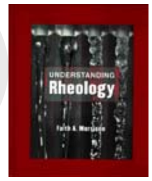
Polymer Rheology

Rhe-

$\underbrace{}$

$\rho\varepsilon\iota$ – *Greek* for flow

CM4650
Polymer Rheology
Michigan Tech



What is rheology anyway?

Rheology = the study of deformation and flow.

"What is Rheology Anyway?" Faith A. Morrison, *The Industrial Physicist*, 10(2) 29-31, April/May 2004.

1

What is rheology anyway?

To the layperson, rheology is

•Mayonnaise does not flow even under stress for a long time; honey always flows

•Silly Putty bounces (is elastic) but also flows (is viscous)

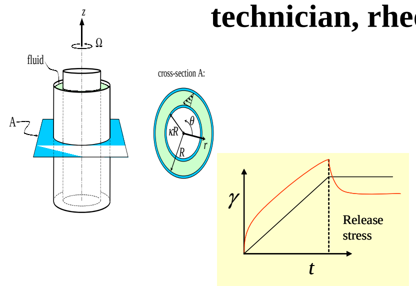
•Dilute flour-water solutions are easy to work with but doughs can be quite temperamental

•Corn starch and water can display strange behavior – poke it slowly and it deforms easily around your finger; punch it rapidly and your fist bounces off of the surface

2

What is rheology anyway?

To the scientist, engineer, or technician, rheology is



- •Yield stresses
- •Viscoelastic effects
- •Memory effects
- •Shear thickening and shear thinning

For both the layperson and the technical person, rheology is a set of problems or observations related to how the stress in a material or force applied to a material is related to deformation (change of shape) of the material.

3

What is rheology anyway?

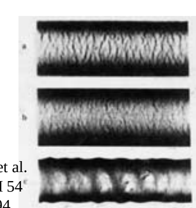
Rheology affects:

•End use (food texture, product pour, motor-oil function)

•Processing (design, costs, production rates)

www.corrugatorman.com/pic/akron%20extruder.JPG

www.math.utwente.nl/mpcm/aamp/examples.html



Pomar et al. JNNFM 54 143 1994

•Product quality (surface distortions, anisotropy, strength, structure development)

4

Goal of the scientist, engineer, or technician:

- **Understand** the kinds of flow and deformation **effects** exhibited by complex systems
- **Apply qualitative** rheological **knowledge** to diagnostic, design, or optimization problems
- In diagnostic, design, or optimization problems, **use or devise quantitative** analytical **tools** that correctly capture rheological effects

How do we reach these goals?

How?

By observing the behavior of different systems

- **Understand** the kinds of flow and deformation **effects** exhibited by complex systems
- **Apply qualitative** rheological **knowledge** to diagnostic, design, or optimization problems
- In diagnostic, design, or optimization problems, **Use or devise quantitative** analytical **tools** that correctly capture rheological effects

By learning which quantitative models apply in what circum-stances

By making calculations with models in appropriate situations

Learning Rheology (bibliography)

Descriptive Rheology

Barnes, H., J. Hutton, and K. Walters, *An Introduction to Rheology* (Elsevier, 1989)

Quantitative Rheology

Morrison, Faith, *Understanding Rheology* (Oxford, 2001)
Bird, R., R. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids, Volume 1* (Wiley, 1987)

Industrial Rheology

Dealy, John and Kurt Wissbrun, *Melt Rheology and Its Role in Plastics Processing* (Van Nostrand Reinhold, 1990)

Polymer Behavior

Larson, Ron, *The Structure and Rheology of Complex Fluids* (Oxford, 1999)
Ferry, John, *Viscoelastic Properties of Polymers* (Wiley, 1980)

Suspension Behavior

Mewis, Jan and Norm Wagner, *Colloidal Suspension* (Cambridge, 2012)
Macosko, Chris, *Rheology: Principles, Measurements, and Applications* (VCH Publishers, 1994)

7

The Physics Behind Rheology:

1. Conservation laws
   - mass
   - momentum
   - energy

Cauchy Momentum Equation

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}$$

2. Mathematics
   - differential equations
   - vectors
   - tensors

3. Constitutive law = law that relates **stress** to **deformation** for a particular fluid

8

Newtonian fluids:
(fluid mechanics)

$$\underbrace{\tau_{21} = -\mu \frac{dv_1}{dx_2}}_{\text{Newton's Law of Viscosity}}$$

material parameter

deformation

Newton's Law of Viscosity

- This is an empirical law (measured or observed)
- May be derived theoretically for some systems

Non-Newtonian fluids:
(rheology)

Need a new law or new <u>laws</u>

- These laws will also either be empirical or will be derived theoretically

Newtonian fluids:
(shear flow only)

$$\tau_{21} = -\mu \frac{dv_1}{dx_2}$$

Constitutive Equation

Non-Newtonian fluids:
(all flows)

$$\overbrace{\underline{\underline{\tau}} = -f\left(\underline{\underline{\dot{\gamma}}}\right)}$$

stress <u>tensor</u>

Rate-of-deformation <u>tensor</u>

non-linear function
(in time and position)

Introduction to Non-Newtonian Behavior

Rheological Behavior of Fluids, National Committee on Fluid Mechanics Films, 1964

Velocity gradient tensor $\underline{\underline{\dot{\gamma}}}$

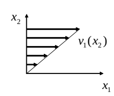
Type of fluid	Momentum balance	Stress –Deformation relationship (constitutive equation)
Inviscid (zero viscosity, μ=0)	Euler equation (Navier-Stokes with zero viscosity)	Stress is isotropic
Newtonian (finite. constant viscosity, μ)	Navier-Stokes (Cauchy momentum equation with Newtonian constitutive equation)	Stress is a function of the **instantaneous** velocity gradient
Non-Newtonian (finite, variable viscosity η plus **memory** effects)	Cauchy momentum equation with memory constitutive equation	Stress is a function of the **history** of the velocity gradient

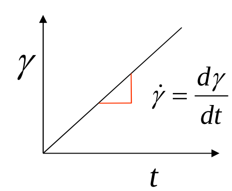
11

Rheological Behavior of Fluids - Newtonian

1. Strain response to imposed shear stress

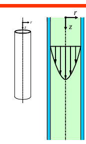
•shear rate is constant





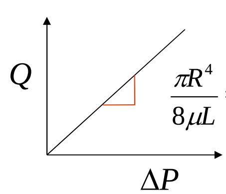
$$\dot{\gamma} = \frac{d\gamma}{dt} = \text{constant}$$

2. Pressure-driven flow in a tube (Poiseuille flow)



•viscosity is constant

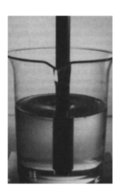
$$Q = \frac{\pi \Delta P R^4}{8\mu L}$$



$$\frac{\pi R^4}{8\mu L} = \text{constant}$$

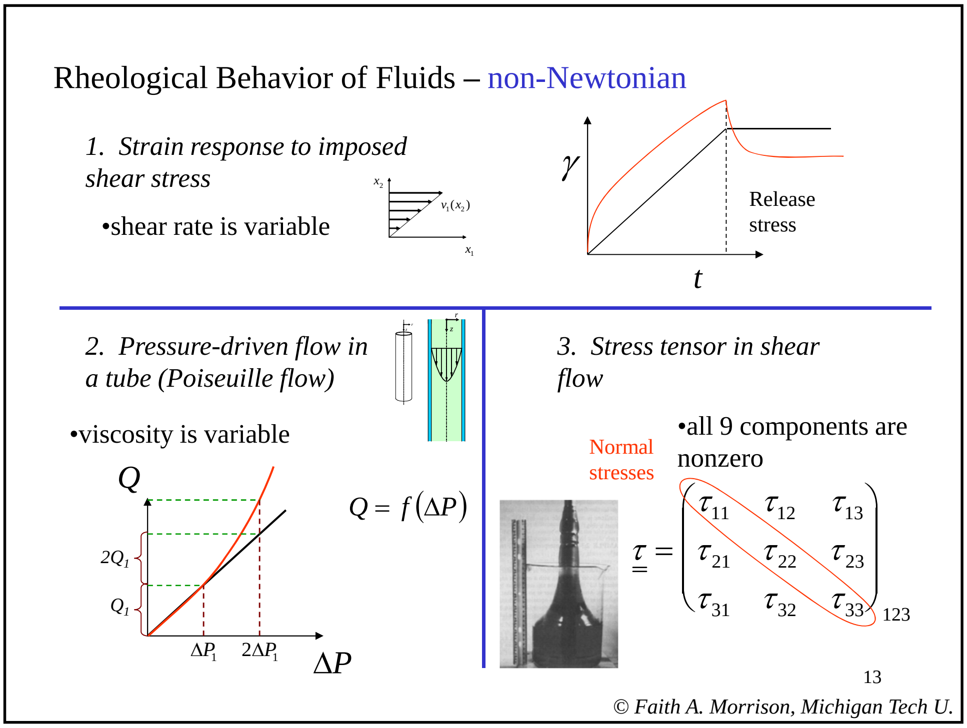
3. Stress tensor in shear flow

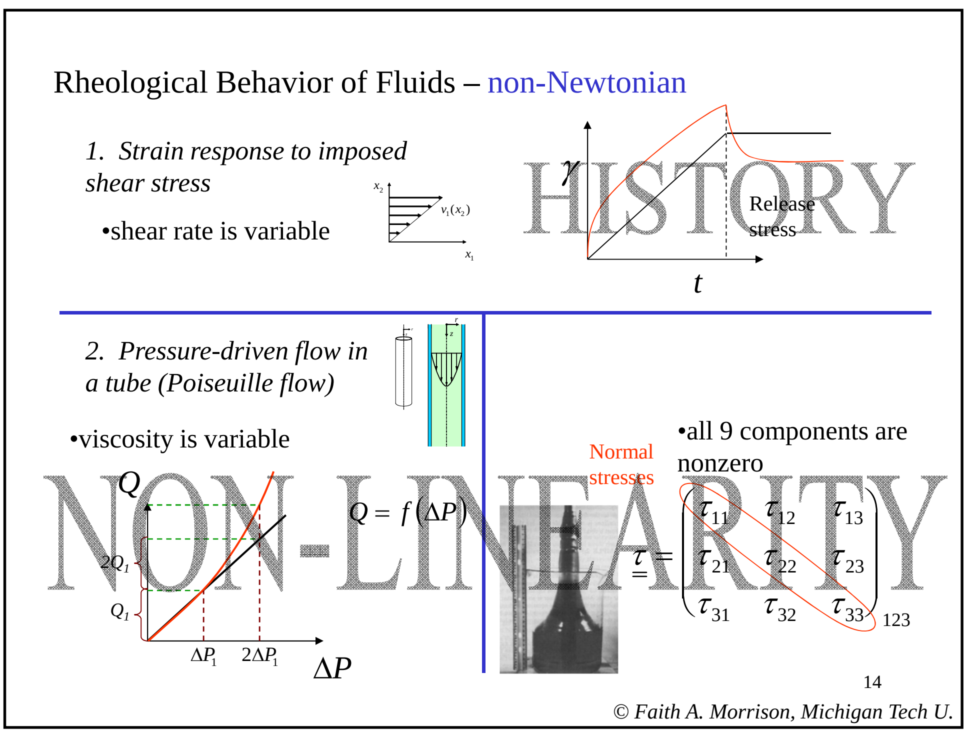
•only two components are nonzero



$$\underline{\underline{\tau}} = \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

12


Rheological Behavior of Fluids – non-Newtonian
1. Strain response to imposed shear stress
•shear rate is variable
γ
t
Release stress
2. Pressure-driven flow in a tube (Poiseuille flow)
•viscosity is variable
Q
$2Q_1$
Q_1
ΔP_1
$2\Delta P_1$
ΔP
$Q = f(\Delta P)$
3. Stress tensor in shear flow
•all 9 components are nonzero
Normal stresses
$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$
13
© Faith A. Morrison, Michigan Tech U.


Rheological Behavior of Fluids – non-Newtonian
1. Strain response to imposed shear stress
•shear rate is variable
HISTORY
γ
t
Release stress
2. Pressure-driven flow in a tube (Poiseuille flow)
•viscosity is variable
NON-LINEARITY
Q
$2Q_1$
Q_1
ΔP_1
$2\Delta P_1$
ΔP
$Q = f(\Delta P)$
•all 9 components are nonzero
Normal stresses
$\underline{\underline{\tau}} = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123}$
14
© Faith A. Morrison, Michigan Tech U.

Examples from the film of . . . .

Dependence on the history of the deformation gradient

•Polymer fluid pours, but springs back
•Elastic ball bounces, but flows if given enough time
•Steel ball dropped in polymer solution "bounces"
•Polymer solution in concentric cylinders – has fading memory
•Quantitative measurements in concentric cylinders show memory and need a finite time to come to steady state

Non-linearity of the function $\underline{\underline{\tau}} = f\left(\dot{\underline{\underline{\gamma}}}\right)$

•Polymer solution draining from a tube is first slower, then faster than a Newtonian fluid
•Double the static head on a draining tube, and the flow rate does not necessarily double (as it does for Newtonian fluids); sometimes more than doubles, sometimes less
•Normal stresses in shear flow
•Die swell

15

Show NCFM Film on *Rheological Behavior of Fluids*

- Search for NCFMF
- web.mit.edu/hml/ncfmf.html
- Also on YouTube

16

Chapter 2: Mathematics Review

1. Scalar – *a mathematical entity that has magnitude only*

e.g.: temperature T
speed v
time t
density ρ

– *scalars may be constant or may be variable*

Laws of Algebra for Scalars:

yes commutative $ab = ba$

yes associative $a(bc) = (ab)c$

yes distributive $a(b+c) = ab+ac$

17

2. Vector – *a mathematical entity that has magnitude and direction*

e.g.: force on a surface $\underline{f}$
velocity $\underline{v}$

– *vectors may be constant or may be variable*

Definitions

magnitude of a vector – a scalar associated with a vector

$$|\underline{v}| = v \qquad |\underline{f}| = f$$

unit vector – a vector of unit length

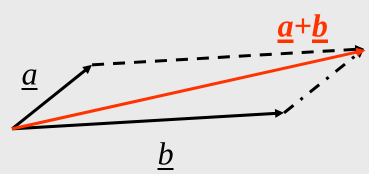
$$\frac{\underline{v}}{|\underline{v}|} = \hat{\underline{v}}$$

a unit vector in the direction of $\underline{v}$

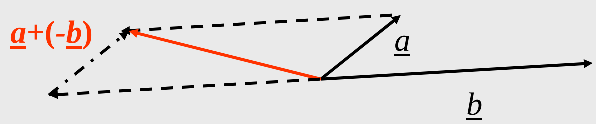
18

Laws of Algebra for Vectors:

1. Addition



2. Subtraction



Laws of Algebra for Vectors (continued):

3. Multiplication by scalar $\alpha \underline{v}$

yes commutative $\alpha \underline{v} = \underline{v}\alpha$

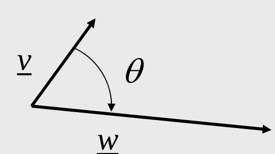
yes associative $\alpha(\beta \underline{v}) = (\alpha\beta)\underline{v} = \alpha\beta\underline{v}$

yes distributive $\alpha(\underline{v} + \underline{w}) = \alpha\underline{v} + \alpha\underline{w}$

4. Multiplication of vector by vector

4a. scalar (dot) (inner) product

$$\underline{v} \cdot \underline{w} = vw\cos\theta$$



Note: we can find magnitude with dot product

$\underline{v} \cdot \underline{v} = vv\cos 0 = v^2$

$v = |\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$

Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con't)

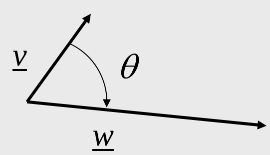
yes commutative $\underline{v}\cdot\underline{w}=\underline{w}\cdot\underline{v}$

NO associative $\cancel{\underline{v}\cdot\underline{w}\cdot\underline{z}}$ no such operation

yes distributive $\underline{z}\cdot\left(\underline{v}+\underline{w}\right)=\underline{z}\cdot\underline{v}+\underline{z}\cdot\underline{w}$

4b. vector (cross) (outer) product

$$\underline{v}\times\underline{w}=vw\sin\theta\,\hat{e}$$



$\hat{e}$ is a unit vector perpendicular to both $\underline{v}$ and $\underline{w}$ following the right-hand rule

21

Laws of Algebra for Vectors (continued):

4b. vector (cross) (outer) product (con't)

NO commutative $\underline{v}\times\underline{w}\neq\underline{w}\times\underline{v}$

NO associative $\underline{v}\times\underline{w}\times\underline{z}\neq\left(\underline{v}\times\underline{w}\right)\times\underline{z}\neq\underline{v}\times\left(\underline{w}\times\underline{z}\right)$

yes distributive $\underline{z}\times\left(\underline{v}+\underline{w}\right)=\left(\underline{z}\times\underline{v}\right)+\left(\underline{z}\times\underline{w}\right)$

22

Coordinate Systems

•Allow us to make actual calculations with vectors

Rule: any three vectors that are *non-zero* and *linearly independent* (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if α, β, and γ can be found such that:

$$\alpha \underline{a} + \beta \underline{b} + \gamma \underline{c} = \underline{0}$$
$$for \quad \alpha, \beta, \gamma \neq 0$$

If α, β, and γ are found to be zero, the vectors are linearly independent.

How can we do actual calculations with vectors?

Rule: *any* vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

coefficient of $\underline{a}$ in the $\hat{e}_y$ direction

any vector

$$\underline{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}_{xyz}$$

$$= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3$$

$$= \sum_{j=1}^{3} a_j \hat{e}_j$$

Trial calculation: dot product of two vectors

$$\begin{aligned}
\underline{a}\cdot\underline{b} &= \left(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3\right)\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\qquad a_2\hat{e}_2\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) + \\
&\qquad\quad a_3\hat{e}_3\cdot\left(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3\right) \\
&= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\qquad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\qquad\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3
\end{aligned}$$

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

If we choose the basis to be orthonormal - mutually perpendicular and of unit length, then we can simplify.

$$\begin{aligned}
\hat{e}_1\cdot\hat{e}_1 &= 1 \\
\hat{e}_1\cdot\hat{e}_2 &= 0 \\
\hat{e}_1\cdot\hat{e}_3 &= 0 \\
&\cdots
\end{aligned}$$

$$\begin{aligned}
\underline{a}\cdot\underline{b} &= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 + \\
&\qquad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 + \\
&\qquad\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3 \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

We can generalize this operation with a technique called Einstein notation.

Einstein Notation

a system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$
$$= \sum_{j=1}^{3} a_j\hat{e}_j$$
$$= a_j\hat{e}_j = a_m\hat{e}_m$$

•the initial choice of subscript letter is *arbitrary*

•the presence of a pair of like subscripts implies a missing summation sign

Einstein Notation (con't)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function

$$\hat{e}_1 \cdot \hat{e}_1 = 1$$
$$\hat{e}_1 \cdot \hat{e}_2 = 0$$
$$\hat{e}_1 \cdot \hat{e}_3 = 0$$
$$\cdots$$

$$\underbrace{\hat{e}_i \cdot \hat{e}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases}}_{\text{Kronecker delta}}$$

Einstein Notation (con't)

To carry out a dot product of two arbitrary vectors . . .

Detailed Notation

$$\begin{aligned}\underline{a}\cdot\underline{b} &= (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)\cdot(b_1\hat{e}_1 + b_2\hat{e}_2 + b_3\hat{e}_3)\\ &= a_1\hat{e}_1\cdot b_1\hat{e}_1 + a_1\hat{e}_1\cdot b_2\hat{e}_2 + a_1\hat{e}_1\cdot b_3\hat{e}_3 +\\ &\quad a_2\hat{e}_2\cdot b_1\hat{e}_1 + a_2\hat{e}_2\cdot b_2\hat{e}_2 + a_2\hat{e}_2\cdot b_3\hat{e}_3 +\\ &\quad a_3\hat{e}_3\cdot b_1\hat{e}_1 + a_3\hat{e}_3\cdot b_2\hat{e}_2 + a_3\hat{e}_3\cdot b_3\hat{e}_3\\ &= a_1b_1 + a_2b_2 + a_3b_3\end{aligned}$$

Einstein Notation

$$\begin{aligned}\underline{a}\cdot\underline{b} &= a_j\hat{e}_j\cdot b_m\hat{e}_m\\ &= a_j\delta_{jm}b_m\\ &= a_jb_j\end{aligned}$$

3. Tensor – *the indeterminate vector product of two (or more) vectors*

e.g.: stress $\underline{\underline{\tau}}$
velocity gradient $\underline{\underline{\dot{\gamma}}}$

– *tensors may be constant or may be variable*

Definitions

dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

$\underline{a}\,\underline{d}$ dyad

$\underline{\underline{A}}$ general representation of a tensor

Laws of Algebra for Indeterminate Product of Vectors:

NO commutative $\underline{a}\,\underline{v} \neq \underline{v}\,\underline{a}$

yes associative $\underline{b}\,(\underline{a}\,\underline{v}) = (\underline{b}\,\underline{a})\,\underline{v} = \underline{b}\,\underline{a}\,\underline{v}$

yes distributive $\underline{a}\,(\underline{v} + \underline{w}) = \underline{a}\,\underline{v} + \underline{a}\,\underline{w}$

How can we represent tensors with respect to a chosen coordinate system?

Just follow the rules of tensor algebra

$$\begin{aligned}
\underline{a}\,\underline{m} &= (a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3)(m_1\hat{e}_1 + m_2\hat{e}_2 + m_3\hat{e}_3) \\
&= a_1\hat{e}_1 m_1\hat{e}_1 + a_1\hat{e}_1 m_2\hat{e}_2 + a_1\hat{e}_1 m_3\hat{e}_3 + \\
&\quad a_2\hat{e}_2 m_1\hat{e}_1 + a_2\hat{e}_2 m_2\hat{e}_2 + a_2\hat{e}_2 m_3\hat{e}_3 + \\
&\quad a_3\hat{e}_3 m_1\hat{e}_1 + a_3\hat{e}_3 m_2\hat{e}_2 + a_3\hat{e}_3 m_3\hat{e}_3 \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k\hat{e}_k m_w\hat{e}_w \\
&= \sum_{k=1}^{3}\sum_{w=1}^{3} a_k m_w\,\hat{e}_k\hat{e}_w
\end{aligned}$$

Any tensor may be written as the sum of 9 dyadic products of basis vectors

Mathematics Review

Polymer Rheology

What about $\underline{\underline{A}}$? Same.

$$\underline{\underline{A}} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_{ij}\, \hat{e}_i \hat{e}_j$$

Einstein notation for tensors: *drop the summation sign; every double index implies a summation sign has been dropped.*

$$\underline{\underline{A}} = A_{ij}\, \hat{e}_i \hat{e}_j = A_{pk}\, \hat{e}_p \hat{e}_k$$

Reminder: the initial choice of subscript letters is arbitrary

33

Mathematics Review

Polymer Rheology

How can we use Einstein Notation to calculate dot products between vectors and tensors?

It's the same as between vectors.

$$\underline{a} \cdot \underline{b} =$$
$$\underline{a} \cdot \underline{u}\, \underline{v} =$$
$$\underline{b} \cdot \underline{\underline{A}} =$$

34

Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index

2. Drop the summation signs

3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:

•In Einstein notation, the presence of repeated indices implies a missing summation sign

•The choice of initial index (i, m, p, etc.) is arbitrary - it merely indicates which indices change together

3. Tensor – *(continued)*

Definitions

Scalar product of two tensors

$$\underline{\underline{A}} : \underline{\underline{M}} = A_{ip}\hat{e}_i\hat{e}_p : M_{km}\hat{e}_k\hat{e}_m$$

$$= A_{ip}M_{km}\hat{e}_i\hat{e}_p : \hat{e}_k\hat{e}_m$$

carry out the dot products indicated

$$= A_{ip}M_{km} \quad (\hat{e}_p \cdot \hat{e}_k)(\hat{e}_i \cdot \hat{e}_m)$$

$$= A_{ip}M_{km} \quad \delta_{pk}\delta_{im}$$

"p" becomes "k"
"i" becomes "m"

$$= A_{mk}M_{km}$$

But, what is a tensor really?

A tensor is a handy representation of a *Linear Vector Function*

scalar function: $y = f(x) = x^2 + 2x + 3$

a mapping of values of *x* onto values of *y*

vector function: $\underline{w} = f(\underline{v})$

a mapping of vectors of $\underline{v}$ into vectors $\underline{w}$

How do we express a vector function?

What is a linear function?

Linear, in this usage, has a precise, mathematical definition.

Linear functions (scalar and vector) have the following two properties:

$$f(\lambda x) = \lambda f(x)$$
$$f(x+w) = f(x) + f(w)$$

It turns out . . .

Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).

Tensors are *Linear Vector Functions*

Let $f(\underline{a}) = \underline{b}$ be a linear vector function.

We can write $\underline{a}$ in Cartesian coordinates.

$$\underline{a} = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

$$f(\underline{a}) = f(a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3) = \underline{b}$$

Using the linear properties of f, we can distribute the function action:

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)} + a_2 \underbrace{f(\hat{e}_2)} + a_3 \underbrace{f(\hat{e}_3)} = \underline{b}$$

These results are just vectors, we will name them $\underline{v}$, $\underline{w}$, and $\underline{m}$.

39

Tensors are *Linear Vector Functions* (continued)

$$f(\underline{a}) = a_1 \underbrace{f(\hat{e}_1)}_{\underline{v}} + a_2 \underbrace{f(\hat{e}_2)}_{\underline{w}} + a_3 \underbrace{f(\hat{e}_3)}_{\underline{m}} = \underline{b}$$

$$f(\underline{a}) = a_1\underline{v} + a_2\underline{w} + a_3\underline{m} = \underline{b}$$

Now we note that the coefficients a_i may be written as,

$$a_1 = \underline{a}\cdot\hat{e}_1 \qquad a_2 = \underline{a}\cdot\hat{e}_2 \qquad a_3 = \underline{a}\cdot\hat{e}_3$$

Substituting,

$$f(\underline{a}) = \underline{a}\cdot\hat{e}_1\,\underline{v} + \underline{a}\cdot\hat{e}_2\,\underline{w} + \underline{a}\cdot\hat{e}_3\,\underline{m} = \underline{b}$$

The indeterminate vector product has appeared!

40

Using the distributive law, we can factor out the dot product with $\underline{a}$:

$$f(\underline{a}) = \underline{a} \cdot \underbrace{\left(\hat{e}_1 \underline{v} + \hat{e}_2 \underline{w} + \hat{e}_3 \underline{m}\right)} = \underline{b}$$

This is just a tensor
(the sum of dyadic
products of vectors)

$$\left(\hat{e}_1 \underline{v} + \hat{e}_2 \underline{w} + \hat{e}_3 \underline{m}\right) \equiv \underline{\underline{M}}$$

$$\boxed{f(\underline{a}) = \underline{a} \cdot \underline{\underline{M}} = \underline{b}}$$

CONCLUSION: Tensor operations are convenient to use to express linear vector functions.

3. Tensor – (continued)

More Definitions

Identity Tensor

$$\underline{\underline{I}} = \hat{e}_i \hat{e}_i = \hat{e}_1 \hat{e}_1 + \hat{e}_2 \hat{e}_2 + \hat{e}_3 \hat{e}_3$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}$$

$$\begin{aligned} \underline{\underline{A}} \cdot \underline{\underline{I}} &= A_{ip} \hat{e}_i \hat{e}_p \cdot \hat{e}_k \hat{e}_k \\ &= A_{ip} \hat{e}_i \delta_{pk} \hat{e}_k \\ &= A_{ik} \hat{e}_i \hat{e}_k \\ &= \underline{\underline{A}} \end{aligned}$$

Mathematics Review

Polymer Rheology

3. Tensor – (continued) More Definitions

Zero Tensor

$$\underline{\underline{0}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123}$$

Magnitude of a Tensor

$$\left|\underline{\underline{A}}\right| \equiv +\sqrt{\frac{\underline{\underline{A}}:\underline{\underline{A}}}{2}}$$

$$\begin{aligned}\underline{\underline{A}}:\underline{\underline{A}} &= A_{ip}\hat{e}_i\hat{e}_p : A_{km}\hat{e}_k\hat{e}_m \\ &= A_{ip}A_{km}\left(\hat{e}_p \cdot \hat{e}_k\right)\left(\hat{e}_i \cdot \hat{e}_m\right) \\ &= A_{mk}A_{km}\end{aligned}$$

products across the diagonal

43

Mathematics Review

Polymer Rheology

3. Tensor – (continued) More Definitions

Tensor Transpose

$$\underline{\underline{M}}^T = \left(M_{ik}\hat{e}_i\hat{e}_k\right)^T = M_{ik}\hat{e}_k\hat{e}_i$$

Exchange the coefficients across the diagonal

CAUTION:

$$\begin{aligned}\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T &= \left(A_{ik}\hat{e}_i\hat{e}_k \cdot C_{pj}\hat{e}_p\hat{e}_j\right)^T = \left(A_{ik}C_{pj}\ \hat{e}_i\hat{e}_j\delta_{kp}\right)^T \\ &= \left(A_{ip}C_{pj}\ \hat{e}_i\hat{e}_j\right)^T \\ &= A_{ip}C_{pj}\ \hat{e}_j\hat{e}_i\end{aligned}$$

It is **not** equal to: $\left(\underline{\underline{A}}\cdot\underline{\underline{C}}\right)^T = \left(A_{ip}C_{pj}\ \hat{e}_i\hat{e}_j\right)^T \neq \cancel{A_{pi}C_{jp}\ \hat{e}_i\hat{e}_j}$

I recommend you always interchange the indices on the basis vectors rather than on the coefficients.

44

3. Tensor – (continued) More Definitions

Symmetric Tensor

$$\underline{\underline{M}} = \underline{\underline{M}}^T$$
$$M_{ik} = M_{ki}$$

e.g.

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}_{123}$$

Antisymmetric Tensor

$$\underline{\underline{M}} = -\underline{\underline{M}}^T$$
$$M_{ik} = -M_{ki}$$

e.g.

$$\begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -5 \\ 3 & 5 & 0 \end{pmatrix}_{123}$$

3. Tensor – (continued) More Definitions

Tensor order

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

order = degree of complexity

scalars	0^{th} -order tensors	3^0
vectors	1^{st} -order tensors	3^1
tensors	2^{nd} -order tensors	3^2
higher-order tensors	3^{rd} -order tensors	3^3

Number of coefficients needed to express the tensor in 3D space

3. Tensor – (continued) More Definitions

Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

vectors: $|\underline{v}| = v$

The magnitude of a vector is a scalar associated with the vector

It is independent of coordinate system, i.e. it is an invariant.

tensors: $\underline{\underline{A}}$

There are three invariants associated with a second-order tensor.

Tensor Invariants

$$I_{\underline{\underline{A}}} \equiv trace\underline{\underline{A}} = tr\underline{\underline{A}}$$

For the tensor written in Cartesian coordinates:

$$trace\underline{\underline{A}} = A_{pp} = A_{11} + A_{22} + A_{33}$$

$$II_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}}\right) = \underline{\underline{A}} : \underline{\underline{A}} = A_{pk}A_{kp}$$

$$III_{\underline{\underline{A}}} \equiv trace\left(\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \underline{\underline{A}}\right) = A_{pj}A_{jh}A_{hp}$$

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.

4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

- To carryout the differentiation with respect to a *single variable*, differentiate each coefficient individually.
- There is no change in order (vectors remain vectors, scalars remain scalars, etc.

$$\frac{\partial \alpha}{\partial t} \qquad \frac{\partial \underline{w}}{\partial t} = \begin{pmatrix} \frac{\partial w_1}{\partial t} \\ \frac{\partial w_2}{\partial t} \\ \frac{\partial w_3}{\partial t} \end{pmatrix}_{123} \qquad \frac{\partial \underline{\underline{B}}}{\partial t} = \begin{pmatrix} \frac{\partial B_{11}}{\partial t} & \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{31}}{\partial t} \\ \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{22}}{\partial t} & \frac{\partial B_{23}}{\partial t} \\ \frac{\partial B_{31}}{\partial t} & \frac{\partial B_{32}}{\partial t} & \frac{\partial B_{33}}{\partial t} \end{pmatrix}_{123}$$

4. *Differential Operations with Vectors, Tensors (continued)*

Del Operator

- To carryout the differentiation with respect to *3D spatial variation*, use the del (nabla) operator.
- This is a vector operator
- Del may be applied in three different ways
- Del may operate on scalars, vectors, or tensors

This is written in Cartesian coordinates

$$\nabla \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}_{123}$$

$$= \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \underbrace{\hat{e}_p \frac{\partial}{\partial x_p}}_{\text{Einstein notation for del}}$$

Mathematics Review Polymer Rheology

4. *Differential Operations with Vectors, Tensors (continued)*

A. Scalars - gradient

Gibbs notation

$$\nabla \beta \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\beta + \hat{e}_2 \frac{\partial}{\partial x_2}\beta + \hat{e}_3 \frac{\partial}{\partial x_3}\beta = \begin{pmatrix} \frac{\partial \beta}{\partial x_1} \\ \frac{\partial \beta}{\partial x_2} \\ \frac{\partial \beta}{\partial x_3} \end{pmatrix}_{123}$$

This is written in Cartesian coordinates

Gradient of a scalar field $= \hat{e}_p \frac{\partial \beta}{\partial x_p}$

The gradient of a scalar field is a *vector*

The gradient operation captures the total spatial variation of a scalar, vector, or tensor field.

•gradient operation <u>increases</u> the order of the entity operated upon

51

Mathematics Review Polymer Rheology

4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient

$$\nabla \underline{w} \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\underline{w} + \hat{e}_2 \frac{\partial}{\partial x_2}\underline{w} + \hat{e}_3 \frac{\partial}{\partial x_3}\underline{w}$$

$$= \hat{e}_1 \frac{\partial}{\partial x_1}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_2 \frac{\partial}{\partial x_2}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_3 \frac{\partial}{\partial x_3}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$= \hat{e}_1\hat{e}_1 \frac{\partial w_1}{\partial x_1} + \hat{e}_1\hat{e}_2 \frac{\partial w_2}{\partial x_1} + \hat{e}_1\hat{e}_3 \frac{\partial w_3}{\partial x_1} + \hat{e}_2\hat{e}_1 \frac{\partial w_1}{\partial x_2} +$$
$$\hat{e}_2\hat{e}_2 \frac{\partial w_2}{\partial x_2} + \hat{e}_2\hat{e}_3 \frac{\partial w_3}{\partial x_2} + \hat{e}_3\hat{e}_1 \frac{\partial w_1}{\partial x_3} + \hat{e}_3\hat{e}_2 \frac{\partial w_2}{\partial x_3} + \hat{e}_3\hat{e}_3 \frac{\partial w_3}{\partial x_3}$$

The basis vectors can move out of the derivatives because they are <u>constant</u> (do not change with position)

This is all written in Cartesian coordinates (basis vectors are <u>constant</u>)

52

4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient *(continued)*

Gradient of a vector field

constants may appear on either side of the differential operator

$$\nabla \underline{w} \equiv \sum_{j=1}^{3}\sum_{k=1}^{3} \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \underbrace{\frac{\partial w_k}{\partial x_j} \overbrace{\hat{e}_j \hat{e}_k}}$$

Einstein notation for gradient of a vector

The gradient of a vector field is a *tensor*

4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence

Divergence of a vector field

$$\nabla \cdot \underline{w} \equiv \left(\hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} \right) \cdot w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3$$

$$= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}$$

$$= \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \underbrace{\frac{\partial w_i}{\partial x_i}}$$

Einstein notation for divergence of a vector

The Divergence of a vector field is a *scalar*

Mathematics Review Polymer Rheology

4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence *(continued)*

This is all written in Cartesian coordinates (basis vectors are constant)

constants may appear on either side of the differential operator

Using Einstein notation

$$\nabla \cdot \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot w_j \hat{e}_j = \frac{\partial w_j}{\partial x_m} \hat{e}_m \cdot \hat{e}_j = \frac{\partial w_j}{\partial x_m} \delta_{mj}$$

$$= \frac{\partial w_j}{\partial x_j}$$

•divergence operation <u>decreases</u> the order of the entity operated upon

55

Mathematics Review Polymer Rheology

4. *Differential Operations with Vectors, Tensors (continued)*

D. Vectors - Laplacian

Using Einstein notation:

$$\nabla \cdot \nabla \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot \hat{e}_p \frac{\partial}{\partial x_p} w_j \hat{e}_j = \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\hat{e}_m \cdot \hat{e}_p\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\delta_{mp}\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} w_j \hat{e}_j$$

The Laplacian of a vector field is a *vector*

$$= \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_2} + \frac{\partial^2 w_1}{\partial x_3} \\ \frac{\partial^2 w_2}{\partial x_1} + \frac{\partial^2 w_2}{\partial x_2} + \frac{\partial^2 w_2}{\partial x_3} \\ \frac{\partial^2 w_3}{\partial x_1} + \frac{\partial^2 w_3}{\partial x_2} + \frac{\partial^2 w_3}{\partial x_3} \end{pmatrix}_{123}$$

•Laplacian operation does not change the order of the entity operated upon

56

4. *Differential Operations with Vectors, Tensors (continued)*

E. Scalar - divergence $\cancel{\nabla \cdot} \alpha$ *(impossible; cannot decrease order of a scalar)*

F. Scalar - Laplacian $\nabla \cdot \nabla \alpha$

G. Tensor - gradient $\nabla \underline{\underline{A}}$

H. Tensor - divergence $\nabla \cdot \underline{\underline{A}}$

I. Tensor - Laplacian $\nabla \cdot \nabla \underline{\underline{A}}$

5. Curvilinear Coordinates

Cylindrical	$\bar{r}, \theta, z$	$\hat{e}_{\bar{r}}, \hat{e}_\theta, \hat{e}_z$
Spherical	r, θ, ϕ	$\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$

See figures 2.11 and 2.12

These coordinate systems are ortho-normal, *but they are not constant* (they vary with position).

This causes some non-intuitive effects when derivatives are taken.

Mathematics Review Polymer Rheology

5. Curvilinear Coordinates (continued)

$$\underline{v} = v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z$$

$$\nabla\cdot\underline{v} = \nabla\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$= \underbrace{\left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)}_{\text{First, we need to write this in cylindrical coordinates.}}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

solve for Cartesian basis vectors and substitute above

$$\left\{\begin{aligned} \hat{e}_r &= \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y \\ \hat{e}_\theta &= -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y \\ \hat{e}_z &= \hat{e}_z \end{aligned}\right.$$

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

substitute above using chain rule (see next slide for details)

59

Mathematics Review Polymer Rheology

$$\hat{e}_x = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$$
$$\hat{e}_y = \sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta$$

$$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}\hat{e}_x + \frac{\partial\psi}{\partial y}\hat{e}_y + \frac{\partial\psi}{\partial z}\hat{e}_z\right)$$

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial\psi}{\partial r}\cos\theta + \frac{\partial\psi}{\partial\theta}\left(\frac{-\sin\theta}{r}\right)$$

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial\psi}{\partial r}\sin\theta + \frac{\partial\psi}{\partial\theta}\left(\frac{\cos\theta}{r}\right)$$

60

Mathematics Review Polymer Rheology

5. Curvilinear Coordinates (continued)

Result:
$$\nabla = \left(\frac{\partial}{\partial x}\hat{e}_x + \frac{\partial}{\partial y}\hat{e}_y + \frac{\partial}{\partial z}\hat{e}_z\right)$$
$$= \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Now, proceed:

$$\nabla\cdot\underline{v} = \left(\hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$
$$= \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

(We cannot use Einstein notation because these are not Cartesian coordinates)

61

Mathematics Review Polymer Rheology

5. Curvilinear Coordinates (continued)

$$\nabla\cdot\underline{v} = \hat{e}_r\frac{\partial}{\partial r}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right) +$$
$$\hat{e}_z\frac{\partial}{\partial z}\cdot\left(v_r\hat{e}_r + v_\theta\hat{e}_\theta + v_z\hat{e}_z\right)$$

$$\hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}\cdot v_r\hat{e}_r = \hat{e}_\theta\cdot\frac{1}{r}\frac{\partial v_r\hat{e}_r}{\partial \theta}$$
$$= \hat{e}_\theta\cdot\frac{1}{r}\left(v_r\frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_r\frac{\partial v_r}{\partial \theta}\right)$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y\right)$$
$$= -\sin\theta\,\hat{e}_x + \cos\theta\,\hat{e}_y$$
$$= \hat{e}_\theta$$

62

5. Curvilinear Coordinates (continued)

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot v_r \hat{e}_r = \hat{e}_\theta \cdot \frac{1}{r}\frac{\partial v_r \hat{e}_r}{\partial \theta}$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \frac{\partial \hat{e}_r}{\partial \theta} + \hat{e}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \hat{e}_\theta \cdot \frac{1}{r}\left(v_r \hat{e}_\theta + \hat{e}_r \frac{\partial v_r}{\partial \theta} \right)$$

$$= \frac{1}{r} v_r$$

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position.

5. Curvilinear Coordinates (continued)

Final result for divergence of a vector in cylindrical coordinates:

$$\nabla \cdot \underline{v} = \hat{e}_r \frac{\partial}{\partial r} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) +$$

$$\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) +$$

$$\hat{e}_z \frac{\partial}{\partial z} \cdot \left(v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right)$$

$$\boxed{\nabla \cdot \underline{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z}}$$

5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

- The basis vectors are ortho-normal
- The basis vectors are non-constant (vary with position)
- These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
- *We cannot use Einstein notation* – must use Tables in Appendix C2 (pp464-468).

6. Vector and Tensor Theorems and definitions

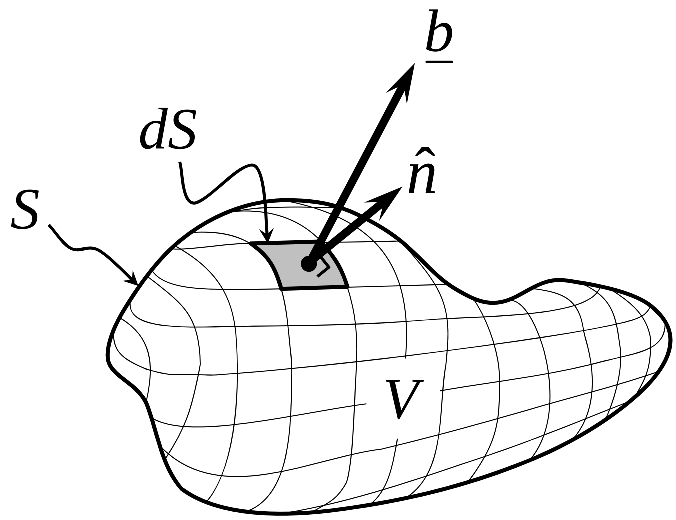
In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

Gauss Divergence Theorem

$$\iiint_V \nabla \cdot \underline{b}\, dV = \iint_S \hat{n} \cdot \underline{b}\, dS$$

outwardly directed unit normal

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.



6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

constant limits

$$I = \int_{\alpha}^{\beta} f(x,t)\, dx$$

$$\frac{dI}{dt} = \frac{d}{dt}\int_{\alpha}^{\beta} f(x,t)\, dx$$

$$= \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t}\, dx$$

one dimension, constant limits

6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx$$

variable limits

$$\frac{dJ}{dt} = \frac{d}{dt}\int_{\alpha(t)}^{\beta(t)} f(x,t)\, dx$$

$$= \int_{\alpha(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t}\, dx + \frac{d\beta}{dt} f(\beta,t) - \frac{d\alpha}{dt} f(\alpha,t)$$

one dimension, *variable* limits

6. Vector and Tensor Theorems *(continued)*

Leibnitz Rule for differentiating integrals

$$J = \iiint_{V(t)} f(x,y,z,t)\, dV$$

$$\frac{dJ}{dt} = \frac{d}{dt}\iiint_{V(t)} f(x,y,z,t)\, dV$$

$$= \iiint_{V(t)} \frac{\partial f(x,y,z,t)}{\partial t}\, dV + \iint_{S(t)} f\left(\underline{v}_{surface} \cdot \hat{\underline{n}}\right) dS$$

velocity of the surface element *dS*

three dimensions, *variable* limits

6. Vector and Tensor Theorems *(continued)*

Substantial Derivative

Consider a function $f(x,y,z,t)$

true for any path:

$$df \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} dx + \left(\frac{\partial f}{\partial y}\right)_{xzt} dy + \left(\frac{\partial f}{\partial z}\right)_{xyt} dz + \left(\frac{\partial f}{\partial t}\right)_{xyz} dt$$

choose special path:

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

time rate of change of *f* along a chosen path

x-component of velocity along that path

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

6. Vector and Tensor Theorems *(continued)* **Substantial Derivative**

When the chosen path is the path of a fluid particle, then the space derivatives are the components of the particle velocities.

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial x}\right)_{yzt} \frac{dx}{dt} + \left(\frac{\partial f}{\partial y}\right)_{xzt} \frac{dy}{dt} + \left(\frac{\partial f}{\partial z}\right)_{xyt} \frac{dz}{dt} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

$$\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \underbrace{\left(\frac{\partial f}{\partial x}\right)_{yzt} v_x + \left(\frac{\partial f}{\partial y}\right)_{xzt} v_y + \left(\frac{\partial f}{\partial z}\right)_{xyt} v_z}_{\underline{v}\cdot\nabla f} + \left(\frac{\partial f}{\partial t}\right)_{xyz}$$

Substantial Derivative

$$\boxed{\left(\frac{df}{dt}\right)_{\substack{along\\ a\ particle\\ path}} \equiv \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f}$$