11} \)
Normal Stress Differences

First normal stress difference
\[ N_1 = \Pi_{11} - \Pi_{22} = \tau_{11} - \tau_{22} \]

Second normal stress difference
\[ N_2 = \Pi_{22} - \Pi_{33} = \tau_{22} - \tau_{33} \]

In shear flow, three stress quantities are measured.
\[ \tau_{21}, N_1, N_2 \]

In elongational flow, two stress quantities are measured.
\[ \tau_{33} - \tau_{11}, \tau_{22} - \tau_{11} \]

Are shear normal stress differences real?

First normal stress effects: rod climbing
\[ \tau_{11} - \tau_{22} < 0 \]
Extra tension in the 1-direction pulls azimuthally and upward (see DPL p65).

Newtonian - glycerin
Viscoelastic - solution of polyacrylamide in glycerin

Bird, et al., *Dynamics of Polymeric Fluids*, vol. 1, Wiley, 1987, Figure 2.3-1 page 63. (DPL)
**Second** normal stress effects: inclined open-channel flow

\[ \tau_{22} - \tau_{33} > 0 \]

Extra tension in the 2-direction pulls down the free surface where \( dv_1/dx_2 \) is greatest (see DPL p65).

Newtonian - glycerin

Viscoelastic - 1% soln of polyethylene oxide in water

\[ N_2 \approx -N_1/10 \]

R. I. Tanner, *Engineering Rheology*, Oxford 1985, Figure 3.6 page 104

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---

**Example**: Can the equation of motion predict rod climbing for typical values of \( N_1, N_2 \)?

\[ \mathbf{v} = \begin{pmatrix} 0 \\ v_\theta \\ 0 \end{pmatrix} \]

cross-section A:

What is \( d\Pi_{zz}/dr \)?

Bird et al. p64

www.chem.mtu.edu/~fmorriso/cm4650/rod_climb.pdf

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What’s next?

Shear

\[ \mathbf{v} = \begin{pmatrix} 
\xi(t)x_2 \\
0 \\
0 
\end{pmatrix} \]

Shear-free (elongational, extensional)

\[ \mathbf{v} = \begin{pmatrix} 
-\frac{1}{2} \xi(t)(1+b)x_1 \\
-\frac{1}{2} \dot{\xi}(t)(1-b)x_2 \\
\dot{\xi}(t)x_3 
\end{pmatrix} \]

Even with just these 2 (or 4) standard flows, we can still generate an infinite number of flows by varying \( \xi(t) \) and \( \dot{\xi}(t) \).

Elongational flow: \( b=0, \ \dot{\xi}(t) > 0 \)

Biaxial stretching: \( b=0, \ \dot{\xi}(t) < 0 \)

Planar elongation: \( b=1, \ \dot{\xi}(t) > 0 \)

We seek to quantify the behavior of non-Newtonian fluids

Procedure:

1. Choose a flow type (shear or a type of elongation).
2. Specify \( \xi(t) \) or \( \dot{\xi}(t) \) as appropriate.
3. Impose the flow on a fluid of interest.
4. Measure stresses.

\[ \tau_{11}, N_1, N_2 \]

shear

\[ \tau_{33} - \tau_{11}, \tau_{22} - \tau_{11} \]

elongation

5. Report stresses in terms of material functions.

6a. Compare measured material functions with predictions of these material functions (from proposed constitutive equations).

7a. Choose the most appropriate constitutive equation for use in numerical modeling.

6b. Compare measured material functions with those measured on other materials.

7a. Draw conclusions on the likely properties of the unknown material based on the comparison.
Chapter 5: Material Functions

Steady Shear Flow Material Functions

Kinematics:
\[ y = \begin{pmatrix} \ddot{z}(0) \\ 0 \\ 0 \end{pmatrix}, \quad \dot{\gamma}(t) = \dot{\gamma}_0 = \text{constant} \]

Material Functions:
- Viscosity: \[ \eta = \frac{-\gamma_{21}}{\dot{\gamma}_0} \]
- First normal-stress coefficient: \[ \psi_1 = \frac{(f_{11} - f_{22})}{\dot{\gamma}_0^2} \]
- Second normal-stress coefficient: \[ \psi_2 = \frac{(f_{22} - f_{33})}{\dot{\gamma}_0^2} \]

Role of Material Functions in Rheological Analysis

**QUALITY CONTROL**
- compare with other in-house data on qualitative basis
- conclude whether or not a material is appropriate for a specific application

**QUALITATIVE ANALYSIS**
- compare data with literature reports on various fluids
- conclude on the probable physical behavior of the fluid based on comparison with known fluid behavior

**MODELING WORK**
- measure material functions, e.g. \( \eta, G'(\omega), G''(\omega), G(t) \)
- compare measured with predicted
- conclude which constitutive equation is best for further modeling calculations

**CALCULATE PREDICTIONS**
- compare with other in-house data on qualitative basis
Role of Material Functions in Rheological Analysis

QUALITY CONTROL
- compare with other in-house data on qualitative basis
- conclude whether or not a material is appropriate for a specific application

QUALITATIVE ANALYSIS
- compare data with literature reports on various fluids
- conclude on the probable physical behavior of the fluid based on comparison with known fluid behavior

We will focus here first

MODELING WORK
- compare measured with predicted
- conclude which constitutive equation is best for further modeling calculations

Material function definitions

1. Choice of flow (shear or elongation)
   \[ \dot{\gamma}(t) x_2 \]
   \[ \dot{\gamma} = \begin{pmatrix} \frac{1}{2} \dot{\gamma}(t)(1 + b)x_1 \\ -1 \dot{\gamma}(t)(1 - b)x_2 \\ \dot{\gamma}(t)x_3 \end{pmatrix} \]
   - Elongational flow: \( b = 0, \dot{\gamma}(t) > 0 \)
   - Biaxial stretching: \( b = 0, \dot{\gamma}(t) < 0 \)
   - Planar elongation: \( b = 1, \dot{\gamma}(t) > 0 \)

2. Choice of details of \( \dot{\gamma}(t) \) or \( \dot{\epsilon}(t) \).

3. Material functions definitions: will be based on
   \( \tau_{21}, N_1, N_2 \) in shear or \( \tau_{33} - \tau_{11}, \tau_{22} - \tau_{11} \)
   in elongational flows.

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Steady Shear Flow Material Functions

Kinematics:
\[
\mathbf{\dot{v}} = \begin{pmatrix} \mathbf{\dot{v}}(t) \mathbf{x}_2 \\ 0 \\ 0 \end{pmatrix}_{123}
\]
\[
\mathbf{\dot{v}} = \dot{\gamma}_0 = \text{constant}
\]

Material Functions:
\[
\begin{align*}
\eta & = \frac{-\tau_{21}}{\dot{\gamma}_0} \\
\psi_1 & = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} \\
\psi_2 & = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}
\end{align*}
\]

Viscosity
First normal-stress coefficient
Second normal-stress coefficient

How do we predict material functions?

**ANSWER:** From the constitutive equation.

\[
\mathbf{\tau} = f(\mathbf{\nu})
\]

What does the **Newtonian** Fluid model predict in steady shearing?

\[
\mathbf{\tau} = -\mu \mathbf{\dot{\gamma}} = -\mu \left[ \nabla \mathbf{\nu} + (\nabla \mathbf{\nu})^T \right]
\]
What does the **Newtonian** Fluid model predict in steady shearing?

\[ \tau = -\mu \dot{\gamma} = -\mu [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \]

**You try.**

*What do we measure for these material functions?*

*(for polymer solutions, for example)*
Steady shear viscosity and first normal stress coefficient

Figure 6.1, p. 170 Menzes and Graessley conc. PB solution

Steady shear viscosity and first normal stress coefficient

Figure 6.2, p. 171 Menzes and Graessley conc. PB solution; $c=0.0676$ g/cm$^3$
Steady shear viscosity for linear and branched PDMS

+ linear 131 kg/mole
▲ branched 156 kg/mole
□ linear 418 kg/mol
◆ branched 428 kg/mol

What have material functions taught us so far?

• Newtonian constitutive equation is inadequate
  1. Predicts constant shear viscosity (not always true)
  2. Predicts no shear normal stresses (these stresses are generated for many fluids)

• Behavior depends on the material (chemical structure, molecular weight, concentration)
Can we fix the Newtonian Constitutive Equation?

\[ \tau = -\mu \left[ \nabla \mathbf{v} + \left( \nabla \mathbf{v} \right)^T \right] \]

Let’s replace \( \mu \) with a function of shear rate because we want to predict a non-constant viscosity in shear.

What does this model predict for steady shear viscosity?

\[ \tau = -M \left( \dot{\gamma} \right)_0 \left[ \nabla \mathbf{v} + \left( \nabla \mathbf{v} \right)^T \right] \]
What does this model predict for steady shear viscosity?

\[ \tau = -M(\dot{\gamma}_0)\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right] \]

You try.

What does this model predict for steady shear viscosity?

\[ \tau = -M(\dot{\gamma}_0)\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right] \]

Answer: \[ \eta = M(\dot{\gamma}_0) \]
If we choose:

\[ M(M_0) = \begin{cases} M_0, & \dot{\gamma}_0 < \dot{\gamma}_c \\ m\dot{\gamma}_0^{n-1}, & \dot{\gamma}_0 \geq \dot{\gamma}_c \end{cases} \]

\[ \log \eta \]

slope = (n-1)

Problem solved!

But what about the normal stresses?

\[ \tau = -M(M_0) \left[ \nabla \nu + (\nabla \nu)^T \right] \]

\[ \nabla \nu = \begin{pmatrix} 0 & 0 & 0 \\ \dot{\gamma}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \dot{\gamma}_c = \begin{pmatrix} 0 & \dot{\gamma}_c & 0 \\ \dot{\gamma}_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

It appears that \( \tau \) should not be simply proportional to \( \dot{\gamma}_c \).

Try something else . . .

\[ \tau = -\mu \gamma + I f(\nu) \]

\[ \tau = f(\nu) \nabla \nu \cdot (\nabla \nu)^T \]

\[ \tau = A \left[ \nabla \nu \cdot (\nabla \nu)^T \right] + B \nabla \nu + C (\nabla \nu)^T \]

\[ \ldots \]
But which ones?

To sort out how to fix the Newtonian equation, we need more observations (to give us ideas).

Let’s try another material function that’s not a steady flow (but stick to shear).

Start-up of Steady Shear Flow Material Functions

Kinematics:

\[
\gamma \equiv \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123}
\]

\[
\dot{\gamma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}
\]

Material Functions:

\[
\eta^+ \equiv -\frac{\tau_{21}(t)}{\dot{\gamma}_0}
\]

Shear stress growth function

First normal-stress growth function

\[
\psi^+_1 \equiv -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2}
\]

Second normal-stress growth function

\[
\psi^+_2 \equiv -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2}
\]
What does the **Newtonian** Fluid model predict in start-up of steady shearing?

\[
\tau = -\mu \dot{\gamma} = -\mu \left[ \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right]
\]

Again, since we know \( \mathbf{v} \), we can just plug it in and calculate the stresses.

**You try.**
Material functions predicted for *start-up of steady shearing* of a Newtonian fluid

\[
\eta^+(t) = \begin{cases} 
0 & t < 0 \\
\mu & t \geq 0
\end{cases}
\]

\[
\Psi_1^+ = \frac{-(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} = 0
\]

\[
\Psi_2^+ = \frac{-(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} = 0
\]

Do these predictions match observations?

---

**Startup of Steady Shearing**

\[
\mathbf{Y} = \begin{bmatrix} \dot{\gamma}(t) \\ 0 \\ 0 \end{bmatrix}, \quad \dot{\mathbf{Y}}(t) = \begin{cases} 
0 & t < 0 \\
\dot{\gamma}_0 & t \geq 0
\end{cases}
\]

---

*Figures 6.49, 6.50, p. 208*

Menezes and Graessley, PB soln

SOR Short Course Beginning Rheology

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What about other non-steady flows?

**Cessation of Steady Shear Flow Material Functions**

**Kinematics:**

\[
\dot{\gamma}(t) = \begin{cases} 
\dot{\gamma}_0 & t < 0 \\
0 & t \geq 0 
\end{cases}
\]

\[
\dot{\gamma}(t) = \begin{pmatrix} 
\dot{\gamma}(t)x_2 \\
0 \\
0 
\end{pmatrix}_{123}
\]

**Material Functions:**

\[
\eta^- \equiv -\frac{\tau_{21}(t)}{\dot{\gamma}_0} \quad \text{Shear stress decay function}
\]

\[
\Psi_1^- \equiv -\frac{(\tau_{11} - \tau_{22})}{\dot{\gamma}_0^2} \quad \text{First normal-stress decay function}
\]

\[
\Psi_2^- \equiv -\frac{(\tau_{22} - \tau_{33})}{\dot{\gamma}_0^2} \quad \text{Second normal-stress decay function}
\]
What does the model we guessed at predict for start-up and cessation of shear?

\[ \tau = -M(\dot{\gamma}_0)\left[\nabla\dot{\gamma} + (\nabla\dot{\gamma})^T\right] \]

\[ M(\dot{\gamma}_0) = \begin{cases} 
M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\
M_0\dot{\gamma}_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c 
\end{cases} \]
What does the model we guessed at predict for start-up and cessation of shear?

\[
\tau = -M(\dot{\gamma}_0) \left[ \nabla \dot{\psi} + (\nabla \dot{\psi})^T \right]
\]

\[
M(\dot{\gamma}_0) = \begin{cases} 
  M_0 & \dot{\gamma}_0 < \dot{\gamma}_c \\
  m_0^{n-1} & \dot{\gamma}_0 \geq \dot{\gamma}_c 
\end{cases}
\]

You try.

Menzes and Graessley, conc. PB solution; 350 kg/mol

\[ M_0 = 18,000 \text{ poise} \]
\[ m = 12,000 \]
\[ n = 0.24 \]
\[ \dot{\gamma}_c = 0.67 \text{s}^{-1} \]
Observations

• The model predicts an instantaneous stress response, and this is not what is observed for polymers.

• The predicted unsteady material functions depend on the shear rate, which is observed for polymers.

\[ \eta^+ = \eta^+ (t, \dot{\gamma}_0) \]

• No normal stresses are predicted.

Lacks memory

Progress here

Related to nonlinearities
To proceed to better-designed constitutive equations, we need to know more about material behavior, i.e. we need more material functions to predict, and we need measurements of these material functions.

- More non-steady material functions (material functions that tell us about memory)
- Material functions that tell us about nonlinearity (strain)

Summary of shear rate kinematics (part 1)
The next three families of material functions incorporate the concept of strain.