Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

- Derive governing equations (mass and momentum balances)
- Solve governing equations for velocity and stress fields

QUICK START

First, before we get deep into derivation, let's do a Navier-Stokes problem to get you started in the mechanics of this type of problem solving.

EXAMPLE: Drag flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Very wide, long
- Uniform pressure

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \]
Chapter 3: Newtonian Fluid Mechanics

TWO GOALS

• Derive governing equations (mass and momentum balances)
• Solve governing equations for velocity and stress fields

**Mass Balance**

Consider an arbitrary control volume \( V \) enclosed by a surface \( S \)

\[
\begin{align*}
\left( \text{rate of increase} \right) &= \left( \text{net flux of mass into } CV \right) \\
\left( \text{of mass in } CV \right) &= \left( \text{net flux of mass into } CV \right)
\end{align*}
\]
Chapter 3: Newtonian Fluid Mechanics

Mass Balance (continued)

Consider an arbitrary volume $V$ enclosed by a surface $S$.

\[
\text{rate of increase of mass in } V = \frac{d}{dt} \left( \iiint_V \rho \, dV \right)
\]

\[
\text{net flux of mass into } V \text{ through surface } S = -\iint_S \rho \hat{n} \cdot \mathbf{v} \, dS
\]

(continued)

Leibnitz rule

\[
\frac{d}{dt} \left( \iiint_V \rho \, dV \right) = -\iint_S \rho \hat{n} \cdot \mathbf{v} \, dS
\]

\[
\iint_V \frac{\partial \rho}{\partial t} \, dV = -\iint_S \hat{n} \cdot (\rho \mathbf{v}) \, dS
\]

Gauss Divergence Theorem

\[
\iiint_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \, dV = 0
\]
Mass Balance (continued)

Since \( V \) is arbitrary,

\[
\iiint_V \left( \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0
\]

Continuity equation: microscopic mass balance

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

Continuity equation (general fluids)

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot \nabla \rho = 0
\]

\[
\frac{D \rho}{D t} + \rho (\nabla \cdot \mathbf{v}) = 0
\]

For \( \rho = \text{constant} \) (incompressible fluids):

\[
\nabla \cdot \mathbf{v} = 0
\]
Consider an arbitrary control volume $V$ enclosed by a surface $S$.

Momentum is conserved.

\[
\text{(rate of increase of momentum in CV)} = \text{(net flux of momentum into CV)} + \text{(sum of forces on CV)}
\]

- resembles the rate term in the mass balance
- resembles the flux term in the mass balance
- Forces: body (gravity) molecular forces

\[
\text{Momentum Balance}
\]

\[
\frac{d}{dt} \int_V \rho \mathbf{v} \, dV = \int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS + \int_{CV} \mathbf{F} \cdot d\mathbf{a}
\]
Momentum Balance

\[ \left( \text{rate of increase} \right) \left( \text{of momentum in} \ V \right) = \frac{d}{dt} \left( \iiint_V \rho \mathbf{v} \, dV \right) = \iiint_V \frac{\partial}{\partial t} (\rho \mathbf{v}) \, dV \]

Leibnitz rule

\[ \left( \text{net flux of} \right) \left( \text{momentum into} \ V \right) = -\iiint_S \mathbf{n} \cdot (\rho \mathbf{v}) \, dS = -\iiint_V \nabla \cdot (\rho \mathbf{v}) \, dV \]

Gauss Divergence Theorem

Forces on \( V \)

Body Forces (non-contact)

\[ \left( \text{force on} \ V \right) \left( \text{due to} \ g \right) = \iiint_V \rho g \, dV \]
Molecular Forces (contact) – this is the tough one

\[ f = \frac{\text{stress at } P}{\text{on } dS} \]

choose a surface through \( P \)

the force on that surface

We need an expression for the state of stress at an arbitrary point \( P \) in a flow.

Molecular Forces (continued)

Think back to the molecular picture from chemistry:

The specifics of these forces, connections, and interactions must be captured by the molecular forces term that we seek.
Molecular Forces  (continued)

• We will concentrate on expressing the molecular forces mathematically;
• We leave to later the task of relating the resulting mathematical expression to experimental observations.

First, choose a surface:
• arbitrary shape
• small

\[ \text{stress at } P \text{ on } dS = f \]

What is \( f \)?

Consider the forces on three mutually perpendicular surfaces through point \( P \):
Molecular Forces (continued)

\[ a = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3 \]
\[ = \Pi_{11} \hat{e}_1 + \Pi_{12} \hat{e}_2 + \Pi_{13} \hat{e}_3 \]
\[ b = b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3 \]
\[ = \Pi_{21} \hat{e}_1 + \Pi_{22} \hat{e}_2 + \Pi_{23} \hat{e}_3 \]
\[ c = c_1 \hat{e}_1 + c_2 \hat{e}_2 + c_3 \hat{e}_3 \]
\[ = \Pi_{31} \hat{e}_1 + \Pi_{32} \hat{e}_2 + \Pi_{33} \hat{e}_3 \]

\[ \Pi_{pk} \]

So far, this is nomenclature; next we relate these expressions to force on an arbitrary surface.
Molecular Forces (continued)

How can we write \( f \) (the force on an arbitrary surface \( dS \)) in terms of the \( \Pi_{pk} \)?

\[
f = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3
\]

- \( f_1 \) is force on \( dS \) in 1-direction
- \( f_2 \) is force on \( dS \) in 2-direction
- \( f_3 \) is force on \( dS \) in 3-direction

There are three \( \Pi_{pk} \) that relate to forces in the 1-direction:

\( \Pi_{11}, \Pi_{21}, \Pi_{31} \)

\( \hat{n} \cdot \hat{e}_1 dS \)
**Molecular Forces (continued)**

$f_i$, the force on $dS$ in 1-direction, is composed of THREE parts:

1. **first part:**
   \[
   (\Pi_{11}) \begin{bmatrix}
   \text{projection of} \\
   dA \text{ onto the} \\
   1-\text{surface}
   \end{bmatrix} = \Pi_{11}\hat{n} \cdot \hat{e}_1 \, dS
   \]

2. **second part:**
   \[
   (\Pi_{21}) \begin{bmatrix}
   \text{projection of} \\
   dA \text{ onto the} \\
   2-\text{surface}
   \end{bmatrix} = \Pi_{21}\hat{n} \cdot \hat{e}_2 \, dS
   \]

3. **third part:**
   \[
   (\Pi_{31}) \begin{bmatrix}
   \text{projection of} \\
   dA \text{ onto the} \\
   3-\text{surface}
   \end{bmatrix} = \Pi_{31}\hat{n} \cdot \hat{e}_3 \, dS
   \]

The sum of these three $= f_i$

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Molecular Forces (continued)

$f_1$, the force in the 1-direction on an arbitrary surface $dS$ is composed of THREE parts.

$$f_1 = \Pi_{11}\hat{n} \cdot \hat{e}_1 \, dS + \Pi_{21}\hat{n} \cdot \hat{e}_2 \, dS + \Pi_{31}\hat{n} \cdot \hat{e}_3 \, dS$$

Using the distributive law:

$$f_1 = \hat{n} \cdot \left( \Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3 \right) \, dS$$

Force in the 1-direction on an arbitrary surface $dS$

The same logic applies in the 2-direction and the 3-direction.

$$f_2 = \hat{n} \cdot \left( \Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3 \right) \, dS$$

$$f_3 = \hat{n} \cdot \left( \Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3 \right) \, dS$$

Assembling the force vector:

$$\mathbf{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3$$

$$= dS \, \hat{n} \cdot \left( \Pi_{11}\hat{e}_1 + \Pi_{21}\hat{e}_2 + \Pi_{31}\hat{e}_3 \right) \hat{e}_1$$

$$+ dS \, \hat{n} \cdot \left( \Pi_{12}\hat{e}_1 + \Pi_{22}\hat{e}_2 + \Pi_{32}\hat{e}_3 \right) \hat{e}_2$$

$$+ dS \, \hat{n} \cdot \left( \Pi_{13}\hat{e}_1 + \Pi_{23}\hat{e}_2 + \Pi_{33}\hat{e}_3 \right) \hat{e}_3$$
Assembling the force vector:

\[ \mathbf{f} = f_1 \hat{\mathbf{e}}_1 + f_2 \hat{\mathbf{e}}_2 + f_3 \hat{\mathbf{e}}_3 \]

\[ = dS \mathbf{n} \cdot \bigg( \Pi_{11} \hat{\mathbf{e}}_1 + \Pi_{21} \hat{\mathbf{e}}_2 + \Pi_{31} \hat{\mathbf{e}}_3 \bigg) \hat{\mathbf{e}}_1 \\
+ dS \mathbf{n} \cdot \bigg( \Pi_{12} \hat{\mathbf{e}}_1 + \Pi_{22} \hat{\mathbf{e}}_2 + \Pi_{32} \hat{\mathbf{e}}_3 \bigg) \hat{\mathbf{e}}_2 \\
+ dS \mathbf{n} \cdot \bigg( \Pi_{13} \hat{\mathbf{e}}_1 + \Pi_{23} \hat{\mathbf{e}}_2 + \Pi_{33} \hat{\mathbf{e}}_3 \bigg) \hat{\mathbf{e}}_3 \]

\[ = dS \mathbf{n} \cdot \bigg[ \Pi_{11} \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \Pi_{21} \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_1 + \Pi_{31} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_1 \\
+ \Pi_{12} \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_2 + \Pi_{22} \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \Pi_{32} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_2 \\
+ \Pi_{13} \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_3 + \Pi_{23} \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_3 + \Pi_{33} \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3 \bigg] \]

linear combination of
dyadic products = tensor

Total stress tensor
(molecular stresses)
Momentum Balance (continued)  

\[
\left( \frac{\text{rate of increase}}{\text{of momentum in } V} \right) = \left( \text{net flux of momentum into } V \right) + \left( \text{sum of forces on } V \right) 
\]

\[
\iiint_v \frac{\partial}{\partial t} (\rho \mathbf{v}) dV = -\iiint_v \nabla \cdot (\rho \mathbf{v}) dV + \iiint_v \rho \mathbf{g} dV + \text{molecular forces} 
\]

\[
\text{molecular forces} = \iiint_s \left( \text{molecular forces on } dS \right) 
\]

\[
= \iiint_s \mathbf{n} \cdot \left( -\tau \right) dS 
\]

\[
= \iiint_v \nabla \cdot \left( -\tau \right) dV 
\]

We use a stress sign convention that requires a negative sign here.

Gauss Divergence Theorem

UR/Bird choice: positive compression (pressure is positive)

Gauss Divergence Theorem

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Momentum Balance

\[ \mathbf{F}_{on} = \sum_{S} \mathbf{n} \cdot (-\mathbf{\Pi}) dS = \sum_{S} \mathbf{n} \cdot (\mathbf{\Pi}) dS \]

\[ \mathbf{\Pi}_{yx} \quad \mathbf{\Pi}_{yx} \]

UR/Bird choice: fluid at lesser \( y \) exerts force on fluid at greater \( y \)

(IFM/Mechanics choice: (opposite))

Final Assembly:

\[ \frac{\partial}{\partial t} (\rho \mathbf{v}) dV = -\nabla \cdot (\rho \mathbf{v} \mathbf{v}) dV + \sum \rho \mathbf{g} dV - \sum \nabla \cdot \mathbf{\Pi} dV \]

\[ \sum \left[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \mathbf{\rho} \mathbf{g} + \nabla \cdot \mathbf{\Pi} \right] dV = 0 \]

Because \( V \) is arbitrary, we may conclude:

Microscopic momentum balance

\[ \frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) - \mathbf{\rho} \mathbf{g} + \nabla \cdot \mathbf{\Pi} = 0 \]
Momentum Balance (continued)  

Microscopic momentum balance  

\[ \frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho v v) - \rho g + \nabla \cdot \Pi = 0 \]

After some rearrangement:

\[ \rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla \cdot \Pi + \rho g \]

Equation of Motion  

\[ \rho \frac{Dv}{Dt} = -\nabla \cdot \Pi + \rho g \]

Now, what to do with \( \Pi \)?

Pressure

definition: An isotropic force/area of molecular origin. Pressure is the same on any surface drawn through a point and acts normally to the chosen surface.

\[ \text{pressure} = p \mathbf{I} = p \hat{e}_1 \hat{e}_1 + p \hat{e}_2 \hat{e}_2 + p \hat{e}_3 \hat{e}_3 = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}_{123} \]

Test: what is the force on a surface with unit normal \( \hat{n} \)?
**Momentum Balance**  
(continued)  
Polymer Rheology

*back to our question,*

Now, what to do with $\Pi$?  
Pressure is part of it.  
There are other, nonisotropic stresses

**Extra Molecular Stresses**

*definition:* The extra stresses are the molecular stresses that are not isotropic

$$\tau \equiv \Pi - p I$$

Extra stress tensor, i.e. everything complicated in molecular deformation

Now, what to do with $\tau$?  
This becomes the central question of rheological study

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**Momentum Balance**  
(continued)  
Polymer Rheology

Stress sign convention affects any expressions with $\Pi, \tilde{\Pi}$ or $\tau, \tilde{\tau}$

$$\Pi \equiv \tau + p I$$

$\tilde{\Pi} \equiv \tilde{\tau} - p I$

<table>
<thead>
<tr>
<th>UR/Bird choice: fluid at lesser $y$ exerts force on fluid at greater $y$</th>
</tr>
</thead>
<tbody>
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<td>(IFM/Mechanics choice: (opposite))</td>
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Constitutive equations for Stress

- are tensor equations
- relate the velocity field to the stresses generated by molecular forces
- are based on observations (empirical) or are based on molecular models (theoretical)
- are typically found by trial-and-error
- are justified by how well they work for a system of interest
- are observed to be symmetric

Observation: the stress tensor is symmetric

Microscopic momentum balance

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

Equation of Motion

In terms of the extra stress tensor:

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

Equation of Motion

Cauchy Momentum Equation

Components in three coordinate systems (our sign convention):

Newtonian Constitutive equation

\[ \tau = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

• for incompressible fluids (see text for compressible fluids)
• is empirical
• may be justified for some systems with molecular modeling calculations

Note: \[ \cdot = +\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

How is the Newtonian Constitutive equation related to Newton’s Law of Viscosity?

\[ \tau = -\mu \left( \nabla \mathbf{v} + (\nabla \mathbf{v})^T \right) \]

\[ \tau_{21} = -\mu \frac{\partial \mathbf{v}_1}{\partial x_2} \]

• incompressible fluids
• rectilinear flow (straight lines)
• no variation in \( x_2 \)-direction
Back to the momentum balance . . .

\[ \rho \left( \frac{\partial \textbf{v}}{\partial t} + \textbf{v} \cdot \nabla \textbf{v} \right) = -\nabla p - \nabla \cdot \tau + \rho \textbf{g} \]

\[ \tau = -\mu \left( \nabla \textbf{v} + (\nabla \textbf{v})^T \right) \]

We can incorporate the Newtonian constitutive equation into the momentum balance to obtain a momentum-balance equation that is specific to incompressible, Newtonian fluids.

Navier-Stokes Equation

\[ \rho \left( \frac{\partial \textbf{v}}{\partial t} + \textbf{v} \cdot \nabla \textbf{v} \right) = -\nabla p + \mu \nabla^2 \textbf{v} + \rho \textbf{g} \]

- incompressible fluids
- Newtonian fluids

Note: The Navier-Stokes is unaffected by the stress sign convention.
Navier-Stokes Equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}
\]

Newtonian Problem Solving

EXAMPLE: Drag flow between infinite parallel plates

- Newtonian
- steady state
- incompressible fluid
- very wide, long
- uniform pressure

from QUICK START
EXAMPLE: Poiseuille flow between infinite parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- Infinitely wide, long

EXAMPLE: Poiseuille flow in a tube

- Newtonian
- Steady state
- Incompressible fluid
- Long tube
EXAMPLE: Torsional flow between parallel plates

- Newtonian
- Steady state
- Incompressible fluid
- \( \nu = zf(r) \)