Final Exam Formulas

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Rate of deformation tensor: $\underline{\underline{\dot{\boldsymbol{\gamma}}}} = \nabla \underline{\boldsymbol{v}} + (\nabla \underline{\boldsymbol{v}})^T$

Rate of deformation: $\dot{\boldsymbol{\gamma}} = \left|\underline{\underline{\dot{\boldsymbol{\gamma}}}}\right|$

Tensor magnitude: $\boldsymbol{A} = \left|\underline{\underline{\boldsymbol{A}}}\right| = +\sqrt{\frac{\underline{\underline{\boldsymbol{A}}}:\underline{\underline{\boldsymbol{A}}}}{2}}$

Total stress tensor: $\underline{\underline{\boldsymbol{\Pi}}} = \boldsymbol{p}\underline{\underline{\boldsymbol{I}}} + \underline{\underline{\boldsymbol{\tau}}}$

(Bird, UR sign convention on stress)

Shear strain: $\boldsymbol{\gamma_{21}(t_a, t_b)} = \int_{t_a}^{t_b} \dot{\boldsymbol{\gamma}}_{21}(\boldsymbol{t''})\boldsymbol{dt''}$

Navier-Stokes Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + \rho \underline{g}$

Cauchy Momentum Equation: $\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v}\right) = -\nabla p - \nabla \cdot \underline{\underline{\tau}} + \rho \underline{g}$

Continuity Equation: $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v})$

Fluid force $\underline{\boldsymbol{F}}$ on a surface S:

$$\underline{\boldsymbol{F}} = \iint_S [\hat{\boldsymbol{n}} \cdot -\underline{\underline{\boldsymbol{\Pi}}}]\Big|_{\boldsymbol{surface}} \boldsymbol{dA}$$

Flow rate $\boldsymbol{Q}$ through a surface S:

$$\boldsymbol{Q} = \iint_S [\hat{\boldsymbol{n}} \cdot \underline{\boldsymbol{v}}]_{\boldsymbol{surface}} \boldsymbol{dA}$$

Fluid torque $\underline{\boldsymbol{T}}$ on a surface S: ($\underline{\boldsymbol{R}}$ is the vector from the axis of rotation to the point of application of the force)

$$\underline{\boldsymbol{T}} = \iint_S [\underline{\boldsymbol{R}} \times (\hat{\boldsymbol{n}} \cdot -\underline{\underline{\boldsymbol{\Pi}}})]_{\boldsymbol{surface}} \boldsymbol{dA}$$

Newtonian, incompressible fluid: $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{\mu}(\nabla \underline{\boldsymbol{v}} + (\nabla \underline{\boldsymbol{v}})^{\boldsymbol{T}})$

Hookean solid (small strain): $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{G}\underline{\underline{\boldsymbol{\gamma}}}(\boldsymbol{t}, \boldsymbol{t}')$

Generalized Newtonian fluid (GNF): $\underline{\underline{\boldsymbol{\tau}}} = -\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}})\underline{\underline{\dot{\boldsymbol{\gamma}}}}$

Power-law GNF model: $\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}}) = \boldsymbol{m}\dot{\boldsymbol{\gamma}}^{\boldsymbol{n}-\mathbf{1}}$

(Note that m and n are parameters of the model and are constants)

Carreau-Yasuda GNF model: $\boldsymbol{\eta}(\dot{\boldsymbol{\gamma}}) = \boldsymbol{\eta}_\infty + (\boldsymbol{\eta}_\mathbf{0} - \boldsymbol{\eta}_\infty)[\mathbf{1} + (\dot{\boldsymbol{\gamma}}\boldsymbol{\lambda})^{\boldsymbol{a}}]^{\frac{\boldsymbol{n}-\mathbf{1}}{\boldsymbol{a}}}$

(Note that a, λ and n, η_o, and η_∞ are parameters of the model and are constants)

Generalized Linear Viscoelastic Model (GLVE) (rate version): $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = -\int_{-\infty}^{\boldsymbol{t}} \boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}')\underline{\underline{\dot{\boldsymbol{\gamma}}}}(\boldsymbol{t}')\boldsymbol{dt}'$

Generalized Linear Viscoelastic Model (GLVE) (strain version) : $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = +\int_{-\infty}^{\boldsymbol{t}} \frac{\partial \boldsymbol{G}(\boldsymbol{t}-\boldsymbol{t}')}{\partial \boldsymbol{t}'}\underline{\underline{\boldsymbol{\gamma}}}(\boldsymbol{t}, \boldsymbol{t}')\boldsymbol{dt}'$

Maxwell GLVE model relaxation function: $\boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}') = \frac{\boldsymbol{\eta}_\mathbf{0}}{\boldsymbol{\lambda}}\boldsymbol{e}^{-(\boldsymbol{t}-\boldsymbol{t}')/\boldsymbol{\lambda}}$

Generalized Maxwell GLVE model relaxation function: $\boldsymbol{G}(\boldsymbol{t} - \boldsymbol{t}') = \sum_{\boldsymbol{k}=\mathbf{1}}^{\boldsymbol{N}} \frac{\boldsymbol{\eta}_{\boldsymbol{k}}}{\boldsymbol{\lambda}_{\boldsymbol{k}}}\boldsymbol{e}^{-(\boldsymbol{t}-\boldsymbol{t}')/\boldsymbol{\lambda}_{\boldsymbol{k}}}$

Lodge Model or Upper Convected Maxwell: (integral version) $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = -\int_{-\infty}^{\boldsymbol{t}} \frac{\boldsymbol{\eta}_\mathbf{0}}{\boldsymbol{\lambda}^2}\boldsymbol{e}^{\frac{-(\boldsymbol{t}-\boldsymbol{t}')}{\boldsymbol{\lambda}}}\underline{\underline{\boldsymbol{C}}}^{-\mathbf{1}}(\boldsymbol{t}', \boldsymbol{t})\boldsymbol{dt}'$

Cauchy-Maxwell Model or Lower Convected Maxwell: (integral version) $\underline{\underline{\boldsymbol{\tau}}}(\boldsymbol{t}) = +\int_{-\infty}^{\boldsymbol{t}} \frac{\boldsymbol{\eta}_\mathbf{0}}{\boldsymbol{\lambda}^2}\boldsymbol{e}^{\frac{-(\boldsymbol{t}-\boldsymbol{t}')}{\boldsymbol{\lambda}}}\underline{\underline{\boldsymbol{C}}}(\boldsymbol{t}, \boldsymbol{t}')\boldsymbol{dt}'$

Elongational flow (uniaxial, biaxial): $\underline{v} = \begin{pmatrix} -\frac{\dot{\varepsilon}(t)}{2} x_1 \\ -\frac{\dot{\varepsilon}(t)}{2} x_2 \\ \dot{\varepsilon}(t) x_3 \end{pmatrix}_{123}$

Shear flow: $\underline{v} = \begin{pmatrix} \dot{\varsigma}(t) x_2 \\ 0 \\ 0 \end{pmatrix}_{123}$

Steady shearing kinematics: $\dot{\varsigma}(t) = \dot{\gamma}_0$ for all values of time t

Start-up of steady shearing kinematics: $\dot{\varsigma}(t) = \begin{cases} 0 & t < 0 \\ \dot{\gamma}_0 & t \geq 0 \end{cases}$

Cessation of steady shearing kinematics: $\dot{\varsigma}(t) = \begin{cases} \dot{\gamma}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$

Small-amplitude oscillatory shear: $\dot{\varsigma}(t) = \gamma_0 \omega \cos \omega t$

Step shear strain kinematics: $\dot{\varsigma}(t) = lim_{\epsilon \to 0} \begin{cases} 0 & t < 0 \\ \gamma_0/\epsilon & 0 \leq t < \epsilon \\ 0 & 0 \leq t \end{cases}$

Steady elongational kinematics: $\dot{\varepsilon}(t) = \dot{\varepsilon}_0$

Start-up of steady elongation kinematics: $\dot{\varepsilon}(t) = \begin{cases} 0 & t < 0 \\ \dot{\varepsilon}_0 & t \geq 0 \end{cases}$

Cessation of steady elongation kinematics: $\dot{\varepsilon}(t) = \begin{cases} \dot{\varepsilon}_0 & t < 0 \\ 0 & t \geq 0 \end{cases}$

Shear viscosity: $\eta = \frac{-(\tau_{21})}{\dot{\gamma}_0}$

Shear normal stress coefficients: $\Psi_1 = \frac{-(\tau_{11}-\tau_{22})}{\dot{\gamma}_0^2}, \Psi_2 = \frac{-(\tau_{22}-\tau_{33})}{\dot{\gamma}_0^2}$

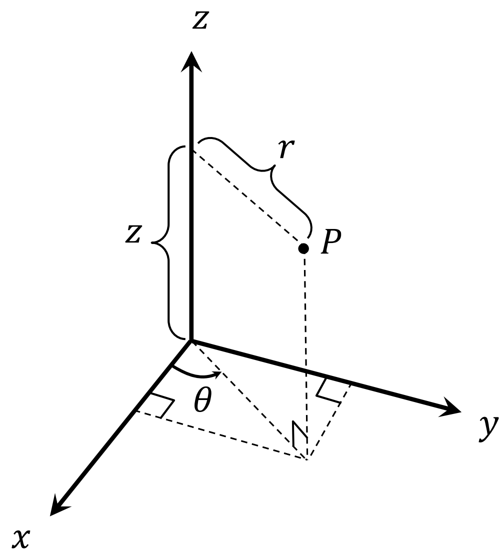
Shear relaxation modulus (step shear strain): $G(t, \gamma_0) = \frac{-\tau_{21}(t)}{\gamma_0}$

Small-amplitude oscillatory shear: $-\tau_{21} = G' \sin \omega t + G'' \cos \omega t$

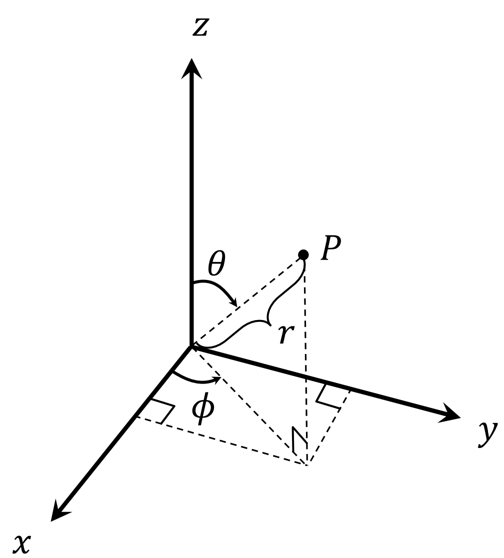
$$G' = \frac{-(\tau_0)}{\gamma_0} \cos \delta \qquad G'' = \frac{-(\tau_0)}{\gamma_0} \sin \delta$$

Elongational viscosity: $\bar{\eta} = \eta_e = \frac{-(\tau_{33}-\tau_{11})}{\dot{\varepsilon}_0}$

Cylindrical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings around the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate is perpendicular to the z-axis.



Spherical Coordinate System: Note that the $\boldsymbol{\theta}$-coordinate swings down from the $\boldsymbol{z}$-axis and the $\boldsymbol{r}$-coordinate emits radially from the origin; these are different from their definitions in the cylindrical system above.



System	Coordinates	Basis vectors
Spherical	$x = r\sin\theta\cos\phi$	$\hat{e}_r = (\sin\theta\cos\phi)\hat{e}_x + (\sin\theta\sin\phi)\hat{e}_y + \cos\theta\hat{e}_z$
Spherical	$y = r\sin\theta\sin\phi$	$\hat{e}_\theta = (\cos\theta\cos\phi)\hat{e}_x + (\cos\theta\sin\phi)\hat{e}_y + (-\sin\theta)\hat{e}_z$
Spherical	$z = r\cos\theta$	$\hat{e}_\phi = (-\sin\phi)\hat{e}_x + \cos\phi\,\hat{e}_y$
Cylindrical	$x = r\cos\theta$	$\hat{e}_r = \cos\theta\,\hat{e}_x + \sin\theta\,\hat{e}_y$
Cylindrical	$y = r\sin\theta$	$\hat{e}_\theta = (-\sin\theta)\hat{e}_x + \cos\theta\,\hat{e}_y$
Cylindrical	$z = z$	$\hat{e}_z = \hat{e}_z$

Differential Areas and Volumes

Coordinate system	*Surface Differential dS*
Cartesian (top, $\hat{n} = \hat{e}_z$)	$dS = dxdy$
Cartesian (top, $\hat{n} = \hat{e}_y$)	$dS = dxdz$
Cartesian (top, $\hat{n} = \hat{e}_x$)	$dS = dydz$
Cylindrical (top, $\hat{n} = \hat{e}_z$)	$dS = rdrd\theta$
Cylindrical (side, $\hat{n} = \hat{e}_r$)	$dS = Rd\theta dz$
Spherical (at $r = R, \hat{n} = \hat{e}_r$)	$dS = R^2\sin\theta d\theta d\phi$

Coordinate system	*Volume Differential dV*
Cartesian	$dV = dxdydz$
Cylindrical	$dV = rdrd\theta dz$
Spherical	$dV = r^2\sin\theta drd\theta d\phi$

Tensor Invariants (p40)

$$I_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3} B_{ii} = trace(\underline{\underline{B}})$$

$$II_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} B_{ij}B_{ji} = trace(\underline{\underline{B}} \cdot \underline{\underline{B}})$$

$$III_{\underline{\underline{B}}} \equiv \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} B_{ij}B_{jk}B_{ki} = trace(\underline{\underline{B}} \cdot \underline{\underline{B}} \cdot \underline{\underline{B}})$$

Note that $\left|\underline{\underline{B}}\right| = +\sqrt{\frac{II_{\underline{\underline{B}}}}{2}}$

Table of Integrals

$$\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1} + C \qquad \alpha \text{ is a constant}$$

$$\int \frac{1}{u} du = \ln u + C$$

$$\int (\ln u) du = u \ln u - u + C$$

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$

$$\int e^u du = e^u + C$$

$$\int u e^u du = e^u(u-1) + C$$

$$\int u^2 e^u du = e^u(u^2 - 2u + 2) + C$$

$$\int \cos(u)\, du = \sin(u) + C$$

$$\int \sin(u)\, du = -\cos(u) + C$$

$$\int u \cos(u) du = u \sin(u) + \cos(u) + C$$

$$\int u \sin(u)\, du = \sin(u) - u\cos(u) + C$$

Miscellaneous

$$\underline{\underline{A}} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}_{123}$$

$$\frac{d}{ds}(uw) = u\frac{dw}{ds} + w\frac{du}{ds}$$

$$\underline{u} \times \underline{w} = \det\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{pmatrix} u_2 w_3 - u_3 w_2 \\ -(u_1 w_3 - u_3 w_1) \\ u_1 w_2 - u_2 w_1 \end{pmatrix}_{123}$$