FINAL REVIEW: KEY

1. Given the force field $\vec{F} = \vec{\imath} + 2\vec{\jmath} - t\,\vec{k}$ and the parametric equations

$$\begin{cases} x = 3\cos\sqrt{t} \\ y = 3\sin\sqrt{t} \\ z = 4\sqrt{t} \end{cases}$$

of a curve C, where $0 \leqslant t \leqslant \pi^2$, find the work over the curve C.

Key:

$$W = \int 1\,dx + 2\,dy - t\,dz = \int_3^{-3} dx + \int_0^0 2\,dy - \int_0^{\pi^2} t\,d(4\sqrt{t})$$

$$= -6 + 0 - \int_0^{\pi^2} t\,\frac{4}{2\sqrt{t}}\,dt = -6 - \frac{4}{3}\pi^3.$$

2. For each of the following force fields that are path-independent, find the corresponding potential function:

(a) $\vec{F} = -y\,\vec{\imath} + x\,\vec{\jmath}$; (b) $\vec{F} = (x^3 - 3xy^2, y^3 - 3x^2y)$.

Key:

(a) $\vec{F} = P\,\vec{\imath} + Q\,\vec{\jmath}$, where $P = -y$ and $Q = x$. One has

$$\frac{\partial P}{\partial y} = -1, \quad \frac{\partial Q}{\partial x} = 1 \neq \frac{\partial P}{\partial y},$$

so that field $\vec{F}$ is not path-independent.

(b) $\vec{F} = P\,\vec{\imath} + Q\,\vec{\jmath}$, where $P = x^3 - 3xy^2$ and $Q = y^3 - 3x^2y$. One has

$$\frac{\partial P}{\partial y} = -6xy, \quad \frac{\partial Q}{\partial x} = -6xy = \frac{\partial P}{\partial y},$$

so that field $\vec{F}$ is path-independent.

If $\vec{F} = \operatorname{grad} f$, then $f_x = x^3 - 3xy^2$ and $f_y = y^3 - 3x^2y$, so that the potential function is

$$\begin{aligned} f(x,y) &= \int f_x\,dx = \int (x^3 - 3xy^2)\,dx = \frac{x^4}{4} - \frac{3x^2y^2}{2} + g(y); \\ f_y &= -3x^2y + g'(y) = y^3 - 3x^2y; \\ g'(y) &= y^3; \\ g(y) &= \int y^3\,dy = \frac{y^4}{4} + C; \\ f(x,y) &= \frac{x^4}{4} - \frac{3x^2y^2}{2} + \frac{y^4}{4} + C = \frac{x^4 - 6x^2y^2 + y^4}{4} + C. \end{aligned}$$

3. For each of the force fields in Problem 4 that are not path-independent, find the work done over the circle of radius 2 centered at point $(0,1)$. Do this in two ways: (i) using Green's theorem and (ii) without using it.

Key:
(i)

$$\int_R \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\, dA = \int_R (1-(-1))\, dA = 2\,\mathrm{Area}(R) = 2 \times \pi 2^2 = 8\pi.$$

(ii) Parametric eqs. are $x = 2\cos t$ and $y = 1 + 2\sin t$, $0 \leqslant t \leqslant 2\pi$.

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} = \int_C -y\,dx + x\,dy &= \int_0^{2\pi} -(1+2\sin t)(-2\sin t)\,dt + 2\cos t\,(2\cos t)\,dt \\ &= \int_0^{2\pi} (2\sin t + 4)\,dt = 8\pi.\end{aligned}$$

4. Find the fluxes of the following vector fields:

(a) $\vec{F} = xyz\,\vec{i} + x\,\vec{j} - y\,\vec{k}$ through the square of side 2 centered on the x-axis in plane $x = 3$ and oriented in the negative x-direction;

Key:
$\vec{n} = -\vec{i}$;

$$\int_S (\vec{F}\cdot\vec{n})\,dA = \int_S (-xyz)\,dA = -3\int_{-1}^{1}\int_{-1}^{1} yz\,dy\,dz = 0.$$

(b) $\vec{F} = -2\vec{r}$ through the sphere of radius 5 centered at the origin and oriented outward;

Key:
$\vec{n} = \vec{r}/\|\vec{r}\| = \frac{1}{5}\,\vec{r}$;

$$\begin{aligned}\int_S (\vec{F}\cdot\vec{n})\,dA &= \int_S (-2\vec{r}\cdot(\tfrac{1}{5}\,\vec{r}))\,dA \\ &= -\frac{2}{5}\int_S \vec{r}\cdot\vec{r}\,dA \\ &= -\frac{2}{5}\times 5^2\int_S dA = -\frac{2}{5}\times 5^2\,\mathrm{Area}(S) = -\frac{2}{5}\times 5^2\, 4\pi\times 5^2 = -1000\pi.\end{aligned}$$

(c) $\vec{F} = y\vec{i} - x\vec{j} + \vec{k}$ through the upper hemisphere of radius 2 centered at the origin and oriented upward;

Key:
With $z = f(x,y) = \sqrt{4 - x^2 - y^2}$,

$$\begin{aligned}\text{Flux} &= \iint_D (F_3 - F_1 f_x - F_2 f_y)\,dx\,dy \\ &= \iint_D \Big(1 + y\frac{x}{\sqrt{4-x^2-y^2}} - x\frac{y}{\sqrt{4-x^2-y^2}}\Big)\,dx\,dy \\ &= \iint_D 1\,dx\,dy = \mathrm{Area}(D) = 2\pi\times 2^2 = 8\pi,\end{aligned}$$

where D is the disk of radius 2 in the xy-plane centered at the origin.

(d) $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k}$ through the sphere of radius 2 centered at the origin and oriented outward;

Key: $\vec{F} = y\vec{i} - x\vec{j} + z\vec{k} = 2\sin\varphi\sin\theta\vec{i} - 2\sin\varphi\cos\theta\vec{j} + 2\cos\varphi\vec{k}$;

$$\begin{aligned}\text{Flux} &= \int_0^{2\pi}\int_0^{\pi} 2^2\sin\varphi(F_1\sin\varphi\cos\theta + F_2\sin\varphi\sin\theta + F_3\cos\varphi)\,d\varphi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} 2^2\sin\varphi(2\sin\varphi\sin\theta\sin\varphi\cos\theta - 2\sin\varphi\cos\theta\sin\varphi\sin\theta + 2\cos\varphi\cos\varphi)\,d\varphi d\theta \\ &= 8\int_0^{2\pi}\int_0^{\pi}\sin\varphi\cos^2\varphi\,d\varphi d\theta = 8\int_0^{2\pi}\Big(-\frac{\cos^3\varphi}{3}\Big)\Big|_{\varphi=0}^{\varphi=\pi}d\theta = \frac{16}{3}\int_0^{2\pi}d\theta = \frac{32\pi}{3}.\end{aligned}$$

(e) $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k}$ through the entire surface (oriented inward) of the solid cylinder defined by the inequalities $x^2 + y^2 \leqslant 4$ and $0 \leqslant z \leqslant 3$;

Key: On the cylinder (side) surface, $\vec{F} = y\vec{i} + x\vec{j} + z\vec{k} = 2\sin\theta\vec{i} + 2\cos\theta\vec{j} + z\vec{k}$; so the flux through the side is

$$\begin{aligned}\text{Flux}_1 &= -\int_0^{2\pi}\int_0^{1} 2(F_1\cos\theta + F_2\sin\theta)\,dzd\theta \\ &= -\int_0^{2\pi}\int_0^{1} 2(2\sin\theta\cos\theta + 2\cos\theta\sin\theta)\,dzd\theta = 0.\end{aligned}$$

At the bottom surface, $z = 0$, the inner unit normal vector $\vec{n} = \vec{k}$ and hence $\vec{F}\cdot\vec{n} = \vec{F}\cdot\vec{k} = z = 0$. So, the flux through the bottom is

$$\text{Flux}_2 = \iint_D \vec{F}\cdot\vec{n}\,dA = 0,$$

where D is the disk of radius 2 in the xy-plane centered at the origin. At the top surface, $z = 3$, the inner unit normal vector $\vec{n} = -\vec{k}$ and hence $\vec{F}\cdot\vec{n} = -\vec{F}\cdot\vec{k} = -z = -3$. So, the flux through the bottom is

$$\text{Flux}_3 = \iint_D \vec{F}\cdot\vec{n}\,dA = \iint_D (-3)\,dA == -3\pi\times 2^2 = -12\pi,$$

which is also the total flux.

(f) $\vec{F} = x\vec{j} + xy\vec{k}$ through the entire surface (oriented outward) of the solid region defined by the inequalities $0 \leqslant z \leqslant 1 - x^2 - y^2$.

Key: The top surface is the graph of the function $f(x,y) = 1 - x^2 - y^2$ over the unit disk D in the xy-plane centered at the origin. So, the flux through the top is

$$\begin{aligned}\text{Flux}_1 &= \iint_D (F_3 - F_1f_x - F_2f_y)\,dx\,dy \\ &= \iint_D \Big(xy + x\,2y\Big)\,dx\,dy \\ &= 3\int_0^{2\pi}\int_0^{1} r\cos\theta\, r\sin\theta\, r\,dr\,d\theta = 0.\end{aligned}$$

The bottom surface is the unit disk D, with $\vec{n} = -\vec{k}$, so that at the bottom $\vec{F} \cdot \vec{n} = -xy$. So, the flux through the bottom is

$$\text{Flux}_2 = \iint_D (-xy)\,dx\,dy = 0,$$

which is also the total flux.

(g) $\vec{F} = x\vec{j} + xy\vec{k}$ through the part of the plane $x + 2y + 3z = 6$ (oriented upward) over the region in the xy-plane defined by the inequalities $x \geqslant 0$, $y \geqslant 0$, $x + 2y \leqslant 6$.

Key: The surface is the graph of the function $f(x, y) = 2 - \frac{x}{3} - \frac{2y}{3}$ over the triangular region, say R, in the xy-plane defined by the inequalities $x \geqslant 0$, $y \geqslant 0$, $x + 2y \leqslant 6$. So,

$$\begin{aligned}
\text{Flux} &= \iint_R (F_3 - F_1 f_x - F_2 f_y)\,dx\,dy \\
&= \iint_R (xy + \tfrac{2}{3}x)\,dx\,dy \\
&= \int_0^6 \int_0^{3-\frac{x}{2}} \left(xy + \tfrac{2x}{3}\right)\,dy\,dx = \frac{51}{2}.
\end{aligned}$$

5. Identify each of the 7 parts in Problem 4 which deals with the flux through a closed surface (that is, through the entire surface of a solid region.) In each such case, use the Divergence Theorem to verify the result you obtained in Problem 4.

Key: Closed surfaces appear in parts (b), (d), (e), and (f).

(b) $\vec{F} = -2\vec{r} = (-2x, -2y, -2z)$; $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = -2-2-2 = -6$. So, the flux is

$$\int_W (-6)\,dV = -6\text{Volume}(W) = -6 \times \frac{4\pi 5^3}{3} = -1000\pi,$$

where W is the ball of radius 5 centered at the origin.

(d) $\vec{F} = (y, -x, z)$; $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 1 = 1$. So, the flux is

$$\int_W 1\,dV = \text{Volume}(W) = \frac{4\pi 2^3}{3} = \frac{32\pi}{3},$$

where W is the ball of radius 2 centered at the origin.

(e) $\vec{F} = (y, x, z)$; $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 1 = 1$. So, the flux (inwards) is

$$-\int_W 1\,dV = -\text{Volume}(W) = -\pi\,2^2 \times 3 = -12\pi,$$

where W is the solid cylindrical region in question.

(f) $\vec{F} = (0, x, xy)$; $\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 0 + 0 + 0 = 0$. So, the flux is 0.