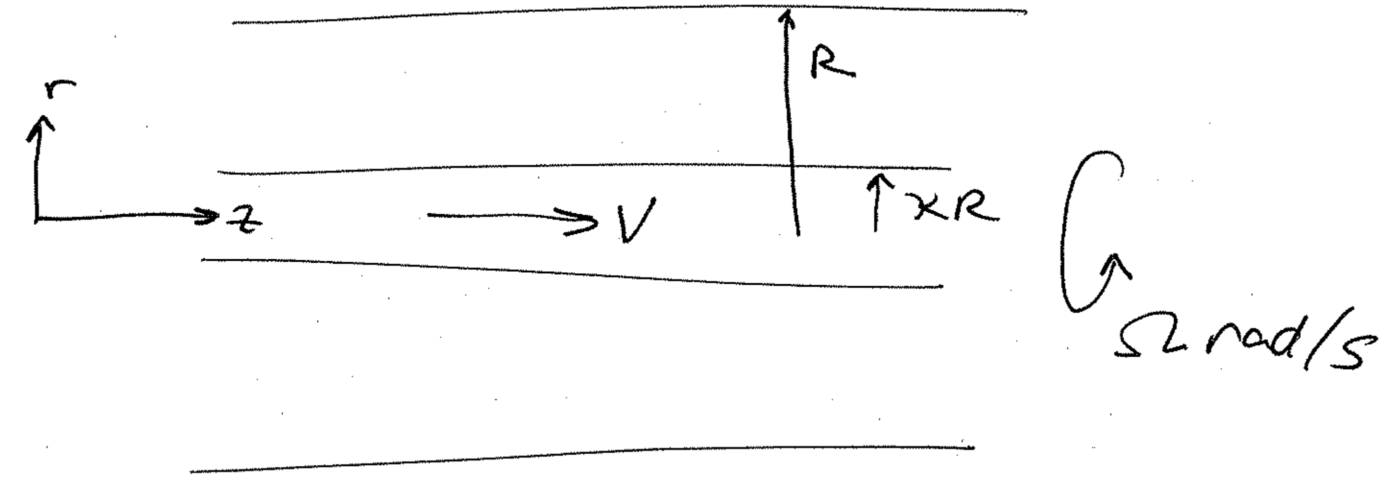
Helical flow



no θ-variation

$\frac{\partial P}{\partial z} = \text{constant} = \lambda$

$r = \kappa R, \; P = P_{\kappa R}$

calculate:

$\underline{v}, P, J$

$$\underline{v} = \begin{pmatrix} v_r \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z} = \begin{pmatrix} 0 \\ v_\theta \\ v_z \end{pmatrix}_{r\theta z}$$

continuity:

$$\underbrace{\cancel{\frac{\partial v_r}{\partial r}}}_{0} + \frac{1}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial v_z}{\partial z} = 0$$

$v_r = 0$ no θ variation

$$\Rightarrow \boxed{\frac{\partial v_z}{\partial z} = 0}$$

Navier Stokes:

$$\rho\left(\underbrace{\cancel{\frac{\partial \underline{v}}{\partial t}}}_{\text{steady}} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{V} + \underbrace{\cancel{\rho\underline{g}}}_{\text{neglect}}$$

$$\boxed{\rho\,\underline{v}\cdot\nabla\underline{v} = -\nabla p + \mu\nabla^2\underline{V}}$$

From Table B.7: $\left(v_r = 0,\ \frac{\partial}{\partial\theta} = 0,\ \frac{\partial v_z}{\partial z} = 0\right)$

r-component:

$$-\rho\frac{v_\theta^2}{r} + v_z \underbrace{\cancel{\frac{\partial v_r}{\partial z}}}_{\substack{\text{long}\\ \text{flow}\\ \text{(no end effects}}} = -\frac{\partial p}{\partial r} + \mu \underbrace{\cancel{\frac{\partial^2 v_r}{\partial z^2}}}_{\substack{\text{no end}\\ \text{effects}}}$$

(also $\frac{\partial v_\theta}{\partial z} = 0$

θ-component:

$$0 = \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial(r v_\theta)}{\partial r}\right)\right)$$

z-component:

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right)$$

r-component: $\boxed{\rho \frac{v_\theta^2}{r} = \frac{\partial p}{\partial r}}$

θ-component:

$$\frac{\partial}{\partial r}\Big(\underbrace{\frac{1}{r} \frac{\partial (r v_\theta)}{\partial r}}_{\psi} \Big) = 0$$

$$\frac{\partial \psi}{\partial r} = 0$$

$$\psi = C_1$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r v_\theta) = C_1$$

$$\frac{\partial}{\partial r}\underbrace{(r v_\theta)}_{\Psi} = C_1 r$$

$$\frac{d \Psi}{\partial r} = C_1 r$$

$$\Psi = C_1 \frac{r^2}{2} + C_2 = r v_\theta$$

$$\boxed{v_\theta = \frac{C_1}{2} r + \frac{C_2}{r}}$$

Boundary conditions:

$$r = R \quad v_\theta = 0$$

$$r = \kappa R \quad v_\theta = \kappa R \Omega$$

$$\left.\begin{aligned} 0 &= \frac{c_1}{2} R + \frac{c_2}{R} \\ \kappa R \Omega &= \frac{c_1}{2} \kappa R + \frac{c_2}{\kappa R} \end{aligned}\right] \text{solve for } c_1, c_2$$

$$\kappa R \Omega = -\frac{\kappa c_2}{R} + \frac{c_2}{\kappa R}$$

$$c_2 = \kappa R \Omega \left(\frac{1}{-\frac{\kappa}{R} + \frac{1}{\kappa R}} \right)$$

$$= \kappa R \Omega \left(\frac{R \kappa}{1-\kappa^2} \right)$$

$$\boxed{c_2 = \frac{R^2 \kappa^2 \Omega}{(1-\kappa^2)}}$$

$$C_1 = -\frac{C_2}{R}\frac{2}{R} = \frac{-2C_2}{R^2}$$

$$\boxed{C_1 = \frac{-2\kappa^2\Omega}{(1-\kappa^2)}}$$

$$v_\theta = \frac{C_1}{2}r + \frac{C_2}{r}$$

$$= \frac{\kappa^2\Omega}{(1-\kappa^2)}\left[\frac{-2}{2}r + R^2\frac{1}{r}\right]$$

$$\boxed{v_\theta = \underbrace{\frac{\kappa^2\Omega R}{(1-\kappa^2)}}_{\equiv B}\left[\frac{R}{r} - \frac{r}{R}\right]}$$

z-component Navier-Stokes:

$$\frac{\partial P}{\partial z} = \lambda = \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right)$$

$$\frac{\lambda}{\mu}r = \frac{\partial}{\partial r}\underbrace{\left(r\frac{\partial v_z}{\partial r}\right)}_{\equiv \phi}$$

$$\frac{\partial \phi}{\partial r} = \frac{\lambda}{\mu} r$$

$$\phi = \frac{\lambda}{\mu}\frac{r^2}{2} + C_3 = r\frac{\partial v_z}{\partial r}$$

$$\frac{\partial v_z}{\partial r} = \frac{\lambda}{2\mu} r + \frac{C_3}{r}$$

$$\boxed{v_z = \frac{\lambda}{2\mu}\frac{r^2}{2} + C_3 \ln r + C_4}$$

Boundary conditions:

$r = R \quad v_z = 0$

$r = KR \quad v_z = V$

$$\left.\begin{aligned} 0 &= \frac{\lambda}{4\mu} R^2 + C_3 \ln R + C_4 \\ V &= \frac{\lambda}{4\mu} K^2R^2 + C_3 \ln KR + C_4 \end{aligned}\right] \text{solve for } C_3, C_4$$

$$V = \frac{\lambda}{4\mu} R^2(\kappa^2-1) + C_3 \ln \frac{\kappa \cancel{R}}{\cancel{R}}$$

$$\boxed{C_3 = \frac{\left[V - \frac{\lambda R^2}{4\mu}(\kappa^2-1)\right]}{\ln \kappa}}$$

$$C_4 = -C_3 \ln R - \frac{\lambda R^2}{4\mu}$$

$$\boxed{C_4 = \frac{\left(V - \frac{\lambda R^2}{4\mu}(\kappa^2-1)\right)}{\ln \kappa}(-1)(\ln R) - \frac{\lambda R^2}{4\mu}}$$

$$v_z = \frac{\lambda}{4\mu} r^2 - \frac{\lambda}{4\mu} R^2 + \left[\frac{V - \frac{\lambda R^2}{4\mu}(\kappa^2-1)}{\ln \kappa}\right] * \ln \frac{r}{R}$$

$$\boxed{v_z = \frac{\lambda R^2}{4\mu}\left(\frac{r^2}{R^2} - 1\right) + \left(V - \frac{\lambda R^2}{4\mu}(\kappa^2-1)\right)\frac{\ln r/R}{\ln \kappa}}$$

Pressure: $\frac{\partial P}{\partial \theta} = 0$

$\frac{\partial P}{\partial z} = \lambda$

NS: $\frac{\partial P}{\partial r} = \rho \frac{v_\theta^2}{r}$

$$= \frac{\rho}{r}\left(\frac{\kappa^2 \Omega R}{1-\kappa^2}\left[\frac{R}{r} - \frac{r}{R}\right]\right)^2$$

$$= \frac{\rho \kappa^4 \Omega^2 \cancel{R^2}}{(1-\kappa^2)^2} \frac{1}{r}\left(\frac{R^2 - r^2}{r\cancel{R}}\right)^2$$

$$= \frac{\rho \kappa^4 \Omega^2}{(1-\kappa^2)^2} \frac{1}{r^3}\left(R^4 - 2R^2 r^2 + r^4\right)$$

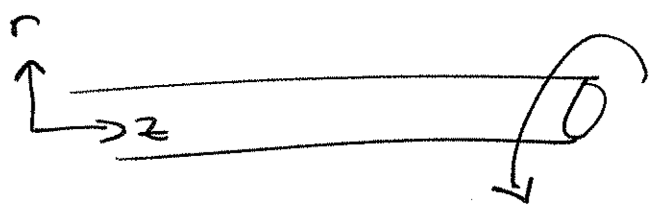
$$\frac{\partial P}{\partial r} = \frac{\rho \kappa^4 \Omega^2}{(1-\kappa^2)^2}\left[\frac{R^4}{r^3} - 2R^2\frac{1}{r} + r\right]$$

$$P = \frac{\rho \kappa^4 \Omega^2}{(1-\kappa^2)^2}\left[R^4 \frac{1}{r^2}\frac{1}{-2} - 2R^2 \ln r + \frac{r^2}{2}\right] + C_5$$

Boundary condition: $\left.\begin{array}{l} r = \kappa R \\ P = P_{\kappa R} \end{array}\right\}$ Solve for C_5 (not done.)

Torque to turn the inner cylinder:

$$\underline{T} = \iint \left(\underline{R} \times \hat{n}\cdot\underline{\underline{\Pi}}\right)_{\substack{\text{at}\\ \text{Surface}}} dS$$



$$\underline{R} = \kappa R\, \hat{e}_r \qquad \text{surface} = r = \kappa R$$

$$\hat{n} = \hat{e}_r$$

$$dS = \kappa R\, d\theta\, dz$$

$$\underline{\underline{\Pi}} = -P\underline{\underline{I}} + \underline{\underline{\tau}}$$

$$= -P\underline{\underline{I}} + \mu\left(\nabla\underline{v} + (\nabla\underline{v})^T\right)$$

Table B.8:

$$\underline{\underline{\tau}} = \mu \begin{pmatrix} 0 & r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) & \frac{\partial v_z}{\partial r} \\ r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) & 0 & 0 \\ \frac{\partial v_z}{\partial r} & 0 & 0 \end{pmatrix}_{r\theta z}$$

$$\underline{\underline{\Pi}} = \begin{pmatrix} -p & \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) & \mu \frac{\partial v_z}{\partial r} \\ \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) & -p & 0 \\ \mu \frac{\partial v_z}{\partial r} & 0 & -p \end{pmatrix}_{r\theta z}$$

$$\hat{n} \cdot \underline{\underline{\Pi}} = \hat{e}_r \cdot \underline{\underline{\Pi}} = \begin{pmatrix} -p \\ \mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) \\ \mu \frac{\partial v_z}{\partial r} \end{pmatrix}_{r\theta z} = \underline{a}$$

$$r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) = Br \frac{\partial}{\partial r}\left(\frac{R}{r^2} - \frac{1}{R}\right)$$

$$= Br\left[\frac{-2R}{r^3}\right] = \frac{-2BR}{r^2}$$

$$= \frac{-2R}{r^2} \frac{\chi^2 \Omega R}{(1-\chi^2)}$$

$$\mu r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\Bigg|_{r=KR} = \mu \frac{-2\cancel{R}}{\cancel{K^2}\cancel{R^2}} \frac{\cancel{K^2}\Omega \cancel{R}}{(1-K^2)}$$

$$= \boxed{-\frac{2\mu\Omega}{1-K^2} = a_\theta}$$

$$\underline{R} \times \hat{n}\cdot\underline{\underline{\Pi}} = \begin{vmatrix} \hat{e}_r & \hat{e}_\theta & \hat{e}_z \\ KR & 0 & 0 \\ a_r & a_\theta & a_z \end{vmatrix}$$

$$= \begin{pmatrix} 0 \\ -KR\,a_z \\ KR\,a_\theta \end{pmatrix}_{r\theta z} \qquad a_z = \mu \frac{\partial v_z}{\partial r}\bigg|_{r=KR}$$

$$\underline{J} = \int_0^L \int_0^{2\pi} \begin{pmatrix} 0 \\ -KR\,a_z \\ KR\,a_\theta \end{pmatrix}_{r\theta z} KR \, d\theta \, dz$$

$$\frac{1}{L}\underline{J} = \int_0^{2\pi} \left(-K^2R^2 a_z \hat{e}_\theta\right) + K^2R^2 a_\theta \hat{e}_z \, d\theta$$

$$\hat{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}_{xyz}$$

$$\int_0^{2\pi} \sin\theta = 0$$

$$\int_0^{2\pi} \cos\theta = 0$$

$\Rightarrow \hat{e}_\theta$ integral $\rightarrow 0$

$$\underline{J} = K^2R^2 L\, 2\pi\, a_\theta\, \hat{e}_z$$

$$= \mu K^2R^2 L\, 2\pi \left(\frac{-2\Omega}{(1-K^2)}\right) \hat{e}_z$$

$$\boxed{\underline{J} = 4\pi K^2R^2 L\mu \left(\frac{\Omega}{1-K^2}\right)(-\hat{e}_z)}$$

TORQUE ON surface w/ $\hat{n} = \hat{e}_r$

Check units:

$$\cancel{m^2}\; m \;\frac{\cancel{kg}}{\cancel{m}\,\cancel{s}}\;\frac{1}{\cancel{s}}\;\frac{N\,\cancel{s^2}}{\cancel{kg}\,\cancel{m}}$$

$$= N\,m \;\checkmark$$