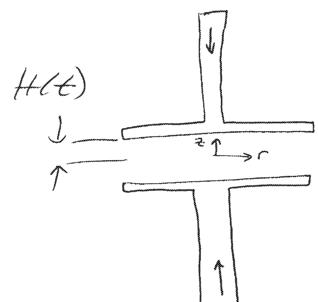
PROBLEM: 7.44 Flow Problem: Squeeze flow An incompressible Newtonian fluid fills the gap between two parallel disks of radius R (Figure 7.63). The disks are subjected to axial forces that cause them to squeeze together. The fluid in the gap responds by producing a combined axial and radial flow that pushes fluid out of the gap. The flow may be assumed to be symmetrical and the azimuthal direction (i.e., no θ variation). A pressure gradient develops in the radial direction; the pressure at the center is p_0 and the pressure at the rim is p_R. Calculate the steady state velocity profile and the radial pressure distribution. If the plates are moving with speed V, calculate the force needed to maintain the motion.

SOLUTION



Assume quasi steady state

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} v_r \\ 0 \\ v_z \end{pmatrix}_{r\theta z}$$

(See also Dynamics of Polymeric Liquids, Bird et al Example 1.3-5)

θ Symmetry $\Rightarrow \boxed{\frac{\partial}{\partial \theta} = 0}$

Continuity: $\nabla \cdot \underline{v} = 0$
(Table B.5)

$$0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$$

momentum: (neglect gravity)
(Table B.6)

r-component:

$$\rho v_r \frac{\partial v_r}{\partial r} + \rho v_z \frac{\partial v_r}{\partial z} = -\frac{\partial P}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \mu \frac{\partial^2 v_r}{\partial z^2}$$

θ-component:

$$0 = -\frac{1}{r}\frac{\partial P}{\partial \theta}$$

z-component:

$$\rho v_r \frac{\partial v_z}{\partial r} + \rho v_z \frac{\partial v_z}{\partial z} = -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \mu \frac{\partial^2 v_z}{\partial z^2}$$

$$\boxed{\frac{\partial P}{\partial \theta} = 0} \qquad P = P(r, z)$$

The other two components are too complex to solve.

Assumptions:

- neglect inertia, i.e. assume slow, viscous-dominated flow

EOM:

$$0 = -\frac{\partial P}{\partial r} + \mu \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right) + \mu \frac{\partial^2 V_r}{\partial z^2}$$

$$0 = -\frac{\partial P}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r}\right) + \mu \frac{\partial^2 V_z}{\partial z^2}$$

V_r is substantial; it is a function of r, z

V_z is nontrivial; it is a function of z, but perhaps a weak function of r.

Assume: $v_z = v_z(z)$

z-component EOM:

$$\frac{\partial P}{\partial z} = \mu \underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)}_{0} + \mu \frac{\partial^2 v_z}{\partial z^2}$$

only a function of z

$$\Rightarrow \quad \frac{\partial^2 v_z}{\partial z^2} = \frac{1}{\mu}\frac{\partial P}{\partial z} = \text{function of } z \text{ only}$$

$$\frac{dv_z}{dz} \equiv \phi(z)$$

Continuity: $$0 = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z}$$

$$\frac{\partial}{\partial r}(r v_r) = -\phi r$$

$$r v_r = -\phi\frac{r^2}{2} + f(z)$$

$$\boxed{v_r = -\frac{\phi(z)}{2} r + \cancel{\frac{f(z)}{r}}}$$

BC: at $r=0$ v_r = finite
$\Rightarrow f(z) = 0$

Back to EOM: (r-component)

$$\frac{\partial P}{\partial r} = \mu \underbrace{\cancel{\frac{\partial}{\partial r}}}_{0}\left(\frac{1}{r}\underbrace{\frac{\partial}{\partial r}(r V_r)}_{-\phi(z)\, r}\right) + \mu \frac{\partial^2 V_r}{\partial z^2}$$

$$= \mu \frac{\partial^2}{\partial z^2}\left(-\phi(z)\frac{r}{2}\right)$$

$$\frac{\partial P}{\partial r} = -\frac{r\mu}{2}\frac{d^2\phi}{dz^2}$$

$$\boxed{P(r,z) = -\frac{\mu}{2}\frac{d^2\phi}{dz^2}\frac{r^2}{2} + \xi(z)}$$

We previously saw that $\frac{\partial P}{\partial z}$ is only a function of z; $\therefore$

$$\frac{\partial P}{\partial z} = \frac{-\mu r^2}{4}\frac{d^3\phi}{dz^3} + \frac{d\xi}{dz}$$

This is only a function of z if

$$\frac{d^3\phi}{dz^3} = 0$$

$$\frac{d^2\phi}{dz^2} = C_1$$

NOTE:

$$\frac{d\phi}{dz} = C_1 z + C_2 = \frac{\partial V_z}{\partial z}$$

$$\boxed{\phi = C_1 \frac{z^2}{2} + \underbrace{C_2}_{0} z + C_3}$$

see below

Recall $\quad V_r = -\frac{\phi}{2} r$

$$\boxed{V_r = -\frac{r}{2}\left(C_1 \frac{z^2}{2} + C_2 z + C_3\right)}$$

Boundary conditions:

$z = 0 \quad \frac{\partial V_r}{\partial z} = 0$

(symmetric)

$$\frac{\partial V_r}{\partial z} = -\frac{r}{2}\left(C_1 \overset{0}{\overset{\shortparallel}{z}} + C_2\right) = 0$$

$$\Rightarrow \boxed{C_2 = 0}$$

$z = \pm h \quad V_r = 0$
(no slip)

$$0 = -\frac{r}{2}\left(c_1 \frac{h^2}{2} + c_3\right)$$

$$\boxed{c_3 = -\frac{c_1 h^2}{2}}$$

$$\boxed{v_r = -\frac{r}{4} c_1 \left(z^2 - h^2\right)}$$

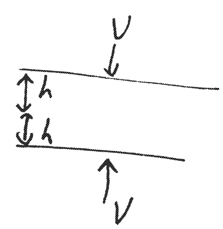
Back to v_z:

$$\frac{dV_z}{dz} = \phi(z) = c_1 \frac{z^2}{2} - \frac{c_1 h^2}{2}$$

$$V_z = \frac{c_1}{2}\left(\frac{z^3}{3} - h^2 z\right) + c_4$$

Boundary Conditions:

$z = \pm h \qquad V_z = \mp V$



$-h \qquad V = \frac{c_1}{2}\left(-\frac{h^3}{3} + h^3\right) + c_4$

$h \qquad -V = \frac{c_1}{2}\left(\frac{h^3}{3} - h^3\right) + c_4$

ADD

$$0 = 2C_4 \Rightarrow \boxed{C_4 = 0}$$

SUBTRACT

$$2V = \frac{C_1}{2}\left(-\frac{2}{3}h^3 + 2h_3\right)$$

$$\cancel{2}V = \frac{C_1}{\cancel{2}}\frac{\cancel{4}}{3}h^3$$

$$\boxed{C_1 = \frac{3V}{h^3}}$$

$$V_z = \frac{3V}{h^3 2}\left(\frac{z^3}{3} - h^2 z\right)$$

$$\boxed{\frac{V_z}{V} = \frac{3}{2}\left(\frac{z^3}{h^3}\frac{1}{3} - \frac{z}{h}\right)}$$

$$V_r = -\frac{r}{4}C_1\left(z^2 - h^2\right)$$

$$= -\frac{r}{4}\frac{3V}{h^3}\left(z^2 - h^2\right)$$

$$\boxed{\frac{V_r}{V} = \frac{-3}{4}\frac{r}{h}\left(\frac{z^2}{h^2} - 1\right)}$$

For the pressure:

$$P = -\frac{\mu}{2}\frac{d^2\phi}{dz^2}\frac{r^2}{2} + \xi(z)$$

$$\frac{\partial P}{\partial z} = \frac{d\xi}{dz} = \overbrace{\mu\frac{\partial^2 V_z}{\partial z^2}}^{\text{from } z\text{-component EOM}}$$

$$\frac{\partial V_z}{\partial z} = \frac{3}{2}V\left(\frac{1}{3h^3}3z^2 - \frac{1}{h}\right)$$

$$\frac{\partial^2 V_z}{\partial z^2} = \frac{3}{\cancel{2}}V\left(\frac{1}{\cancel{3}h^3}\cancel{3}\cancel{2}z\right)$$

$$= \frac{3Vz}{h^3}$$

$$\frac{d\xi}{dz} = \mu\frac{3V}{h^3}z$$

$$\boxed{\xi = \frac{3\mu V}{h^3}\frac{z^2}{2} + C_5}$$

$$\frac{d\phi}{dz} = \frac{\cancel{2}C_1}{\cancel{2}}z$$

$$\boxed{\frac{d^2\phi}{dz^2} = C_1 = \frac{3V}{h^3}}$$

$$P = -\frac{\mu}{2}\frac{r^2}{2}\frac{3V}{h^3} + \frac{3\mu V}{2h^3} z^2 + C_5$$

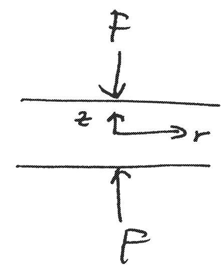
$$P = \frac{3\mu V}{h^3}\left(-\frac{r^2}{4} + \frac{1}{2} z^2\right) + C_5$$

B.C. $r = R \quad z = 0 \quad P = P_R$

$$\Rightarrow \frac{3\mu V}{h^3}\left(-\frac{R^2}{4}\right) + C_5 = P_R$$

$$\boxed{P - P_R = \frac{3\mu V}{h^3}\left(\frac{R^2 - r^2}{4} + \frac{z^2}{2}\right)}$$

Force to maintain the motion:



$$\underline{F} = \iint_S \left(\hat{n}\cdot\underline{\underline{\Pi}}\right)_{\text{surface}} dS$$

Top surface: $z = h$

$$\hat{n} = -\hat{e}_z$$

$$\underline{\underline{\Pi}} = -p\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$= -p\underline{\underline{I}} + \left(\nabla \underline{v} + (\nabla \underline{v})^T\right)\mu$$

(Table B.8)

$$\hat{n}\cdot\underline{\underline{\Pi}} = (0 \quad 0 \quad -1)_{r\theta z} \begin{pmatrix} 2\frac{\partial v_r}{\partial r} - p & 0 & \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \\ 0 & 2\frac{v_r}{r} - p & 0 \\ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} & 0 & 2\frac{\partial v_z}{\partial z} - p \end{pmatrix}_{r\theta z}$$

$$\hat{n}\cdot\underline{\underline{\Pi}} = \begin{pmatrix} -\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \\ 0 \\ -2\frac{\partial v_z}{\partial z} + p \end{pmatrix}_{r\theta z}$$

← radial component $\hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$ (integrates to zero)

← z-direction force

$$F_z = \int_0^{2\pi}\int_0^{R} \left(P - 2\,\frac{\partial v_z}{\partial z}\right)\Bigg|_{z=h} r\,dr\,d\theta$$

$$\frac{\partial v_z}{\partial z} = \frac{3}{2}V\left(\frac{z^2}{h^3} - \frac{1}{h}\right)$$

$$\frac{\partial v_z}{\partial z}\bigg|_{z=h} = \left(\frac{1}{h} - \frac{1}{h}\right)\frac{3V}{2} = 0$$

$$F_z = 2\pi\int_0^{R} rP_R + \frac{3\mu V}{4h^3}(R^2 - r^2)r + \frac{3}{2}\frac{\mu V}{\cancel{h^3}}\cancel{h^2}\,r\,dr$$

$$\frac{F_z}{2\pi} = P_R\frac{r^2}{2}\bigg|_0^{R} + \frac{3\mu V}{4h^3}\left(R^2\frac{r^2}{2} - \frac{r^4}{4}\right)\bigg|_0^{R}$$

$$+\frac{3}{2}\frac{\mu V}{h}\frac{r^2}{2}\bigg|_0^{R}$$

$$F_z = P_R\frac{R^2}{\cancel{2}}\cancel{2}\pi + \frac{3}{\cancel{4}_{2}}\frac{\mu V\cancel{\pi}}{h^3}\left(\frac{R^4}{2} - \frac{R^4}{4}\right) + \frac{3}{\cancel{4}_{2}}\frac{\mu V R^2}{h}\cancel{2}\pi$$

$$\boxed{F_z = P_R\pi R^2 + \frac{3}{8}\frac{\mu V\pi R^4}{h^3} + \frac{3\mu V R^2\pi}{2h}}$$

Check soln:

$$\frac{\partial p}{\partial r} = \frac{3\mu V}{h^3}\,\frac{1}{\cancel{4}_2}(-1)(\cancel{2}r)$$

$$\frac{\partial v_r}{\partial z} = -\frac{3}{\cancel{4}_2}V\,\frac{r}{h}\,\frac{1}{h^2}\,\cancel{2}z$$

$$\frac{\partial^2 v_r}{\partial z^2} = -\frac{3}{2}\,\frac{Vr}{h^3}$$

$$r v_r = -\frac{3V}{4h}\,r^2\left(\frac{z^2}{h^2}-1\right)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) = \frac{-3V}{4h}\left(\frac{z^2}{h^2}-1\right)\,2\cancel{r}\,\frac{1}{\cancel{r}}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) = 0$$

r-component EOM:

$$0 = -\frac{\partial p}{\partial r} + \mu\,\underbrace{\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)}}_{0} + \mu\,\frac{\partial^2 v_r}{\partial z^2}$$

$$0 = +\frac{3\mu V}{2h^3}\,r + \mu\left(-\frac{3}{2}V\frac{r}{h^3}\right) \checkmark$$

$$\frac{\partial P}{\partial z} = \frac{3\mu V}{h^3}\,\frac{1}{\cancel{2}}\,\cancel{2}z$$

$$\frac{\partial v_z}{\partial z} = \frac{3}{2}\left(\frac{1}{\cancel{3}h^3}\cancel{3}z^2 - \frac{1}{h}\right)V$$

$$\frac{\partial^2 v_z}{\partial z^2} = \frac{3V}{\cancel{2}}\,\frac{1}{\cancel{3}h^3}\,\cancel{3}\,\cancel{2}z = \frac{3zV}{h^3}$$

$$\frac{\partial v_z}{\partial r} = 0$$

z-component EOM:

$$0 = -\frac{\partial P}{\partial z} + \mu\,\cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial v_z}{\partial r}\right)} + \mu\,\frac{\partial^2 v_z}{\partial z^2}$$

$$0 = -\frac{3\cancel{\mu}V}{h^3}z + \cancel{\mu}\,\frac{3zV}{h^3} \quad \checkmark$$

Continuity: $\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{\partial v_z}{\partial z} = 0$

$$-\frac{3}{2}\frac{V}{h}\left(\frac{z^2}{h^2} - 1\right) + \frac{3}{2}V\left(\frac{z^2}{h^3} - \frac{1}{h}\right) = 0 \quad \checkmark$$