4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

- To carryout the differentiation with respect to a *single variable*, differentiate each coefficient individually.
- There is no change in order (vectors remain vectors, scalars remain scalars, etc.

$$\frac{\partial \alpha}{\partial t} \qquad \frac{\partial \underline{w}}{\partial t} = \begin{pmatrix} \frac{\partial w_1}{\partial t} \\ \frac{\partial w_2}{\partial t} \\ \frac{\partial w_3}{\partial t} \end{pmatrix}_{123} \qquad \frac{\partial \underline{\underline{B}}}{\partial t} = \begin{pmatrix} \frac{\partial B_{11}}{\partial t} & \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{31}}{\partial t} \\ \frac{\partial B_{21}}{\partial t} & \frac{\partial B_{22}}{\partial t} & \frac{\partial B_{23}}{\partial t} \\ \frac{\partial B_{31}}{\partial t} & \frac{\partial B_{32}}{\partial t} & \frac{\partial B_{33}}{\partial t} \end{pmatrix}_{123}$$

4. *Differential Operations with Vectors, Tensors (continued)*

Del Operator

- To carryout the differentiation with respect to *3D spatial variation*, use the del (nabla) operator.
- This is a vector operator
- Del may be applied in three different ways
- Del may operate on scalars, vectors, or tensors

This is written in Cartesian coordinates

$$\nabla \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix}_{123}$$

$$= \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \hat{e}_p \frac{\partial}{\partial x_p}$$

Einstein notation for del

4. *Differential Operations with Vectors, Tensors (continued)*

A. Scalars - gradient

Gibbs notation

$$\nabla \beta \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\beta + \hat{e}_2 \frac{\partial}{\partial x_2}\beta + \hat{e}_3 \frac{\partial}{\partial x_3}\beta = \begin{pmatrix} \frac{\partial \beta}{\partial x_1} \\ \frac{\partial \beta}{\partial x_2} \\ \frac{\partial \beta}{\partial x_3} \end{pmatrix}_{123}$$

This is written in Cartesian coordinates

Gradient of a scalar field $= \hat{e}_p \frac{\partial \beta}{\partial x_p}$

The gradient of a scalar field is a *vector*

The gradient operation captures the total spatial variation of a scalar, vector, or tensor field.

•gradient operation increases the order of the entity operated upon

4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient

$$\nabla \underline{w} \equiv \hat{e}_1 \frac{\partial}{\partial x_1}\underline{w} + \hat{e}_2 \frac{\partial}{\partial x_2}\underline{w} + \hat{e}_3 \frac{\partial}{\partial x_3}\underline{w}$$

$$= \hat{e}_1 \frac{\partial}{\partial x_1}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_2 \frac{\partial}{\partial x_2}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$
$$+ \hat{e}_3 \frac{\partial}{\partial x_3}\left(w_1\hat{e}_1 + w_2\hat{e}_2 + w_3\hat{e}_3\right)$$

$$= \hat{e}_1\hat{e}_1 \frac{\partial w_1}{\partial x_1} + \hat{e}_1\hat{e}_2 \frac{\partial w_2}{\partial x_1} + \hat{e}_1\hat{e}_3 \frac{\partial w_3}{\partial x_1} + \hat{e}_2\hat{e}_1 \frac{\partial w_1}{\partial x_2} +$$
$$\hat{e}_2\hat{e}_2 \frac{\partial w_2}{\partial x_2} + \hat{e}_2\hat{e}_3 \frac{\partial w_3}{\partial x_2} + \hat{e}_3\hat{e}_1 \frac{\partial w_1}{\partial x_3} + \hat{e}_3\hat{e}_2 \frac{\partial w_2}{\partial x_3} + \hat{e}_3\hat{e}_3 \frac{\partial w_3}{\partial x_3}$$

The basis vectors can move out of the derivatives because they are constant (do not change with position)

This is all written in Cartesian coordinates (basis vectors are constant)

4. *Differential Operations with Vectors, Tensors (continued)*

B. Vectors - gradient *(continued)*

Gradient of a vector field

constants may appear on either side of the differential operator

$$\nabla \underline{w} \equiv \sum_{j=1}^{3}\sum_{k=1}^{3} \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \hat{e}_j \hat{e}_k \frac{\partial w_k}{\partial x_j} = \frac{\partial w_k}{\partial x_j} \hat{e}_j \hat{e}_k$$

Einstein notation for gradient of a vector

The gradient of a vector field is a *tensor*

4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence

Divergence of a vector field

$$\nabla \cdot \underline{w} \equiv \left(\hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} \right) \cdot w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3$$

$$= \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3}$$

$$= \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \frac{\partial w_i}{\partial x_i}$$

Einstein notation for gradient of a vector

The Divergence of a vector field is a *scalar*

4. *Differential Operations with Vectors, Tensors (continued)*

C. Vectors - divergence *(continued)*

This is all written in Cartesian coordinates (basis vectors are <u>constant</u>)

constants may appear on either side of the differential operator

Using Einstein notation

$$\nabla \cdot \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot w_j \hat{e}_j = \frac{\partial w_j}{\partial x_m} \hat{e}_m \cdot \hat{e}_j = \frac{\partial w_j}{\partial x_m} \delta_{mj}$$

$$= \frac{\partial w_j}{\partial x_j}$$

•divergence operation <u>decreases</u> the order of the entity operated upon

4. *Differential Operations with Vectors, Tensors (continued)*

D. Vectors - Laplacian

Using Einstein notation:

$$\nabla \cdot \nabla \underline{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} \cdot \hat{e}_p \frac{\partial}{\partial x_p} w_j \hat{e}_j = \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\hat{e}_m \cdot \hat{e}_p\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_m} \frac{\partial}{\partial x_p} w_j \left(\delta_{mp}\right) \hat{e}_j$$

$$= \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_p} w_j \hat{e}_j$$

The Laplacian of a vector field is a *vector*

$$= \begin{pmatrix} \frac{\partial^2 w_1}{\partial x_1} + \frac{\partial^2 w_1}{\partial x_2} + \frac{\partial^2 w_1}{\partial x_3} \\ \frac{\partial^2 w_2}{\partial x_1} + \frac{\partial^2 w_2}{\partial x_2} + \frac{\partial^2 w_2}{\partial x_3} \\ \frac{\partial^2 w_3}{\partial x_1} + \frac{\partial^2 w_3}{\partial x_2} + \frac{\partial^2 w_3}{\partial x_3} \end{pmatrix}_{123}$$

•Laplacian operation does not change the order of the entity operated upon

4. *Differential Operations with Vectors, Tensors (continued)*

E. Scalar - divergence ~~$\nabla \cdot \alpha$~~ *(impossible; cannot decrease order of a scalar)*

F. Scalar - Laplacian $\nabla \cdot \nabla \alpha$

G. Tensor - gradient $\nabla \underline{\underline{A}}$

H. Tensor - divergence $\nabla \cdot \underline{\underline{A}}$

I. Tensor - Laplacian $\nabla \cdot \nabla \underline{\underline{A}}$