Exam 1 Formulas

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Rate of deformation tensor: $\dot{\underline{\underline{\boldsymbol{\gamma}}}} = \nabla \underline{\boldsymbol{v}} + (\nabla \underline{\boldsymbol{v}})^{\boldsymbol{T}}$

Rate of deformation: $\dot{\boldsymbol{\gamma}} = \left|\dot{\underline{\underline{\boldsymbol{\gamma}}}}\right|$

Tensor magnitude: $\boldsymbol{A} = \left|\underline{\underline{\boldsymbol{A}}}\right| = +\sqrt{\frac{\underline{\underline{\boldsymbol{A}}}:\underline{\underline{\boldsymbol{A}}}}{\mathbf{2}}}$

Total stress tensor: $\underline{\underline{\boldsymbol{\Pi}}} = \boldsymbol{p}\underline{\underline{\boldsymbol{I}}} + \underline{\underline{\boldsymbol{\tau}}}$

(Bird, UR sign convention on stress)

Navier-Stokes Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p + \mu\nabla^2\underline{v} + \rho\underline{g}$

Cauchy Momentum Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v}\right) = -\nabla p - \nabla\cdot\underline{\underline{\tau}} + \rho\underline{g}$

(Bird, UR sign convention on stress)

Continuity Equation: $\frac{\partial \rho}{\partial t} = -\nabla\cdot\left(\rho\underline{v}\right)$

Newtonian, incompressible constitutive equation: $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{\mu}(\nabla\underline{\boldsymbol{v}} + (\nabla\underline{\boldsymbol{v}})^{\boldsymbol{T}})$

(Bird sign convention on stress)

Fluid force $\underline{\boldsymbol{F}}_{\boldsymbol{on}}$ on a surface **S**:

(Bird, UR sign convention on stress)

$$\underline{\boldsymbol{F}}_{\boldsymbol{on}} = \iint_{\boldsymbol{S}} \left[\hat{\boldsymbol{n}}\cdot -\underline{\underline{\boldsymbol{\Pi}}}\right]\Big|_{\boldsymbol{surface}} \boldsymbol{dS}$$

Flow rate $\boldsymbol{Q}$ through a surface **S**:

$$\boldsymbol{Q} = \iint_{\boldsymbol{S}} \left[\hat{\boldsymbol{n}}\cdot\underline{\boldsymbol{v}}\right]_{\boldsymbol{surface}} \boldsymbol{dS}$$

Fluid torque $\underline{\boldsymbol{T}}_{\boldsymbol{on}}$ on a surface **S**: ($\tilde{\underline{\boldsymbol{R}}}$ is the lever arm vector from the axis of rotation to the point of application of the force)

(Bird, UR sign convention on stress)

$$\underline{\boldsymbol{T}}_{\boldsymbol{on}} = \iint_{\boldsymbol{S}} \left[\tilde{\underline{\boldsymbol{R}}} \times (\hat{\boldsymbol{n}}\cdot -\underline{\underline{\boldsymbol{\Pi}}})\right]_{\boldsymbol{surface}} \boldsymbol{dS}$$

Table of Integrals

$$\int u^n du = \frac{u^{n+1}}{n+1} + C \qquad n \text{ is a constant}$$

$$\int \frac{1}{u} du = \ln u + C$$

$$\int (\ln u) du = u \ln u - u + C$$

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

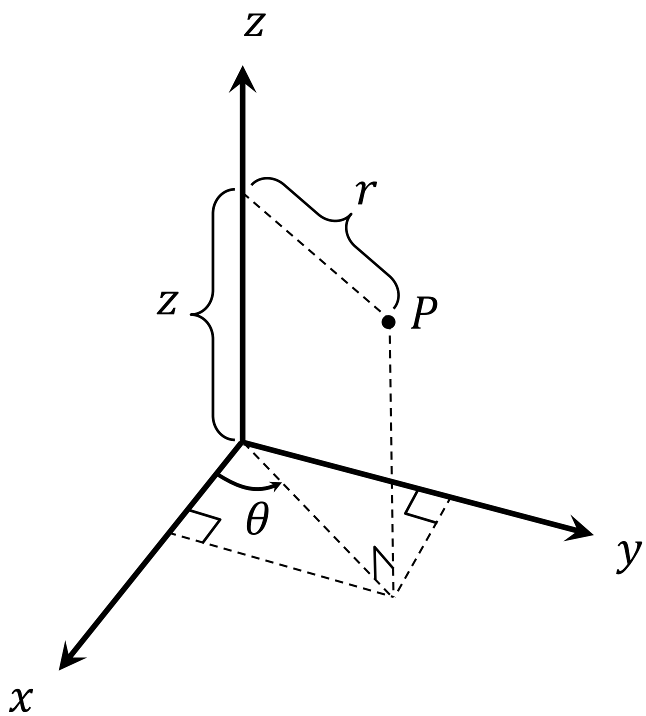
Miscellaneous

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123}$$

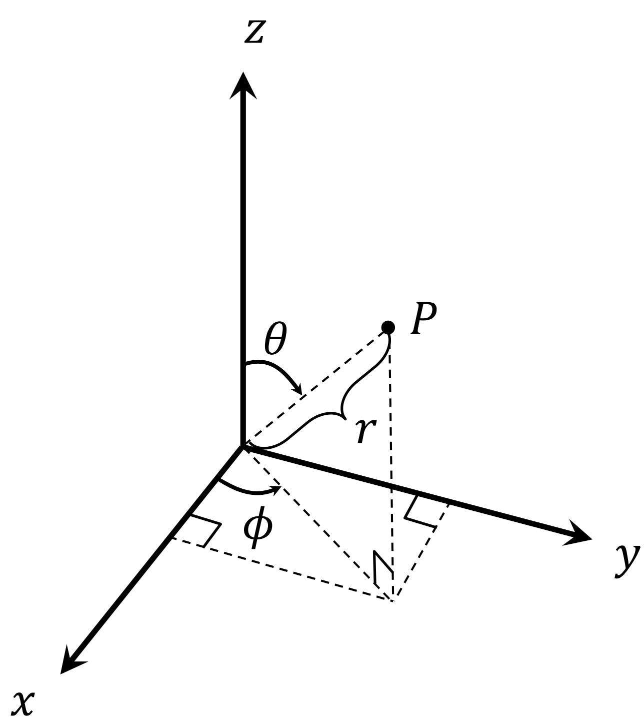
$$\frac{d}{ds}(uw) = u\frac{dw}{ds} + w\frac{du}{ds}$$

$$\underline{u} \times \underline{w} = \det \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ -(u_1 w_3 - u_3 w_1) \\ u_1 w_2 - u_2 w_1 \end{pmatrix}_{123}$$

Cylindrical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings around the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate is perpendicular to the z-axis.



Spherical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings <u>down</u> from the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate emits radially from the origin to the point; these are different from their definitions in the cylindrical system above.



Cylindrical Coordinates

System	Coordinates	Basis vectors
Cylindrical	$\boldsymbol{r = \sqrt{x^2 + y^2}}$	$\boldsymbol{\hat{e}_r = \cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y}$
Cylindrical	$\boldsymbol{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$	$\boldsymbol{\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\, \hat{e}_y}$
Cylindrical	$\boldsymbol{z = z}$	$\boldsymbol{\hat{e}_z = \hat{e}_z}$
Cylindrical	$\boldsymbol{x = r\cos\theta}$	$\boldsymbol{\hat{e}_x = \cos\theta\, \hat{e}_r + (-\sin\theta)\hat{e}_\theta}$
Cylindrical	$\boldsymbol{y = r\sin\theta}$	$\boldsymbol{\hat{e}_y = \sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta}$
Cylindrical	$\boldsymbol{z = z}$	$\boldsymbol{\hat{e}_z = \hat{e}_z}$

Spherical Coordinates

System	Coordinates	Basis vectors
Spherical	$\boldsymbol{x = r\sin\theta\cos\phi}$	$\boldsymbol{\hat{e}_x = (\sin\theta\cos\phi)\hat{e}_r + (\cos\theta\cos\phi)\hat{e}_\theta + (-\sin\phi\,)\,\hat{e}_\phi}$
Spherical	$\boldsymbol{y = r\sin\theta\sin\phi}$	$\boldsymbol{\hat{e}_y = (\sin\theta\sin\phi)\hat{e}_r + (\cos\theta\sin\phi)\hat{e}_\theta + \cos\phi\, \hat{e}_\phi}$
Spherical	$\boldsymbol{z = r\cos\theta}$	$\boldsymbol{\hat{e}_z = \cos\theta\ \ \hat{e}_r + (-\sin\theta)\hat{e}_\theta}$
Spherical	$\boldsymbol{r = \sqrt{x^2 + y^2 + z^2}}$	$\boldsymbol{\hat{e}_r = (\sin\theta\cos\phi)\hat{e}_x + (\sin\theta\sin\phi)\hat{e}_y + \cos\theta\hat{e}_z}$
Spherical	$\boldsymbol{\theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right)}$	$\boldsymbol{\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z}$
Spherical	$\boldsymbol{\phi = \tan^{-1}\left(\frac{y}{x}\right)}$	$\boldsymbol{\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\, \hat{e}_y}$

Coordinate system	surface differential dS
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (side a, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (side b, $\hat{n} = \hat{e}_x$)	$dS = dydz$
cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
spherical, ($\hat{n} = \hat{e}_r$)	$dS = R^2 \sin\theta d\theta d\phi$

Coordinate system	volume differential dV
Cartesian	$dV = dxdydz$
cylindrical	$dV = rdrd\theta dz$
spherical	$dV = r^2 \sin\theta drd\theta d\phi$

C.2 Differential Operations in Curvilinear Coordinates

TABLE C.3
Differential Operations in the Cylindrical Coordinate System r, θ, z

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_z \end{pmatrix}_{r\theta z}$$

$$\nabla = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{\partial a}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} + \frac{\partial^2 a}{\partial z^2}$$

$$\nabla \cdot \underline{w} = \frac{1}{r}\frac{\partial}{\partial r}(r w_r) + \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{\partial w_z}{\partial z}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r}\frac{\partial w_z}{\partial \theta} - \frac{\partial w_\theta}{\partial z} \\ \frac{\partial w_r}{\partial z} - \frac{\partial w_z}{\partial r} \\ \frac{1}{r}\frac{\partial (r w_\theta)}{\partial r} - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta z}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{pmatrix}_{r\theta z}$$

$$\nabla \underline{w} = \begin{pmatrix} \frac{\partial w_r}{\partial r} & \frac{\partial w_\theta}{\partial r} & \frac{\partial w_z}{\partial r} \\ \frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r} & \frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r} & \frac{1}{r}\frac{\partial w_z}{\partial \theta} \\ \frac{\partial w_r}{\partial z} & \frac{\partial w_\theta}{\partial z} & \frac{\partial w_z}{\partial z} \end{pmatrix}_{r\theta z}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_r)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_r}{\partial \theta^2} + \frac{\partial^2 w_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial w_\theta}{\partial \theta} \\ \frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial (r w_\theta)}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 w_\theta}{\partial \theta^2} + \frac{\partial^2 w_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial w_r}{\partial \theta} \\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w_z}{\partial \theta^2} + \frac{\partial^2 w_z}{\partial z^2} \end{pmatrix}_{r\theta z}$$

F. A. Morrison, Understanding Rheology (Oxford, 2001)

$$\nabla \cdot \underline{\underline{A}} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(rA_{rr}) + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\partial A_{zr}}{\partial z} - \frac{A_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_{r\theta}) + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\partial A_{z\theta}}{\partial z} + \frac{A_{\theta r} - A_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(rA_{rz}) + \frac{1}{r}\frac{\partial A_{\theta z}}{\partial \theta} + \frac{\partial A_{zz}}{\partial z} \end{pmatrix}_{r\theta z} \quad \text{(C.3-10)}$$

$$\underline{u} \cdot \nabla \underline{w} = \begin{pmatrix} u_r\left(\frac{\partial w_r}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_r}{\partial \theta} - \frac{w_\theta}{r}\right) + u_z\left(\frac{\partial w_r}{\partial z}\right) \\ u_r\left(\frac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_\theta}{\partial \theta} + \frac{w_r}{r}\right) + u_z\left(\frac{\partial w_\theta}{\partial z}\right) \\ u_r\left(\frac{\partial w_z}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial w_z}{\partial \theta}\right) + u_z\left(\frac{\partial w_z}{\partial z}\right) \end{pmatrix}_{r\theta z} \quad \text{(C.3-11)}$$

TABLE C.4
Differential Operations in the Spherical Coordinate System r, θ, ϕ

$$\underline{w} = \begin{pmatrix} w_r \\ w_\theta \\ w_\phi \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-1)}$$

$$\nabla = \hat{e}_r\frac{\partial}{\partial r} + \hat{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \quad \text{(C.4-2)}$$

$$\nabla a = \begin{pmatrix} \frac{\partial a}{\partial r} \\ \frac{1}{r}\frac{\partial a}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial a}{\partial \phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-3)}$$

$$\nabla \cdot \nabla a = \nabla^2 a = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial a}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial a}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 a}{\partial \phi^2} \quad \text{(C.4-4)}$$

$$\nabla \cdot \underline{w} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial w_\phi}{\partial \phi} \quad \text{(C.4-5)}$$

$$\nabla \times \underline{w} = \begin{pmatrix} \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(w_\phi\sin\theta) - \frac{1}{r\sin\theta}\frac{\partial w_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial w_r}{\partial \phi} - \frac{1}{r}\frac{\partial}{\partial r}(rw_\phi) \\ \frac{1}{r}\frac{\partial}{\partial r}(rw_\theta) - \frac{1}{r}\frac{\partial w_r}{\partial \theta} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-6)}$$

$$\underline{\underline{A}} = \begin{pmatrix} A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi} \end{pmatrix}_{r\theta\phi} \quad \text{(C.4-7)}$$

continued

$$\nabla \underline{w} = \begin{pmatrix} \dfrac{\partial w_r}{\partial r} & \dfrac{\partial w_\theta}{\partial r} & \dfrac{\partial w_\phi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial w_r}{\partial \theta} - \dfrac{w_\theta}{r} & \dfrac{1}{r}\dfrac{\partial w_\theta}{\partial \theta} + \dfrac{w_r}{r} & \dfrac{1}{r}\dfrac{\partial w_\phi}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial w_r}{\partial \phi} - \dfrac{w_\phi}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial w_\theta}{\partial \phi} - \dfrac{w_\phi}{r}\cot\theta & \dfrac{1}{r\sin\theta}\dfrac{\partial w_\phi}{\partial \phi} + \dfrac{w_r}{r} + \dfrac{w_\theta}{r}\cot\theta \end{pmatrix}_{r\theta\phi} \tag{C.4-8}$$

$$\nabla^2 \underline{w} = \begin{pmatrix} \left\{ \dfrac{\partial}{\partial r}\left[\dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 w_r)\right] + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial w_r}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_r}{\partial\phi^2} \right. \\ \left. - \dfrac{2}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}(w_\theta\sin\theta) - \dfrac{2}{r^2\sin\theta}\dfrac{\partial w_\phi}{\partial\phi} \right\} \\ \left\{ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial w_\theta}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial\theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(w_\theta\sin\theta)\right] + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_\theta}{\partial\phi^2} \right. \\ \left. + \dfrac{2}{r^2}\dfrac{\partial w_r}{\partial\theta} - \dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial w_\phi}{\partial\phi} \right\} \\ \left\{ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial w_\phi}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial\theta}\left[\dfrac{1}{\sin\theta}\dfrac{\partial}{\partial\theta}(w_\phi\sin\theta)\right] + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 w_\phi}{\partial\phi^2} \right. \\ \left. + \dfrac{2}{r^2\sin\theta}\dfrac{\partial w_r}{\partial\phi} + \dfrac{2\cot\theta}{r^2\sin\theta}\dfrac{\partial w_\theta}{\partial\phi} \right) \end{pmatrix}_{r\theta\phi} \tag{C.4-9}$$

$$\nabla\cdot\underline{\underline{A}} = \begin{pmatrix} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 A_{rr}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta r}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi r}}{\partial\phi} - \dfrac{A_{\theta\theta}+A_{\phi\phi}}{r} \\ \dfrac{1}{r^3}\dfrac{\partial}{\partial r}(r^3 A_{r\theta}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta\theta}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi\theta}}{\partial\phi} + \dfrac{(A_{\theta r}-A_{r\theta}) - A_{\phi\phi}\cot\theta}{r} \\ \dfrac{1}{r^3}\dfrac{\partial}{\partial r}(r^3 A_{r\phi}) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(A_{\theta\phi}\sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial A_{\phi\phi}}{\partial\phi} + \dfrac{(A_{\phi r}-A_{r\phi}) + A_{\phi\theta}\cot\theta}{r} \end{pmatrix}_{r\theta\phi} \tag{C.4-10}$$

$$\underline{u}\cdot\nabla\underline{w} = \begin{pmatrix} u_r\left(\dfrac{\partial w_r}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_r}{\partial\theta} - \dfrac{w_\theta}{r}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_r}{\partial\phi} - \dfrac{w_\phi}{r}\right) \\ u_r\left(\dfrac{\partial w_\theta}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_\theta}{\partial\theta} + \dfrac{w_r}{r}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_\theta}{\partial\phi} - \dfrac{w_\phi}{r}\cot\theta\right) \\ u_r\left(\dfrac{\partial w_\phi}{\partial r}\right) + u_\theta\left(\dfrac{1}{r}\dfrac{\partial w_\phi}{\partial\theta}\right) + u_\phi\left(\dfrac{1}{r\sin\theta}\dfrac{\partial w_\phi}{\partial\phi} + \dfrac{w_r}{r} + \dfrac{w_\theta}{r}\cot\theta\right) \end{pmatrix}_{r\theta\phi} \tag{C.4-11}$$