Chapter 4: Standard Flows

Newtonian fluids: \( \tau = -\mu \gamma \)

non-Newtonian fluids: \( \tau \neq -\mu \gamma \)

How can we investigate non-Newtonian behavior?

Chapter 4: Standard Flows for Rheology

\( \nu_i(H) = \dot{\gamma}_i H \)

\( \dot{\gamma}_0 = \text{constant} \)

shear

elongation

© Faith A. Morrison, Michigan Tech U.
On to . . . Polymer Rheology . . .

We now know how to model Newtonian fluid motion, \( \mathbf{v}(x, t), p(x, t) \):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]
Continuity equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \tau + \rho \mathbf{g}
\]
Cauchy momentum equation

\[
\tau = -\mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T)
\]
Newtonian constitutive equation

Rheological Behavior of Fluids – Non-Newtonian

How do we model the motion of Non-Newtonian fluid fluids?

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]
Continuity equation

\[
\rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \cdot \tau + \rho \mathbf{g}
\]
Cauchy Momentum Equation

\[
\tau = f(x, t)
\]
Non-Newtonian constitutive equation
Rheological Behavior of Fluids – Non-Newtonian

How do we model the motion of Non-Newtonian fluid fluids?

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]

Continuity equation

\[ \rho \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p - \nabla \tau + \rho \mathbf{g} \]

Cauchy Momentum Equation

\[ \tau = f(x, t) \]

Non-Newtonian constitutive equation

This is the missing piece

Chapter 4: Standard Flows for Rheology

Chapter 4: Standard flows
Chapter 5: Material Functions
Chapter 6: Experimental Data

To get to constitutive equations, we must first quantify how non-Newtonian fluids behave

\{ New Constitutive Equations \}

\[ \tau = f(x, t) \]
What do we observe?

Rheological Behavior of Fluids – **Newtonian**

1. **Strain response to imposed shear stress**
   - shear rate is constant

   \[ \dot{\gamma} = \frac{d\gamma}{dt} = \text{constant} \]

2. **Pressure-driven flow in a tube (Poiseuille flow)**
   - viscosity is constant

   \[ Q = \frac{\pi \Delta P R^4}{8 \mu L} = \text{constant} \]

3. **Stress tensor in shear flow**
   - only two components are nonzero

   \[ \tau = \begin{pmatrix} 0 & \tau_{12} & 0 \\ \tau_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{123} \]

---

What do we observe?

Rheological Behavior of Fluids – **Non-Newtonian**

1. **Strain response to imposed shear stress**
   - shear rate is variable

2. **Pressure-driven flow in a tube (Poiseuille flow)**
   - viscosity is variable

3. **Stress tensor in shear flow**
   - all 9 components are nonzero

\[ \tau = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix}_{123} \]
Non-Newtonian Constitutive Equations

- We have observations that some materials are not like Newtonian fluids.
- How can we be systematic about developing new, unknown models for these materials?

Need measurements

For Newtonian fluids, measurements were **easy**:
- independent of flow (use shear flow)
- one stress, $\tau_{21}$
- one material constant, $\mu$ (viscosity)

\[ \tau = -\mu (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \]

For non-Newtonian fluids, measurements are **not easy**:
- Depends on the flow (shear flow is not the only choice)
- Four non-zero stresses even in shear, $\tau_{21}, \tau_{11}, \tau_{22}, \tau_{33}$
- Unknown number of material constants in $\tau(\mathbf{v})$
- Unknown number of material functions in $\tau(\mathbf{v})$

$\tau = ???$
Non-Newtonian Constitutive Equations

We know we need to make measurements to know more,

For non-Newtonian fluids, measurements are not easy:

- Depends on the flow (shear flow is not the only choice)
- Four non-zero stresses even in shear, \( \tau_{21}, \tau_{11}, \tau_{22}, \tau_{33} \)
- Unknown number of material constants in \( \mathbf{T}(\mathbf{v}) \)
- Unknown number of material functions in \( \mathbf{T}(\mathbf{v}) \)

\[ \mathbf{T} = ??? \]
Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab

Chapter 4: Standard flows
Non-Newtonian Constitutive Equations

What should we do?

1. Pick a small number of simple flows Chapter 4: Standard flows
   - Standardize the flows
   - Make them easy to calculate with
   - Make them easy to produce in the lab

2. Make calculations
3. Make measurements
   Chapter 5: Material Functions
   Chapter 6: Experimental Data

4. Try to deduce $\tau(\dot{\gamma})$
   Chapter 7: GNF
   Chapter 8: GLVE
   Chapter 9: Advanced

© Faith A. Morrison, Michigan Tech U.
**Standard flows** – choose a velocity field (not an apparatus or a procedure)

- For model predictions, calculations are straightforward
- For experiments, design can be optimized for accuracy and fluid variety

**Material functions** – choose a common vocabulary of stress and kinematics to report results

- Make it easier to compare model/experiment
- Record an “inventory” of fluid behavior (expertise)
How can we investigate non-Newtonian behavior?

**Newtonian fluids:**
\[ \tau = -\mu \dot{\gamma} \]

**Non-Newtonian fluids:**
\[ \tau \neq -\mu \dot{\gamma} \]

**Simple Shear Flow**

velocity field
\[ v_i(H) = V = \dot{\gamma} H \]
\[ \dot{\gamma}_0 = \text{constant} \]

path lines
\[ v = \begin{pmatrix} \dot{\gamma}(t)x_2 \\ 0 \\ 0 \end{pmatrix} \]
Near solid surfaces, the flow is shear flow.
Standard Nomenclature for Shear Flow

Why is shear a standard flow?

- simple velocity field
- represents all sliding flows
- simple stress tensor
How do particles move apart in shear flow?

Consider two particles in the same \( x_1-x_2 \) plane, initially along the \( x_2 \) axis.

\[ \begin{align*}
  t = 0 & \\
  & \\
  t > 0 &
\end{align*} \]

Each particle has a different velocity depending on its \( x_2 \) position:

\[ \begin{align*}
  v_1 &= \gamma_0 x_2 \\
  v_2 &= \gamma_0 l_1 \\
  P_1 : & \quad v_1 = \gamma_0 l_1 \\
  P_2 : & \quad v_1 = \gamma_0 l_2
\end{align*} \]

The initial \( x_1 \) position of each particle is \( x_1 = 0 \). After \( t \) seconds, the two particles are at the following positions:

\[ \begin{align*}
  P_1(t) : & \quad x_1 = \gamma_0 l_1 t \\
  P_2(t) : & \quad x_1 = \gamma_0 l_2 t
\end{align*} \]

location = initial + \( \begin{array}{c}
  \text{length} \\
  \text{time}
\end{array} \)
What is the separation of the particles after time $t$?

$$l^2 = l_0^2 + \left[ \dot{\gamma}_0 t (l_2 - l_1) \right]^2$$

$$= l_0^2 + \dot{\gamma}_0^2 t^2 l_1^2$$

$$= l_0^2 \left( 1 + \dot{\gamma}_0^2 t^2 \right)$$

$$l = l_0 \sqrt{1 + \dot{\gamma}_0^2 t^2} \approx l_0 \dot{\gamma}_0 t$$

negligible as $t \to \infty$

In shear the distance between points is directly proportional to time.

Uniaxial Elongational Flow

$$v = \begin{pmatrix} \dot{\epsilon}(t) x_1 \\ -\frac{\dot{\epsilon}(t)}{2} x_2 \\ -\frac{\dot{\epsilon}(t)}{2} x_3 \end{pmatrix}$$

$\dot{\epsilon}(t) > 0$
Uniaxial Elongational Flow

\[ y = \begin{bmatrix} \dot{\varepsilon}(t) x_1 \\ -\frac{1}{2} \dot{\varepsilon}(t) x_2 \\ \dot{\varepsilon}(t) x_3 \end{bmatrix} \]

\[ \dot{\varepsilon}(t) > 0 \]

path lines

Elongational flow occurs when there is stretching - die exit, flow through contractions

© Faith A. Morrison, Michigan Tech U.
Experimental Elongational Geometries

$X_1 \rightarrow X_2$

$X_1 \rightarrow X_2$

Air-bed to support sample

$X_1 \rightarrow X_2$

Fluid

$t_0$

$t_0 + \Delta t$

$t_0 + 2\Delta t$

Thin, lubricating layer on each plate

Sentmanat Extension Rheometer (2005)

- Originally developed for rubbers, good for melts
- Measures elongational viscosity, startup, other material functions
- Two counter-rotating drums
- Easy to load; reproducible

http://www.xpansioninstruments.com/rheo-optics.htm
Why is elongation a standard flow?

- simple velocity field
- represents all stretching flows
- simple stress tensor

How do particles move apart in elongational flow?

Consider two particles in the same $x_1$-$x_3$ plane, initially along the $x_3$ axis.

At $t = 0$:

- $P_1(0,0,\frac{l_o}{2})$
- $P_2(0,0,\frac{-l_o}{2})$

© Faith A. Morrison, Michigan Tech U.
How do particles move apart in elongational flow?

Consider two particles in the same $x_1$-$x_3$ plane, initially along the $x_3$ axis.

\[
\begin{align*}
  x_1 &= 0 \\
  x_2 &= 0 \\
  x_3 &\text{ varies}
\end{align*}
\]

\[
v = \begin{pmatrix}
  \dot{\varepsilon}_0 x_1 \\
  \dot{\varepsilon}_0 x_2 \\
  \dot{\varepsilon}_0 x_3
\end{pmatrix}_{123}
\]

\[
v = \begin{pmatrix}
  0 \\
  0 \\
  \dot{\varepsilon}_0 x_3
\end{pmatrix}_{123}
\]

\[
v_x = \frac{dx_3}{dt} = \dot{\varepsilon}_0 x_3
\]

\[
ln x_3 = \dot{\varepsilon}_0 t + C_1
\]

\[
x_3 = x_3(0)e^{\dot{\varepsilon}_0 t}
\]

\[
l = l_0 e^{\dot{\varepsilon}_0 t}
\]

Particles move apart exponentially fast.

A second type of shear-free flow: Biaxial Stretching

\[
y = \begin{pmatrix}
  -\dot{\varepsilon}(t)x_1 \\
  \dot{\varepsilon}(t)x_2 \\
  \dot{\varepsilon}(t)x_3
\end{pmatrix}_{123}
\]

\[
\dot{\varepsilon}(t) < 0
\]
How do uniaxial and biaxial deformations differ?

Consider a uniaxial flow in which a particle is doubled in length in the flow direction.

How do uniaxial and biaxial deformations differ?

Consider a biaxial flow in which a particle is doubled in length in the flow direction.
A third type of shear-free flow:
**Planar Elongational Flow**

\[
\mathbf{v} = \begin{pmatrix}
-\dot{\epsilon}(t)x_1 \\
0 \\
\dot{\epsilon}(t)x_3
\end{pmatrix}_{123}, \quad \dot{\epsilon}(t) > 0
\]

All three shear-free flows can be written together as:

\[
\mathbf{v} = \begin{pmatrix}
-\frac{1}{2}\dot{\epsilon}(t)(1+b)x_1 \\
-\frac{1}{2}\dot{\epsilon}(t)(1-b)x_2 \\
\frac{1}{2}\dot{\epsilon}(t)x_3
\end{pmatrix}_{123}
\]

Elongational flow: \( b=0, \quad \dot{\epsilon}(t) > 0 \)
Biaxial stretching: \( b=0, \quad \dot{\epsilon}(t) < 0 \)
Planar elongation: \( b=1, \quad \dot{\epsilon}(t) > 0 \)
Why have we chosen these flows?

**Answer:** Because these simple flows have **symmetry**.

And symmetry allows us to draw conclusions about the stress tensor that is associated with these flows **for any fluid** subjected to that flow.

---

**In general:**

\[ \tau = \begin{pmatrix}
\tau_{11} & \tau_{12} & \tau_{13} \\
\tau_{21} & \tau_{22} & \tau_{23} \\
\tau_{31} & \tau_{32} & \tau_{33}
\end{pmatrix}_{123} \]

But the stress tensor is **symmetric** – leaving 6 independent stress components.

**Can we choose a flow to use in which there are fewer than 6 independent stress components?**

Yes we can – **symmetric flows**
How does the stress tensor simplify for shear (and later, elongational) flow?

\[
\begin{pmatrix}
P(3,1,0)_{123}
\end{pmatrix}
\]

\[
\text{What would the velocity function be for a Newtonian fluid in this coordinate system?}
\]

\[
v = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix}_{123}
\]
What would the velocity function be for a Newtonian fluid in this coordinate system?

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} \]

Vectors are independent of coordinate system, but in general the coefficients will be different when the same vector is written in two different coordinate systems:

\[ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{123} \]

For shear flow and the two particular coordinate systems we have just examined, however:

\[ \mathbf{v} = \begin{pmatrix} \frac{V}{2H}x_2 \\ 0 \\ 0 \end{pmatrix}_{123} = \begin{pmatrix} \frac{V}{2H}x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]
If we plug in the same number in for $x_2$ and $\bar{x}_2$, we will NOT be asking about the same point in space, but we WILL get the same exact velocity vector.

Since stress is calculated from the velocity field, we will get the same exact stress components when we calculate them from either vector representation.

This is an unusual circumstance only true for the particular coordinate systems chosen.

What do we learn if we formally transform $\mathbf{v}$ from one coordinate system to the other?
What do we learn if we formally transform $\tau$ from one coordinate system to the other?

$\hat{e}_1 = -\bar{e}_1$
$\hat{e}_2 = -\bar{e}_2$
$\hat{e}_3 = \bar{e}_3$

What do we learn if we formally transform $\mathbf{v}$ from one coordinate system to the other?

$\tau = \tau_m \hat{e}_m \hat{e}_s = \bar{\tau}_m \bar{e}_m \bar{e}_s$

(now, substitute from previous slide and simplify)

You try.
Because of symmetry, there are only 5 nonzero components of the extra stress tensor in shear flow.

\[
\tau = \begin{pmatrix}
\tau_{11} & \tau_{12} & 0 \\
\tau_{21} & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}
\]

This greatly simplifies the experimentalists tasks as only four stress components must be measured instead of 6 (recall \(\tau_{21} = \tau_{12}\)).

Conclusion:

Summary:

We have found a coordinate system (the shear coordinate system) in which there are only 5 non-zero coefficients of the stress tensor. In addition, \(\tau_{21} = \tau_{12}\).

This leaves only four stress components to be measured for this flow, expressed in this coordinate system.
How does the stress tensor simplify for elongational flow?

There is 180° of symmetry around all three coordinate axes.

Because of symmetry, there are only 3 nonzero components of the extra stress tensor in elongational flows.

\[
\tau = \begin{pmatrix}
\tau_{11} & 0 & 0 \\
0 & \tau_{22} & 0 \\
0 & 0 & \tau_{33}
\end{pmatrix}
\]

This greatly simplifies the experimentalists tasks as only three stress components must be measured instead of 6.
Standard Flows Summary

Choose velocity field:
\[ \mathbf{v} = \begin{pmatrix} \xi(t) x_2 \\ 0 \\ 0 \end{pmatrix} \]

Symmetry alone implies:
(no constitutive equation needed yet)
\[ \mathbf{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} \]

\[ \mathbf{\tau} = \begin{pmatrix} \frac{1}{2} \dot{\xi}(t)(1 + b)x_1 \\ \frac{1}{2} \dot{\xi}(t)(1 - b)x_2 \\ \dot{\xi}(t)x_3 \end{pmatrix} \]

By choosing these symmetric flows, we have reduced the number of stress components that we need to measure.
Next, build and assume this

Choose velocity field:

\[ \mathbf{v} = \begin{pmatrix} \xi(t)x_2 \\ 0 \\ 0 \end{pmatrix}_{123} \]

\[ \mathbf{v} = \begin{pmatrix} -\frac{1}{2} \dot{\xi}(t)(1+b)x_1 \\ -\frac{1}{2} \dot{\xi}(t)(1-b)x_2 \\ \dot{\xi}(t)x_3 \end{pmatrix}_{123} \]

Symmetry alone implies:

(no constitutive equation needed yet)

\[ \mathbf{\tau} = \begin{pmatrix} \tau_{11} & \tau_{12} & 0 \\ \tau_{21} & \tau_{22} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}_{123} \]

Measure and predict this

One final comment on measuring stresses...

What is measured is the total stress, \( \Pi \):

\[ \Pi = \begin{pmatrix} p + \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & p + \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & p + \tau_{33} \end{pmatrix}_{123} \]

For the normal stresses we are faced with the difficulty of separating \( p \) from \( \tau_{ii} \).

Compressible fluids: Get \( p \) from measurements of \( T \) and \( V \).

Incompressible fluids: ?
Density does not vary (much) with pressure for polymeric fluids.

\[ \rho = \frac{M}{RT} \]

For incompressible fluids it is not possible to separate \( p \) from \( t_{ii} \).

Luckily, this is not a problem since we only need

\[ \nabla \cdot \mathbf{\Pi} = \nabla p + \nabla \cdot \mathbf{\tau} \]

Equation of motion

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla \mathbf{\Pi} + \rho \mathbf{g} \\
= -\nabla p - \nabla \cdot \mathbf{\tau} + \rho \mathbf{g}
\]

We do not need \( t_{ii} \) directly to solve for velocities

Solution? Normal stress differences
Normal Stress Differences

First normal stress difference
\[ N_1 \equiv \Pi_{11} - \Pi_{22} = \tau_{11} - \tau_{22} \]
Second normal stress difference
\[ N_2 \equiv \Pi_{22} - \Pi_{33} = \tau_{22} - \tau_{33} \]

In shear flow, three stress quantities are measured
\[ \tau_{21}, N_1, N_2 \]
In elongational flow, two stress quantities are measured
\[ \tau_{33} - \tau_{11}, \tau_{22} - \tau_{11} \]

Are shear normal stress differences real?
**First** normal stress effects: rod climbing

\[ \tau_{11} - \tau_{22} < 0 \]

Extra tension in the 1-direction pulls azimuthally and upward (see DPL p65).

Newtonian - glycerin  
Viscoelastic - solution of polyacrylamide in glycerin

Bird, et al., *Dynamics of Polymeric Fluids*, vol. 1, Wiley, 1987, Figure 2.3-1 page 63. (DPL)

© Faith A. Morrison, Michigan Tech U.

---

**Second** normal stress effects: inclined open-channel flow

\[ \tau_{22} - \tau_{33} > 0 \]

Extra tension in the 2-direction pulls down the free surface where \( \frac{dv_1}{dx_2} \) is greatest (see DPL p65).

Newtonian - glycerin  
Viscoelastic - 1% soln of polyethylene oxide in water

\[ N_2 \approx -N_1 /10 \]

R. I. Tanner, *Engineering Rheology*, Oxford 1985, Figure 3.6 page 104

© Faith A. Morrison, Michigan Tech U.
Example: Can the equation of motion predict rod climbing for typical values of $N_1$, $N_2$?

\[
v = \begin{pmatrix}
0 \\
\dot{\psi} \\
0
\end{pmatrix}
\]

cross-section $A$:

What is $\frac{d\Pi_{zz}}{dr}$?

www.chem.mt.edu/~fmorriso/cm4650/rod_climb.pdf

What’s next?

Shear-free (elongational, extensional)

Even with just these 2 (or 4) standard flows, we can still generate an infinite number of flows by varying $\dot{\varepsilon}(t)$ and $\dot{\varepsilon}(t)$.

\[
v = \begin{pmatrix}
\dot{\varepsilon}(t)x_2 \\
0 \\
0
\end{pmatrix}
\]

Elongational flow: $b=0$, $\dot{\varepsilon}(t) > 0$
Biaxial stretching: $b=0$, $\dot{\varepsilon}(t) < 0$
Planar elongation: $b=1$, $\dot{\varepsilon}(t) > 0$

Shear

\[
v = \begin{pmatrix}
-\frac{1}{2}\varepsilon(t)(1+b)x_1 \\
-\frac{1}{2}\varepsilon(t)(1-b)x_2 \\
\dot{\varepsilon}(t)x_3
\end{pmatrix}
\]
We seek to quantify the behavior of non-Newtonian fluids

Procedure:
1. Choose a flow type (shear or a type of elongation).
2. Specify \( \dot{\gamma}(t) \) or \( \dot{\varepsilon}(t) \) as appropriate.
3. Impose the flow on a fluid of interest.
4. Measure stresses.
5. Report stresses in terms of material functions.

6a. Compare measured material functions with predictions of these material functions (from proposed constitutive equations).
7a. Choose the most appropriate constitutive equation for use in numerical modeling.

6b. Compare measured material functions with those measured on other materials.
7a. Draw conclusions on the likely properties of the unknown material based on the comparison.

Procedure:
1. Choose a flow type (shear or a type of elongation).
2. Specify \( \dot{\gamma}(t) \) or \( \dot{\varepsilon}(t) \) as appropriate.
3. Impose the flow on a fluid of interest.
4. Measure stresses.
5. Report stresses in terms of material functions.

6a. Compare measured material functions with predictions of these material functions (from proposed constitutive equations).
7a. Choose the most appropriate constitutive equation for use in numerical modeling.

6b. Compare measured material functions with those measured on other materials.
7a. Draw conclusions on the likely properties of the unknown material based on the comparison.

© Faith A. Morrison, Michigan Tech U.

Done with Standard Flows.

Let’s move on to Material Functions

© Faith A. Morrison, Michigan Tech U.
Chapter 5: Material Functions

Steady Shear Flow Material Functions

Kinematics:
\[
\dot{\gamma} = \begin{pmatrix} 2(0\tau_{12}) \\ 0 \\ 0 \end{pmatrix} \\
\ddot{\gamma}(t) = \dot{\gamma}_0 = \text{constant}
\]

Material Functions:

\[\eta = \frac{-221}{\dot{\gamma}_0}\]

Viscosity

\[\eta_1 = \frac{-\left(\tau_{11} - \tau_{22}\right)}{\dot{\gamma}_0^2}\]

First normal-stress coefficient

\[\eta_2 = \frac{-\left(\tau_{33} - \tau_{11}\right)}{\dot{\gamma}_0^2}\]

Second normal-stress coefficient