We:
• Defined rheology
• Contrasted with Newtonian and non-Newtonian behavior
• Saw demonstrations (film)

Now...

Key to deformation and flow is the momentum balance:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g} \]

Newtonian fluids:
• Linear
• Instantaneous
• \( \mathbf{\tau}(t) = -\mu \mathbf{\dot{v}}(t) \)

Non-Newtonian fluids:
• Non-linear
• Non-instantaneous
• \( \mathbf{\tau}(t) = ? \)

(missing piece)
Newtonian fluids:
- Linear
- Instantaneous
\[ \tau(\boldsymbol{e}) = -\mu \dot{\gamma}(t) \]

Non-Newtonian fluids:
- Non-linear
- Non-instantaneous
\[ \tau(\boldsymbol{e}) = ? \]

Key to deformation and flow is the momentum balance:
\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{e} + \rho g \]

We're going to be trying to identify the constitutive equation \( \tau(t) \) for non-Newtonian fluids.

We're going to need to calculate how different guesses affect the predicted behavior.
Key to deformation and flow is the momentum balance:

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g} \]

Newtonian fluids:
- Linear
- Instantaneous
\[ \tau_\text{eff}(t) = -\mu \dot{\gamma}(t) \]

Non-Newtonian fluids:
- Non-linear
- Non-instantaneous
\[ \tau_\text{eff}(t) = f(\gamma, \dot{\gamma}) \]

We’re going to be trying to identify the constitutive equation \( \tau(t) \) for non-Newtonian fluids.

We’re going to need to calculate how different guesses affect the predicted behavior.

We need to understand and be able to manipulate this mathematical notation.

---

Chapter 2: Mathematics Review

1. Vector review
2. Einstein notation
3. Tensors

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Chapter 2: Mathematics Review

1. Scalar – a mathematical entity that has magnitude only
   
   e.g.:  
   - temperature $T$
   - speed $v$
   - time $t$
   - density $r$

   - scalars may be constant or may be variable

   **Laws of Algebra for Scalars:**
   - yes commutative $ab = ba$
   - yes associative $a(bc) = (ab)c$
   - yes distributive $a(b+c) = ab + ac$

2. Vector – a mathematical entity that has magnitude and direction
   
   e.g.:  
   - force on a surface $f$
   - velocity $v$

   - vectors may be constant or may be variable

   **Definitions**
   - magnitude of a vector – a scalar associated with a vector $|v| = v$
   - unit vector – a vector of unit length $\hat{v} = \frac{v}{|v|}$

   This notation $(v, \hat{v}, f)$ is called **Gibbs notation**.
Laws of Algebra for Vectors:

1. Addition

\[ \mathbf{a} + \mathbf{b} = \mathbf{c} \]

2. Subtraction

\[ \mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \]

3. Multiplication by scalar

- Commutative
  \[ \alpha \mathbf{v} = \mathbf{v} \alpha \]
- Associative
  \[ \alpha (\beta \mathbf{v}) = (\alpha \beta) \mathbf{v} = \alpha \beta \mathbf{v} \]
- Distributive
  \[ \alpha (\mathbf{v} + \mathbf{w}) = \alpha \mathbf{v} + \alpha \mathbf{w} \]

4. Multiplication of vector by vector

4a. Scalar (dot) inner product

\[ \mathbf{v} \cdot \mathbf{w} = \mathbf{v} \mathbf{w} \cos \theta \]

Note: we can find magnitude with dot product

\[ \mathbf{v} \cdot \mathbf{v} = \mathbf{v} \mathbf{v} \cos 0 = \mathbf{v}^2 \]
\[ \mathbf{v} = |\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} \]
Laws of Algebra for Vectors (continued):

4a. scalar (dot) (inner) product (con’t)
- yes commutative $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- NO associative $\mathbf{y} \cdot (\mathbf{w} \cdot \mathbf{z})$ no such operation
- yes distributive $\mathbf{z} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{z} \cdot \mathbf{v} + \mathbf{z} \cdot \mathbf{w}$

4b. vector (cross) (outer) product

$\mathbf{v} \times \mathbf{w} = \mathbf{vw} \sin \theta \hat{e}$

$\hat{e}$ is a unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$ following the right-hand rule
Coordinate Systems

• Allow us to make actual calculations with vectors

Rule: any three vectors that are non-zero and linearly independent (non-coplanar) may form a coordinate basis

Three vectors are linearly dependent if \( a, b, \) and \( c \) can be found such that:

\[
\alpha a + \beta b + \gamma c = 0
\]

for \( \alpha, \beta, \gamma \neq 0 \)

If \( a, b, \) and \( c \) are found to be zero, the vectors are linearly independent.

How can we do actual calculations with vectors?

Rule: any vector may be expressed as the linear combination of three, non-zero, non-coplanar basis vectors

\[
\mathbf{a} = a_x \hat{e}_x + a_y \hat{e}_y + a_z \hat{e}_z = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}
\]

\[
= a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3
\]

\[
= \sum_{j=1}^{3} a_j \hat{e}_j
\]
Trial calculation: dot product of two vectors

\[ \mathbf{a} \cdot \mathbf{b} = (a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3) \cdot (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \]

\[ = a_1 \mathbf{e}_1 \cdot (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) + \]

\[ a_2 \mathbf{e}_2 \cdot (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) + \]

\[ a_3 \mathbf{e}_3 \cdot (b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3) \]

\[ = a_1 \mathbf{e}_1 \cdot \mathbf{b}_1 + a_1 \mathbf{e}_1 \cdot \mathbf{b}_2 + a_1 \mathbf{e}_1 \cdot \mathbf{b}_3 + \]

\[ a_2 \mathbf{e}_2 \cdot \mathbf{b}_1 + a_2 \mathbf{e}_2 \cdot \mathbf{b}_2 + a_2 \mathbf{e}_2 \cdot \mathbf{b}_3 + \]

\[ a_3 \mathbf{e}_3 \cdot \mathbf{b}_1 + a_3 \mathbf{e}_3 \cdot \mathbf{b}_2 + a_3 \mathbf{e}_3 \cdot \mathbf{b}_3 \]

If we choose the basis to be orthonormal - mutually perpendicular and of unit length - then we can simplify.

We can generalize this operation with a technique called Einstein notation.
Einstein Notation

A system of notation for vectors and tensors that allows for the calculation of results in Cartesian coordinate systems.

\[ \mathbf{a} = a_1 \mathbf{\hat{e}}_1 + a_2 \mathbf{\hat{e}}_2 + a_3 \mathbf{\hat{e}}_3 = \sum_{j=1}^{3} a_j \mathbf{\hat{e}}_j = a_j \mathbf{\hat{e}}_j = a_{mj} \mathbf{\hat{e}}_m \]

- The initial choice of subscript letter is arbitrary.
- The presence of a pair of like subscripts implies a missing summation sign.

Einstein Notation (con’t)

The result of the dot products of basis vectors can be summarized by the Kronecker delta function:

\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_1 = 1 \]
\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_2 = 0 \]
\[ \mathbf{\hat{e}}_1 \cdot \mathbf{\hat{e}}_3 = 0 \]
\[ \ldots \]

\[ \mathbf{\hat{e}}_i \cdot \mathbf{\hat{e}}_p = \delta_{ip} = \begin{cases} 1 & i = p \\ 0 & i \neq p \end{cases} \]

Kronecker delta
Einstein Notation (con’t)

To carry out a dot product of two arbitrary vectors . . .

**Detailed Notation**

\[
\begin{align*}
\vec{a} \cdot \vec{b} &= (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) \cdot (b_1 \hat{e}_1 + b_2 \hat{e}_2 + b_3 \hat{e}_3) \\
&= a_1 b_1 \hat{e}_1 \cdot \hat{e}_1 + a_1 b_2 \hat{e}_1 \cdot \hat{e}_2 + a_1 b_3 \hat{e}_1 \cdot \hat{e}_3 + \\
&\quad a_2 b_1 \hat{e}_2 \cdot \hat{e}_1 + a_2 b_2 \hat{e}_2 \cdot \hat{e}_2 + a_2 b_3 \hat{e}_2 \cdot \hat{e}_3 + \\
&\quad a_3 b_1 \hat{e}_3 \cdot \hat{e}_1 + a_3 b_2 \hat{e}_3 \cdot \hat{e}_2 + a_3 b_3 \hat{e}_3 \cdot \hat{e}_3 \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3
\end{align*}
\]

**Einstein Notation**

\[
\begin{align*}
\vec{a} \cdot \vec{b} &= a_j \hat{e}_j \cdot b_m \hat{e}_m \\
&= a_j \delta_{jm} b_m \\
&= a_j b_j
\end{align*}
\]

3. Tensor – the indeterminate vector product of two (or more) vectors

**e.g.:**

- Stress tensor \( \mathbf{T} \)
- Velocity gradient \( \vec{\gamma} \)

- Tensors may be constant or may be variable.

**Definitions**

dyad or dyadic product – a tensor written explicitly as the indeterminate vector product of two vectors

\[
\begin{align*}
\vec{a} \cdot \vec{b} &= a_d \\
\Delta &= \text{general representation of a tensor}
\end{align*}
\]

This notation \((a \cdot \vec{b}, \Delta)\) is also part of Gibbs notation.
Laws of Algebra for Indeterminate Product of Vectors:

- NO commutative: \( \vec{a} \vec{v} \neq \vec{v} \vec{a} \)
- yes associative: \( b(\vec{a} \vec{v}) = (b \vec{a}) \vec{v} = \vec{b} \vec{a} \vec{v} \)
- yes distributive: \( \vec{a}(\vec{v} + \vec{w}) = \vec{a} \vec{v} + \vec{a} \vec{w} \)

How can we represent tensors with respect to a chosen coordinate system?

Just follow the rules of tensor algebra

\[
\begin{align*}
\vec{a} \vec{m} &= (a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3)(m_1 \hat{e}_1 + m_2 \hat{e}_2 + m_3 \hat{e}_3) \\
&= a_1 m_1 \hat{e}_1 \hat{e}_1 + a_1 m_2 \hat{e}_1 \hat{e}_2 + a_1 m_3 \hat{e}_1 \hat{e}_3 + \\
&\quad a_2 m_1 \hat{e}_2 \hat{e}_1 + a_2 m_2 \hat{e}_2 \hat{e}_2 + a_2 m_3 \hat{e}_2 \hat{e}_3 + \\
&\quad a_3 m_1 \hat{e}_3 \hat{e}_1 + a_3 m_2 \hat{e}_3 \hat{e}_2 + a_3 m_3 \hat{e}_3 \hat{e}_3 \\
&= \sum_{k=1}^{3} \sum_{w=1}^{3} a_k m_w \hat{e}_k \hat{e}_w \\
&= \sum_{k=1}^{3} \sum_{w=1}^{3} a_k m_w \hat{e}_k \hat{e}_w
\end{align*}
\]

Any tensor may be written as the sum of 9 dyadic products of basis vectors.
What about $A$? 

Same.

$$A = \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} \hat{e}_i \hat{e}_j$$

Einstein notation for tensors: drop the summation sign; every double index implies a summation sign has been dropped.

$$A = A_{ij} \hat{e}_i \hat{e}_j = A_{pk} \hat{e}_p \hat{e}_k$$

Reminder: the initial choice of subscript letters is arbitrary.

---

How can we use Einstein Notation to calculate dot products between vectors and tensors?

It’s the same as between vectors.

$$a \cdot b =$$

$$a \cdot \nu =$$

$$b \cdot A =$$
Summary of Einstein Notation

1. Express vectors, tensors, (later, vector operators) in a Cartesian coordinate system as the sums of coefficients multiplying basis vectors - each separate summation has a different index
2. Drop the summation signs
3. Dot products between basis vectors result in the Kronecker delta function because the Cartesian system is orthonormal.

Note:
• In Einstein notation, the presence of repeated indices implies a missing summation sign
• The choice of initial index (i, m, p, etc.) is arbitrary - it merely indicates which indices change together

3. Tensor – (continued)

Definitions

Scalar product of two tensors

\[ A : M = A_{ip} \hat{e}_p : M_{km} \hat{e}_k \hat{e}_m \]

\[ = A_{ip} M_{km} \hat{e}_p : \hat{e}_k \hat{e}_m \]

\[ = A_{ip} M_{km} (\hat{e}_p : \hat{e}_k) (\hat{e}_i : \hat{e}_m) \]

\[ = A_{ip} M_{km} \delta_{pk} \delta_{im} \]

\[ = A_{ik} M_{km} \]

“p” becomes “k”
“i” becomes “m”
But, what is a tensor really?

A tensor is a handy representation of a **Linear Vector Function**

**Scalar function:** \[ y = f(x) = x^2 + 2x + 3 \]

a mapping of values of \( x \) onto values of \( y \)

**Vector function:** \[ w = f(y) \]

a mapping of vectors of \( y \) into vectors \( w \)

How do we express a vector function?

---

What is a linear function?

*Linear, in this usage, has a precise, mathematical definition.*

Linear functions (scalar and vector) have the following two properties:

\[
\begin{align*}
    f(\lambda x) &= \lambda f(x) \\
    f(x + w) &= f(x) + f(w)
\end{align*}
\]

It turns out . . .

Multiplying vectors and tensors is a convenient way of representing the actions of a linear vector function (as we will now show).
Tensors are Linear Vector Functions

Let \( f(a) = b \) be a linear vector function.

We can write \( g \) in Cartesian coordinates.

\[
a = a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3
\]

\[
f(a) = f(a_1 \hat{e}_1 + a_2 \hat{e}_2 + a_3 \hat{e}_3) = b
\]

Using the linear properties of \( f \), we can distribute the function action:

\[
f(a) = a_1 f(\hat{e}_1) + a_2 f(\hat{e}_2) + a_3 f(\hat{e}_3) = b
\]

These results are just vectors, we will name them \( \mathbf{v}, \mathbf{w}, \) and \( \mathbf{m} \).

Now we note that the coefficients \( a_i \) may be written as,

\[
a_1 = a \cdot \hat{e}_1 \quad a_2 = a \cdot \hat{e}_2 \quad a_3 = a \cdot \hat{e}_3
\]

Substituting,

\[
f(a) = a \cdot \hat{e}_1 \mathbf{v} + a \cdot \hat{e}_2 \mathbf{w} + a \hat{e}_3 \mathbf{m} = b
\]

The indeterminate vector product has appeared!
Using the distributive law, we can factor out the dot product with \( \mathbf{a} \):

\[
f(\mathbf{a}) = \mathbf{a} \cdot (\hat{\mathbf{e}}_1 \mathbf{v} + \hat{\mathbf{e}}_2 \mathbf{w} + \hat{\mathbf{e}}_3 \mathbf{m}) = \mathbf{b}
\]

This is just a tensor (the sum of dyadic products of vectors)

\[
(\hat{\mathbf{e}}_1 \mathbf{v} + \hat{\mathbf{e}}_2 \mathbf{w} + \hat{\mathbf{e}}_3 \mathbf{m}) = \mathbf{M}
\]

\[
f(\mathbf{a}) = \mathbf{a} \cdot \mathbf{M} = \mathbf{b}
\]

**CONCLUSION:** Tensor operations are convenient to use to express linear vector functions.

---

3. Tensor – (continued)

**More Definitions**

**Identity Tensor**

\[
\mathbf{I} = \hat{\mathbf{e}}_1 \hat{\mathbf{e}}_1 + \hat{\mathbf{e}}_2 \hat{\mathbf{e}}_2 + \hat{\mathbf{e}}_3 \hat{\mathbf{e}}_3
\]

\[
= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{123}
\]

\[
\mathbf{A} \cdot \mathbf{I} = A_{ij} \hat{\mathbf{e}}_i \hat{\mathbf{e}}_j \cdot \hat{\mathbf{e}}_k \hat{\mathbf{e}}_k
\]

\[
= A_{ij} \hat{\mathbf{e}}_i \delta_{jk} \hat{\mathbf{e}}_k
\]

\[
= A_{i} \hat{\mathbf{e}}_i
\]

\[
= \mathbf{A}
\]
3. Tensor – (continued)  

### More Definitions

#### Zero Tensor

\[
\mathbf{0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}_{123}
\]

#### Magnitude of a Tensor

\[
\mathbf{A} : \mathbf{A} = \frac{1}{2} \sqrt{\mathbf{A} : \mathbf{A} - \sum \mathbf{e}_i : \left( \sum \mathbf{e}_i \times \mathbf{e}_j \right) \mathbf{e}_j}
\]

Note that the book has a typo on this equation: the "2" is under the square root.

#### Tensor Transpose

\[
\mathbf{A}^T = \left( A_{ik} \mathbf{e}_k \mathbf{e}_j \right)^T = A_{jk} \mathbf{e}_j \mathbf{e}_k
\]

CAUTION:

\[
\left( \mathbf{A} : \mathbf{C} \right)^T = \left( \mathbf{A}_{ik} \mathbf{e}_k \mathbf{e}_j \mathbf{C}_{pj} \mathbf{e}_p \mathbf{e}_j \right)^T = \mathbf{A}_{jk} \mathbf{C}_{pj} \mathbf{e}_j \mathbf{e}_k \delta_{kp}
\]

It is not equal to:

\[
\left( \mathbf{A} : \mathbf{C} \right)^T = \left( A_{ik} \mathbf{e}_k \mathbf{e}_j \right)^T \mathbf{C}_{pj} \mathbf{e}_j
\]

I recommend you always interchange the indices on the basis vectors rather than on the coefficients.
3. Tensor – (continued)  

**Symmetric Tensor**

For a symmetric tensor, the components are invariant under interchange of indices:

\[ \mathbf{M} = \mathbf{M}^T \]

\[ M_{ik} = M_{ki} \]

**Example**

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 5 \\
3 & 5 & 6
\end{pmatrix}_{123}
\]

**Antisymmetric Tensor**

For an antisymmetric tensor, the components change sign under interchange of indices:

\[ \mathbf{M} = -\mathbf{M}^T \]

\[ M_{ik} = -M_{ki} \]

**Example**

\[
\begin{pmatrix}
0 & -2 & -3 \\
2 & 0 & -5 \\
3 & 5 & 0
\end{pmatrix}_{123}
\]

---

**Tensor Order**

Scalars, vectors, and tensors may all be considered to be tensors (entities that exist independent of coordinate system). They are tensors of different orders, however.

**Order** = degree of complexity

<table>
<thead>
<tr>
<th>Scalars</th>
<th>0th-order tensors</th>
<th>3^0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vectors</td>
<td>1st-order tensors</td>
<td>3^1</td>
</tr>
<tr>
<td>Tensors</td>
<td>2nd-order tensors</td>
<td>3^2</td>
</tr>
<tr>
<td>Higher-order tensors</td>
<td>3rd-order tensors</td>
<td>3^3</td>
</tr>
</tbody>
</table>

(Number of coefficients needed to express the tensor in 3D space)
3. Tensor — (continued)  

**More Definitions**

### Tensor Invariants

Scalars that are associated with tensors; these are numbers that are independent of coordinate system.

**vectors:** \( \|v\| = v \)  
The magnitude of a vector is a scalar associated with the vector.  
It is independent of coordinate system, i.e. it is an invariant.

**tensors:** \( A \)  
There are three invariants associated with a second-order tensor.

**Tensor Invariants**

\[
I_4 = \text{trace} A = tr A \\
II_4 = \text{trace}(A \cdot A) = A : A = A_{pk} A_{kp} \\
III_4 = \text{trace}(A \cdot A \cdot A) = A_{pj} A_{jh} A_{hp}
\]

For the tensor written in Cartesian coordinates:

\[
\text{trace} A = A_{pp} = A_{11} + A_{22} + A_{33}
\]

Note: the definitions of invariants written in terms of coefficients are only valid when the tensor is written in Cartesian coordinates.
4. Differential Operations with Vectors, Tensors

Scalars, vectors, and tensors are differentiated to determine rates of change (with respect to time, position)

- To carry out the differentiation with respect to a single variable, differentiate each coefficient individually.
- There is no change in order (vectors remain vectors, scalars remain scalars, etc.

\[
\frac{\partial \alpha}{\partial t} = \frac{\partial w}{\partial t} = \frac{\partial B}{\partial t} = \begin{pmatrix}
\frac{\partial B_{1}}{\partial t} \\
\frac{\partial B_{2}}{\partial t} \\
\frac{\partial B_{3}}{\partial t}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial B_{1}}{\partial t} \\
\frac{\partial B_{2}}{\partial t} \\
\frac{\partial B_{3}}{\partial t}
\end{pmatrix}
\]

- To carry out the differentiation with respect to 3D spatial variation, use the del (nabla) operator.
- This is a vector operator.
- Del may be applied in three different ways.
- Del may operate on scalars, vectors, or tensors.

\[
\nabla \equiv \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\frac{\partial}{\partial x_2} \\
\frac{\partial}{\partial x_3}
\end{pmatrix}
\]

This is written in Cartesian coordinates

\[
\nabla = \sum_{p=1}^{3} \hat{e}_p \frac{\partial}{\partial x_p} = \hat{e}_p \frac{\partial}{\partial x_p}
\]

Einstein notation for del
4. Differential Operations with Vectors, Tensors (continued)

A. Scalars - gradient

Gradient of a scalar field

\[ \nabla \beta = \hat{e}_1 \frac{\partial \beta}{\partial x_1} + \hat{e}_2 \frac{\partial \beta}{\partial x_2} + \hat{e}_3 \frac{\partial \beta}{\partial x_3} \]

This is written in Cartesian coordinates

This gradient operation increases the order of the entity operated upon.

B. Vectors - gradient

The basis vectors can move out of the derivatives because they are constant (do not change with position).

\[ \nabla \mathbf{w} = \hat{e}_1 \frac{\partial \mathbf{w}}{\partial x_1} + \hat{e}_2 \frac{\partial \mathbf{w}}{\partial x_2} + \hat{e}_3 \frac{\partial \mathbf{w}}{\partial x_3} \]

This is all written in Cartesian coordinates (basis vectors are constant)
B. Vectors - gradient (continued)

Gradient of a vector field

\[ \nabla \mathbf{w} = \sum_{j=1}^{3} \sum_{k=1}^{3} \hat{e}_j \frac{\partial w_k}{\partial x_j} = \hat{e}_j \frac{\partial w_k}{\partial x_j} = \sum_{j=1}^{3} \hat{e}_j \frac{\partial w_k}{\partial x_j} \]

Constants may appear on either side of the differential operator.

The gradient of a vector field is a tensor.

C. Vectors - divergence

Divergence of a vector field

\[ \nabla \cdot \mathbf{w} = \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) \cdot (w_1 \hat{e}_1 + w_2 \hat{e}_2 + w_3 \hat{e}_3) \]

\[ = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} \]

\[ = \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} \]

The divergence of a vector field is a scalar.

Einstein notation for divergence of a vector.
Chapter 2: Vectors and Tensors

Mathematics Review

4. Differential Operations with Vectors, Tensors (continued)

C. Vectors - divergence (continued)

This is all written in Cartesian coordinates (basis vectors are constant)

Using Einstein notation:

\[ \nabla \cdot \mathbf{w} \equiv \hat{e}_m \frac{\partial}{\partial x_m} w_j \hat{e}_j = \frac{\partial w_j}{\partial x_m} \epsilon^m_j = \frac{\partial w_j}{\partial x_j} \delta^j_m \]

\[ = \frac{\partial w_j}{\partial x_j} \]

• divergence operation decreases the order of the entity operated upon

D. Vectors - Laplacian

Using Einstein notation:

\[ \nabla \cdot \nabla \mathbf{w} \equiv \epsilon^m_p \frac{\partial}{\partial x_p} \frac{\partial w_j}{\partial x_m} \hat{e}_j = \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_m} w_j \left( \epsilon^m_n \hat{e}_n \right) \]

\[ = \frac{\partial}{\partial x_p} \frac{\partial w_j}{\partial x_m} \hat{e}_j \]

\[ = \frac{\partial}{\partial x_p} \frac{\partial w_j}{\partial x_p} \hat{e}_j \]

The Laplacian of a vector field is a vector

\[ \left( \frac{\partial^2 w_1}{\partial x_1^2} + \frac{\partial^2 w_1}{\partial x_2^2} + \frac{\partial^2 w_1}{\partial x_3^2} \right) \]

\[ + \left( \frac{\partial^2 w_2}{\partial x_1^2} + \frac{\partial^2 w_2}{\partial x_2^2} + \frac{\partial^2 w_2}{\partial x_3^2} \right) \]

\[ + \left( \frac{\partial^2 w_3}{\partial x_1^2} + \frac{\partial^2 w_3}{\partial x_2^2} + \frac{\partial^2 w_3}{\partial x_3^2} \right) \]

• Laplacian operation does not change the order of the entity operated upon

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4. Differential Operations with Vectors, Tensors (continued)

E. Scalar - divergence
\[ \nabla \cdot \alpha \]
(Impossible; cannot decrease order of a scalar)

F. Scalar - Laplacian
\[ \nabla \cdot \nabla \alpha \]

G. Tensor - gradient
\[ \nabla A \]

H. Tensor - divergence
\[ \nabla \cdot A \]

I. Tensor - Laplacian
\[ \nabla \cdot \nabla A \]

5. Curvilinear Coordinates

Cylindrical
\[ \bar{r}, \theta, z \] \[ \hat{e}_r, \hat{e}_\theta, \hat{e}_z \]

Spherical
\[ r, \theta, \phi \] \[ \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \]

Note: my spherical \( \theta \) comes from the \( z \)-axis.
5. Curvilinear Coordinates

Cylindrical \( \bar{r}, \theta, z \) \( \hat{e}_r, \hat{e}_\theta, \hat{e}_z \)

Spherical \( r, \theta, \phi \) \( \hat{e}_r, \hat{e}_\theta, \hat{e}_\phi \)

These coordinate systems are ortho-normal, but they are not constant (they vary with position).

This causes some non-intuitive effects when derivatives are taken.

Note: my spherical \( \theta \) comes from the \( z \)-axis.
First, we need to write this in cylindrical coordinates.

\[ \mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \]

\[ \nabla \cdot \mathbf{v} = \nabla \cdot \left( v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right) \]

\[ = \left( \frac{\partial}{\partial x} \mathbf{e}_x + \frac{\partial}{\partial y} \mathbf{e}_y + \frac{\partial}{\partial z} \mathbf{e}_z \right) \cdot \left( v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right) \]

\[ = \left( \frac{\partial}{\partial r} \mathbf{e}_r + \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r} \frac{\partial}{\partial z} \mathbf{e}_z \right) \cdot \left( v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \right) \]

\[ = \frac{1}{r} \left( \frac{\partial v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r} \frac{\partial v_z}{\partial z} \right) \]

Solve for Cartesian basis vectors and substitute above using chain rule (see next slide for details).

\[ \mathbf{e}_r = \cos \theta \mathbf{e}_x + \sin \theta \mathbf{e}_y \]

\[ \mathbf{e}_\theta = -\sin \theta \mathbf{e}_x + \cos \theta \mathbf{e}_y \]

\[ \mathbf{e}_z = \mathbf{e}_z \]

\[ x = r \cos \theta \]

\[ y = r \sin \theta \]

\[ z = z \]
Chapter 2: Vectors and Tensors

5. Curvilinear Coordinates (continued)

Result: \[ \nabla = \left( \frac{\partial}{\partial x} \hat{e}_x + \frac{\partial}{\partial y} \hat{e}_y + \frac{\partial}{\partial z} \hat{e}_z \right) = \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \]

Now, proceed:

\[ \nabla \cdot \mathbf{v} = \left( \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z} \right) \cdot \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) = \hat{e}_r \frac{\partial}{\partial r} \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) + \hat{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) + \hat{e}_z \frac{\partial}{\partial z} \left( v_r \hat{e}_r + v_\theta \hat{e}_\theta + v_z \hat{e}_z \right) \]

(We cannot use Einstein notation because these are not Cartesian coordinates)
5. Curvilinear Coordinates (continued)

This term is not intuitive, and appears because the basis vectors in the curvilinear coordinate systems vary with position.

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5. Curvilinear Coordinates (continued)

Curvilinear Coordinates (summary)

- The basis vectors are ortho-normal
- The basis vectors are non-constant (vary with position)
- These systems are convenient when the flow system mimics the coordinate surfaces in curvilinear coordinate systems.
- *We cannot* use Einstein notation – must use Tables in Appendix C2 (pp464-468).

6. Vector and Tensor Theorems and definitions

In Chapter 3 we review Newtonian fluid mechanics using the vector/tensor vocabulary we have learned thus far. We just need a few more theorems to prepare us for those studies. These are presented without proof.

Gauss Divergence Theorem

\[ \iiint_V \nabla \cdot \mathbf{b} \, dV = \iint_S \mathbf{n} \cdot \mathbf{b} \, dS \]

This theorem establishes the utility of the divergence operation. The integral of the divergence of a vector field over a volume is equal to the net outward flow of that property through the bounding surface.
6. Vector and Tensor Theorems (continued)

**Leibnitz Rule** for differentiating integrals

\[
I = \int_{\alpha}^{\beta} f(x,t) \, dx
\]

\[
\frac{dI}{dt} = \frac{d}{dt} \int_{\alpha}^{\beta} f(x,t) \, dx
\]

\[
= \int_{\alpha}^{\beta} \frac{\partial f(x,t)}{\partial t} \, dx
\]
### 6. Vector and Tensor Theorems (continued)

**Leibnitz Rule** for differentiating integrals

For **variable limits** in one dimension:

\[
J = \int_{a(t)}^{\beta(t)} f(x,t) \, dx
\]

\[
\frac{dJ}{dt} = \frac{d}{dt} \int_{a(t)}^{\beta(t)} f(x,t) \, dx = \int_{a(t)}^{\beta(t)} \frac{\partial f(x,t)}{\partial t} \, dx + \frac{d\beta}{dt} f(\beta,t) - \frac{d\alpha}{dt} f(\alpha,t)
\]

For **variable limits** in three dimensions:

\[
J = \iiint_{V(t)} f(x,y,z,t) \, dV
\]

\[
\frac{dJ}{dt} = \frac{d}{dt} \iiint_{V(t)} f(x,y,z,t) \, dV = \iiint_{V(t)} \frac{\partial f(x,y,z,t)}{\partial t} \, dV + \iint_{S(t)} f(x,y,z,t) \, dS
\]

where \( dS \) is the **velocity of the surface element**.
Consider a function \( f(x, y, z, t) \)

true for any path:

\[
    df \equiv \left( \frac{\partial f}{\partial x} \right)_y \, dx + \left( \frac{\partial f}{\partial y} \right)_x \, dy + \left( \frac{\partial f}{\partial z} \right)_y \, dz + \left( \frac{\partial f}{\partial t} \right)_y \, dt
\]

choose special path:

\[
    \frac{df}{dt} \equiv \left( \frac{\partial f}{\partial x} \right)_y \, \frac{dx}{dt} + \left( \frac{\partial f}{\partial y} \right)_x \, \frac{dy}{dt} + \left( \frac{\partial f}{\partial z} \right)_y \, \frac{dz}{dt} + \left( \frac{\partial f}{\partial t} \right)_y \, \frac{dt}{dt}
\]

When the chosen path is the path of a fluid particle, then these are the components of the particle velocities.

Substantial Derivative

\[
    \frac{df}{dt} \quad \text{along a particle path}
\]

\[
    \text{Gibbs notation}
\]

\[
    \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f
\]
Notation Summary:

Gibbs—no reference to coordinate system \((a, A, \nabla p, \nabla \cdot a)\)

Einstein—references to Cartesian coordinate system (ortho-normal, constant) \((a_1 \hat{e}_1, A_{p k} \hat{e}_p \hat{e}_k)\)

Matrix—uses column or row vectors for vectors and \(3 \times 3\) matrix of coefficients for tensors

\[
\begin{pmatrix}
a_1 \\ a_2 \\ a_3
\end{pmatrix}, \begin{pmatrix}
A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}
\end{pmatrix}_{123}
\]

Curvilinear coordinate—references to curvilinear coordinate system (ortho-normal, vary with position)

\[
\begin{pmatrix}
a_r \\ a_\theta \\ a_\phi
\end{pmatrix}, \begin{pmatrix}
A_{rr} & A_{r\theta} & A_{r\phi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\phi} \\ A_{\phi r} & A_{\phi\theta} & A_{\phi\phi}
\end{pmatrix}_{r\theta\phi}
\]

Done with Math background.

Let’s use it with Newtonian fluids