Mass Balance *(continued)*

Consider an arbitrary volume V enclosed by a surface S

$$\begin{pmatrix} \textit{rate of increase} \\ \textit{of mass in } V \end{pmatrix} = \frac{d}{dt}\left(\iiint_V \rho \, dV \right)$$

$$\begin{pmatrix} \textit{net flux of} \\ \textit{mass into } V \\ \textit{through surface } S \end{pmatrix} = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

outwardly pointing unit normal

Mass Balance *(continued)*

$$\frac{d}{dt}\left(\iiint_V \rho \, dV \right) = -\iint_S \rho \, \hat{n} \cdot \underline{v} \, dS$$

Leibnitz rule

$$\iiint_V \frac{\partial \rho}{\partial t} dV = -\iint_S \hat{n} \cdot \left(\rho \underline{v} \right) dS$$

Gauss Divergence Theorem

$$= -\iiint_V \nabla \cdot \left(\rho \underline{v} \right) dV$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \underline{v} \right) \right) dV = 0$$

Mass Balance *(continued)*

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) \right) dV = 0$$

Since V is *arbitrary*,

Continuity equation:
microscopic mass balance

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

Mass Balance *(continued)*

Continuity equation (general fluids)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho (\nabla \cdot \underline{v}) + \underline{v} \cdot \nabla \rho = 0$$

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \underline{v}) = 0$$

For ρ=constant (incompressible fluids):

$$\nabla \cdot \underline{v} = 0$$