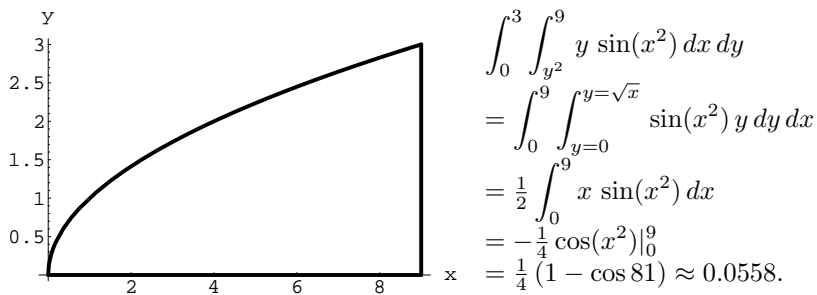


REVIEW 3: Key

1. Given the iterated integral $\int_0^3 \int_{y^2}^9 y \sin(x^2) dx dy$, draw the corresponding region of integration on the xy -plane, reverse the order of integration, and evaluate the integral.

Solution:



$$\begin{aligned} & \int_0^3 \int_{y^2}^9 y \sin(x^2) dx dy \\ &= \int_0^9 \int_{y=0}^{y=\sqrt{x}} \sin(x^2) y dy dx \\ &= \frac{1}{2} \int_0^9 x \sin(x^2) dx \\ &= -\frac{1}{4} \cos(x^2) \Big|_0^9 \\ &= \frac{1}{4} (1 - \cos 81) \approx 0.0558. \end{aligned}$$

2. Find the volume under the graph of the function $f(x, y) = 3xy + y^2$ over the region R defined by the inequalities $x \geq 0$, $y \geq 0$, and $x + 2y \leq 2$.

Solution:

The region of integration is the triangle bounded by lines $x = 0$, $y = 0$, and $x + 2y = 2$.

$$\begin{aligned} V &= \int_0^2 \int_{y=0}^{y=1-x/2} (3xy + y^2) dy dx \\ &= \int_0^2 \left(\frac{3}{2} xy^2 + \frac{y^3}{3} \right) \Big|_{y=0}^{y=1-x/2} dx \\ &= \int_0^2 \left(\frac{3}{2} x(1-x/2)^2 + \frac{(1-x/2)^3}{3} \right) dx \\ &= \int_0^2 \frac{1}{24} (4x^3 - 15x^2 + 12x + 4) dx = \frac{2}{3}. \end{aligned}$$

3. Find the mass of the pyramid bounded by the planes $x = 0$, $y = 0$, $z = 1$, and $3x + 2y + z = 7$ given the mass density function $\delta(x, y, z) = 2z$.

Key:

$$\int_0^2 \int_0^{3-3x/2} \int_1^{7-3x-2y} 2z dz dy dx = 30.$$

4. Find the double integral

$$\int_{\pi/4}^{3\pi/4} \int_0^{4/\sin \theta} r dr d\theta$$

in polar coordinates and sketch the region of integration. Interpret the double integral geometrically as an area or volume.

Key:

$$\int_{\pi/4}^{3\pi/4} \int_0^{4/\sin\theta} r \, dr \, d\theta = \int_{\pi/4}^{3\pi/4} \frac{8}{\sin^2\theta} \, d\theta = 16.$$

The region is the triangle bounded by the lines $y = x$, $y = -x$, and $y = 4$. The double integral is the area of the region; it is also the volume of the solid between the planes $z = 0$ and $z = 1$ over the region.

5. Using cylindrical coordinates, find the mass of the upper half (above the xy -plane) of the unit ball $x^2 + y^2 + z^2 \leq 1$ given the mass density function $\delta(x, y, z) = z$. Find the same mass using spherical coordinates.

Key:

$$\int_0^{2\pi} \int_0^1 \left(\int_0^{\sqrt{1-r^2}} z \, dz \right) r \, dr \, d\theta = \frac{\pi}{4} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho \cos\varphi \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta.$$

6. Given the parametric equations

$$\begin{cases} x = 3 \cos \sqrt{t} \\ y = 3 \sin \sqrt{t} \\ z = 4\sqrt{t} \end{cases}$$

of a curve C , where $0 \leq t \leq \pi^2$, find the velocity and acceleration vectors as they depend on t . Find also the speed and the length of curve C . Describe the motion and its trajectory in words.

Key:

$$\begin{aligned} \vec{v} &= \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \left(-\frac{3}{2\sqrt{t}} \sin \sqrt{t}, \frac{3}{2\sqrt{t}} \cos \sqrt{t}, \frac{2}{\sqrt{t}} \right); \\ \vec{a} &= \left(\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt} \right) \\ &= \left(\frac{3}{4t\sqrt{t}} \sin \sqrt{t} - \frac{3}{4t} \cos \sqrt{t}, -\frac{3}{4t\sqrt{t}} \cos \sqrt{t} - \frac{3}{4t} \sin \sqrt{t}, -\frac{1}{t\sqrt{t}} \right); \\ \|\vec{v}\| &= \frac{5}{2\sqrt{t}}; \end{aligned}$$

$$\text{length} = \int_0^{\pi^2} \|\vec{v}\| \, dt = \int_0^{\pi^2} \frac{5}{2\sqrt{t}} \, dt = 5\pi.$$

The motion starts at time $t = 0$ at point $(3, 0, 0)$ and proceeds along the helix trajectory with decreasing speed $\|\vec{v}\| = \frac{5}{2\sqrt{t}}$, to arrive at point $(-3, 0, 4\pi)$ at time $t = \pi^2$.