A geometric non-existence proof of an extremal additive code

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Abstract

We use a geometric approach to solve an extremal problem in coding theory. Expressed in geometric language we show the non-existence of a system of 12 lines in $PG(8,2)$ with the property that no hyperplane contains more than 5 of the lines. In coding-theoretic terms this is equivalent with the non-existence of an additive quaternary code of length 12, binary dimension 9 and minimum distance 7.

1 Introduction

The main purpose of the present paper is a geometric proof of non-existence of an additive quaternary $[12, 4.5, 7]_4$-code. While the geometric approach to linear codes is a classical branch of algebraic coding theory (see for example Chapter 16 of [1] and the survey [8]) its generalization to additive codes seems to have been considered only quite recently. Blokhuis-Brouwer [5] first studied additive quaternary codes from a geometric point of view. We concentrate on the quaternary case as well and use the following definition:

Definition 1. *Let k be such that $2k$ is a positive integer. An additive quaternary $[n, k]$-code $\mathcal{C}$ (length n, dimension k) is a $2k$-dimensional subspace of $\mathbb{F}_2^{2n}$, where the coordinates come in pairs of two. We view the codewords as n-tuples where the coordinate entries are elements of $\mathbb{F}_2^2$.*

A **generator matrix** *G of $\mathcal{C}$ is a binary $(2k, 2n)$-matrix whose rows form a basis of the binary vector space $\mathcal{C}$.*

Definition 2. *Let $\mathcal{C}$ be an additive quaternary $[n, k]$-code. The* **weight** *of a codeword is the number of its n coordinates where the entry is different from 00. The minimum weight (equal to* **minimum distance***) d of $\mathcal{C}$ is the smallest weight of its nonzero codewords. The parameters are then also written $[n, k, d]$.*

The strength of $\mathcal{C}$ is the largest number t such that all $(2k, 2t)$-submatrices of a generator matrix whose columns correspond to some t quaternary coordinates have full rank $2t$.

Here notation for length and dimension has been chosen to facilitate comparison with quaternary **linear** codes. In fact it is clear that each linear $[n, k]_4$-code is also an additive $[n, k]$-code (where k of course is an integer) and the notations of minimum distance and strength of the linear code coincide with the corresponding additive notions.

While the geometric description of a linear $[n, k]_4$-code is in terms of a multiset of n points in $PG(k-1, 4)$, the geometric description of an additive $[n, k]$-code is based on lines in $PG(2k-1, 2)$. In fact, consider a generator matrix G. For each quaternary coordinate $i \in \{1, 2, \ldots, n\}$ we are given points $P_i, Q_i \in PG(2k-1, 2)$. Let L_i be the line determined by P_i, Q_i. The geometric description of code $\mathcal{C}$ as in Definition 2 is based on this multiset of lines (the **codelines**) $\{L_1, L_2, \ldots, L_n\}$. Code $\mathcal{C}$ has minimum distance $\geq d$ if and only if for each hyperplane H of $PG(2k-1, 2)$ we find at least d codelines (in the multiset sense), which are not contained in H. Strength t means that any set of t codelines is in general position. Duality is based on the Euclidean bilinear form, the dot product for binary spaces. The dual of an additive $[n, k]$-code $\mathcal{C}$ is an $[n, n-k]$-code, and $\mathcal{C}$ has strength t if and only if $\mathcal{C}^\perp$ has minimum distance $> t$.

The optimal minimum distances d of quaternary additive codes of lengths $n \leq 12$ have been determined by Blokhuis-Brouwer [5], with two exceptions.

Our main results are sketched in [4]: additive codes $[12, 7, 5]$ and $[12, 4.5, 7]$ do not exist whereas a code $[13, 7.5, 5]$ does exist. As a result the only existence question that remains open in length $n \leq 13$ concerns $[13, 6.5, 6]$. In [3] a similar geometric approach is applied to the study of quantum stabilizer codes in the sense of [6].

In the present paper we give a detailed account of our geometric proof that an additive $[12, 4.5, 7]$-code cannot exist:

Theorem 1. *There is no additive quaternary* $[12, 4.5, 7]$*-code.*

The following concept, which is encountered in the proof, also is of independent interest:

Definition 3. *An* $[(l, r), k]_{(4,2)}$*-code is a* $2k$*-dimensional vector space of binary* $(2l+r)$*-tuples, where the coordinates are divided into* l *pairs (written on the left) and* r *single coordinates. We view each codeword as an* $(l+r)$*-tuple, where the left coordinates are quaternary, the right ones are binary.*

A code $[(l, r), k]_{(4,2)}$ is described geometrically by a multiset of l lines and r points (codelines and codepoints) in $PG(2k-1, 2)$. The code has strength $\geq t$ if any set of t objects (codepoints or codelines) are in general position. The definition of minimum distance (equal to the minimum weight) is obvious. A generator matrix is a binary $(2k, 2l + r)$-matrix whose rows form a binary basis of the code. The dual of an additive $[(l, r), k]_{(4,2)}$-code of strength t is an additive $[(l, r), l + r/2 - k, t + 1]_{(4,2)}$-code.

The geometric work happens in $PG(8, 2)$. As we find it often more convenient to work with vector space dimensions we denote i-dimensional vector subspaces by V_i $(= PG(i-1, 2))$. The following obvious observation is often useful:

Proposition 1. *Let* C *be an additive* $[n, k, d]$*-code. Assume some* i *codelines generate a subspace* V_{2i-j}*. Then the subcode of* C *consisting of the codewords with vanishing entry in those* i *coordinates is an* $[n - i, k - i + j/2, d]$*-code.*

In Section 2 we start with a synthetic construction of a self-dual $[7, 3.5, 4]$-code. This construction has a design-theoretic flavour. The proof of Theorem 1 starts in Section 3. It is geometric and coding-theoretic in nature and also relies heavily on computer searches.

2 A self-dual $[7, 3.5, 4]$-code

A computer construction of a cyclic additive $[7, 3.5, 4]$-code was given in [5]. We start with a synthetic construction of this code.

Let $\Omega = \{1, 2, 3, 4, 5, 6, 7\}$. There are exactly 30 Fano planes that can be constructed on the point set Ω. Any two different Fano planes on Ω have either $0, 1$ or 3 lines in common. In particular there exist pairs of Fano planes that do not have lines in common. For these facts see for example [2]. The result that there exist 2 but not 3 linewise disjoint Fano planes on a given 7-element ground set is attributed to Cayley [7]. Let F_1, F_2 be a pair of linewise disjoint Fano planes and D_1, D_2 codes $[7, 3, 4]_2$, that is binary codes geometrically described by F_1 and F_2, respectively. For D_1 use as alphabet $00, 01$, for D_2 the alphabet $00, 10$ and consider both as embedded in Q^7, where $Q = \mathbb{F}_2^2$. Let D be the (direct) sum of D_1 and D_2. Then D is a $[7, 3, 4]$-code. Because of the linewise disjointness of F_1, F_2 we have that $(11)^7$ has distance 4 from D and together with D generates an additive $[7, 3.5, 4]$-code. This code is self-orthogonal with respect to the dot-product.

Observe that additive codes which are self-dual or self-orthogonal with respect to the symplectic form correspond to quantum codes. A length 7 distance 4 quantum code cannot exist, see [6].

A computer program showed that 7 points can be appended to our code. This leads to a mixed $[(7, 7), 3.5]_{(4,2)}$-code of strength 3.

3 Nonexistence of an additive $[12, 4.5, 7]$-code

We work in $PG(8, 2)$. There can be no 5 codelines in a V_6 as we would find a hyperplane V_8 containing more than 5 such lines.

Assume there is a V_6 containing 4 codelines. Each of the remaining 8 codelines generates, together with the fixed V_6, either a V_7 or a hyperplane. This shows that V_6 must be contained in at least 8 hyperplanes, which is not the case. We conclude that each V_6 contains at most 3 codelines. In particular there can be no repeated codelines and any three codelines generate either a V_5 or a V_6. Any two codelines are skew as otherwise Proposition 1 would yields a $[10, 3, 7]$-code which does not exist.

Lemma 1. *There are no repeated codelines. Each V_6 contains at most 3 codelines and any three codelines generate V_5 or V_6. Any two codelines are mutually skew.*

Let M be the union of the points on the codelines. We know by now that M is a set of 36 points, at most 22 on each hyperplane. This describes a binary code $[36, 9, 14]_2$, obtained from the hypothetical $[12, 4.5, 7]$ by concatenation.

Lemma 2. *Any four codelines generate either V_7 or V_8.*

Proof. Assume they generate a V_6. By Proposition 1 this yields a subcode $[8, 1.5, 7]$ and a linear $[24, 3, 14]_2$, which by the Griesmer bound cannot exist. □

Definition 4. *Let M be the union of the points on the codelines. Let $V \subset V_9$ be a V_i-subspace of our ambient space V_9. The factor space V_9/V is a $PG(8-i, 2)$, which we denote by $\Pi(V)$. We speak of an $m-V_i$ if $|M \cap V| = m$. The* **weight** *$w(P)$ of a point $P \in \Pi(V)$ is the number of points of M which are contained in its preimage (a V_{i+1}) and outside V.*

Observe that in the situation of Definition 4 we have

$$|M| = 36 = |M \cap V| + \sum_{P \in \Pi(V)} w(P).$$

Lemma 3. *The following are upper bounds for the number of points of M on subspaces: 22 on a hyperplane, 15 on a V_7, 11 on a V_6 and 9 on a V_5.*

Proof. The first two statements are obvious. Assume V is a V_6 containing 12 points of M. Then $w(P) \leq 3$ for each $P \in \Pi(V)$, but $\sum_P w(P) = 36 - 12 = 24$, contradiction.

Assume V is an $10 - V_5$. This time the factor space is a $PG(3, 2)$. Each of its 15 points has weight at most 1 and the sum of the weights is 26, contradiction. □

Lemma 4. *In the factor space of an $11 - V_6$ all points have weights 3 or 4. Those of weight 3 form a line of the factor space.*

In the factor space of a $9 - V_5$ all points have weights 1 or 2. There are three points of weight 1 and they form a line R_0 of the factor space.

Proof. Consider an $11 - V_6$. The weights in the factor plane are ≤ 4, the sum of weights along each line is ≤ 11 and the sum of all seven weights is 25. Consider the points of weight < 4. They form a blocking set and there are only 3 such points. It follows that they are collinear and have weights $= 3$.

Consider now a $9-V_5$ and its factor space $PG(3,2)$. All weights are ≤ 2, the sum of all weights is 27. This shows that there are at least 12 points of weight 2 in the factor space. By the first part of the lemma each point P of weight 2 has the following property: 4 of the lines containing P have weights summing to 6, the remaining 3 have weights summing to 5. Let x be the number of points of weight 2 in the factor plane. The number of lines all of whose points are of this type equals $4x/3$. As $x \geq 12$ is divisible by 3 it follows $x = 12$. All the weights are 1 or 2. There are $x = 12$ points of weight 2 and 3 of weight 1. There are 16 lines containing three points of weight 2 and $12 \times 3/2 = 18$ lines containing two of them. It follows that the three points of weight 1 of the factor space form a line R_0. □

Lemma 5. *Our code has strength* 3 : *any three codelines are in general position.*

Proof. Assume some three codelines are not in general position. They generate a $9-V_5$ which we call U. Lemma 4 shows that we have precise information on the distribution of points of M on spaces containing U. In particular each hyperplane containing the special line R_0 of $\Pi(U)$ contains precisely 4 codelines, and each of the remaining hyperplanes containing U contains the maximum of 5 codelines.

We study the distribution of codelines on the preimages of lines in $\Pi(U)$. Observe that each codeline aside of the three contained in U together with U generates a V_7 and therefore describes a line in $\Pi(U)$.

Define the **heavyness** $h(g)$ of a line g of $\Pi(U)$ to be 3 less than the number of codelines contained in the preimage of g. We have that the heavinesses of lines of $\Pi(U)$ sum to 9. The special line R_0 and the lines of type $(1,2,2)$ have heavyness 0 or 1, those of type $(2,2,2)$ have heavyness $0, 1$ or 2.

Case 1: assume $h(R_0) = 1$, in other words the V_7 corresponding to R_0 contains 4 codelines. The Fano planes of $F(U)$ containing R_0 show that all lines of type $(1,2,2)$ have heavyness 0. The other Fano planes yield the condition: the sum of the heavynesses of lines of type $(2,2,2)$ on any Fano plane of $F(U)$ not containing R_0 must equal 2. We should solve this combinatorial question: Given a $PG(3,2)$ and a line R_0, is it possible to assign weights $0, 1, 2$ to the 16 lines skew to R_0 in such a way that the following are satisfied:

- The sum of all weights is 8,
- For each Fano plane not on R_0 the sum of the weights of the lines contained in it is $= 2$.

All solutions are related to spreads in $PG(3,2)$. In order to understand them here is some basic data: there are 35 lines, 16 lines are skew to a given line and 6 lines are skew to two given skew lines. There is a total of 56 spreads, 8 through any given line, two through any given pair of two skew lines and precisely one through each triple of mutually skew lines. Each Fano plane shares precisely one line with any spread.

Given three skew lines, there are exactly three other skew lines (the **inverse**) covering the same set of 9 points. This follows from what has been said above: consider the unique completion of the three lines to a spread. The inverse is the second completion of the remaining two lines to a spread.

The first solution (**spread doubling**) fixes a spread through R_0 and assigns heaviness 2 to each line $\neq R_0$ of the spread. The second (**mixed**) solution fixes a second line R_1 skew to R_0 and assigns heaviness 2 to R_1, heaviness 1 to each further line skew to both. The third (**pure**) solution fixes two spreads having only R_0 in common and assigns heaviness 1 to each line $\neq R_0$ of any of those two spreads. A little computer program shows that these three solutions are uniquely determined and there are no others.

Case 2: assume $h(R_0) = 0$. There must be exactly three lines of type $(1,2,2)$, which have heavyness 1, one through each point of R_0, one on each Fano plane containing R_0. Consequently the heavynesses of lines of type $(2,2,2)$ must add to 6. The main condition remains unaltered: the sum of all heavynesses of lines contained in any Fano plane not through R_0 is 2. This combinatorial problem has 5 inequivalent solutions. They are related to spreads as well.

Fix a spread R_0, R_1, R_2, T_1, T_2 containing R_0 and let S_0, S_1, S_2 be the inverse of R_0, R_1, R_2 (meaning that S_0, S_1, S_2, T_1, T_2 is the second spread containing T_1, T_2). The first solution consists of S_0, S_1, S_2, R_1, R_2 and of T_1, T_2, each with heaviness 2. The second solution uses S_0, S_1, S_2, T_1 with heaviness 2 and T_2 with heaviness 1 as well as the dual of R_1, R_2, T_2.

All remaining solutions uses only single lines (heaviness 1). The third solution consists of S_0, S_1, S_2, the dual of R_2, T_1, T_2 and the dual of R_1, T_1, T_2. The fourth solution uses S_0, S_1, S_2, T_1, T_2 and the dual of R_2, T_1, T_2. The fifth and last solution is hardest to describe. It uses S_0, S_1, S_2, T_1, T_2 and the lines $\neq R_0$ of a spread containing R_0, which does not contain R_1 or R_2. A little computer program shows that these are the only solutions. Another computer search revealed that in none of these cases the corresponding additive code exists. □

Any five codelines generate either a V_7 or a V_8 or the whole space. We need some information on the corresponding codes.

Codes generated by 5 lines

Let $\mathcal{L}$ be a set of 5 lines generating V_8 such that any 3 of the lines are in general position. Equivalently this describes an additive $[5,4]_4$-code of strength 4, the dual of a $[5,1,4]$-code. This code is uniquely determined: the ambient space is a line so we have to take it 5 times. We can choose the $[5,1,4]$-code as follows:

$$\left(\begin{array}{c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 \\ 10 & 10 & 00 & 00 & 00 \\ 00 & 10 & 10 & 00 & 00 \\ 00 & 00 & 10 & 10 & 00 \\ 00 & 00 & 00 & 10 & 10 \\ \hline 01 & 01 & 00 & 00 & 00 \\ 00 & 01 & 01 & 00 & 00 \\ 00 & 00 & 01 & 01 & 00 \\ 00 & 00 & 00 & 01 & 01 \end{array}\right).$$

The symmetry group of the code is $S_3 \times S_5$ of order $6! = 720$. Clearly it has a permutation representation on 5 objects, the lines. Let the basis of V_8 be $v_1, \ldots, v_4, w_1, \ldots, w_4$. Let K be the kernel of this representation. It is S_3, generated by $\prod(v_i, w_i)$ and $\prod(v_i, w_i, v_i + w_i)$. The factor group is the full S_5, generated by an element of order 5 :

$$\left(\begin{array}{cccccccc} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

permuting $(L_1, L_2, L_3, L_4, L_5)$, and

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of order 2, inducing the permutation (L_1, L_2).

Proposition 2. *There is a uniquely determined additive* $[5, 4]$*-code of strength* 3. *Each* V_5 *containing two of its lines has either* 6 *or* 7 *points in common with the union of the codelines. The space generated by* 3 *of the* 5 *codelines meets the union of those lines in precisely* 9 *points.*

Proof. Let U be a V_5 contained in a $[5, 4]$-code of strength 3. As the code is uniquely determined and its automorphism group is 2-transitive on the lines, we can choose U as containing L_1 and L_2 above. Each of the remaining 9 points on the union of the remaining lines generates a different V_5 with L_1 and L_2. □

Consider now a set $\mathcal{L}$ of 5 lines generating V_7 such that any 3 lines are in general position. This describes a code $[5, 3.5]$ of strength 3, whose dual is a $[5, 1.5, 4]$, in other words a set of 5 different lines of the Fano plane. Because of double transitivity there is essentially only one choice for this set of 5 lines. We can choose this dual code as

$$\left(\begin{array}{c|c|c|c|c} 10 & 00 & 10 & 01 & 01 \\ 01 & 01 & 00 & 01 & 10 \\ 00 & 10 & 01 & 10 & 01 \end{array}\right)$$

and our strength 3 additive $[5, 3.5]$-code as

$$\left(\begin{array}{c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 \\ 01 & 01 & 00 & 00 & 00 \\ 10 & 00 & 10 & 00 & 00 \\ 00 & 10 & 01 & 00 & 00 \\ 00 & 10 & 00 & 10 & 00 \\ 11 & 00 & 00 & 01 & 00 \\ 01 & 00 & 00 & 00 & 10 \\ 10 & 10 & 00 & 00 & 01 \end{array}\right)$$

As the stabilizer of 2 points in $GL(3,2)$ has order 8 we expect the automorphism group of our code to be a dihedral group D_8 of order 8. In fact, let

$$g = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} h = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then $D = \langle g, h \rangle$ is a D_8 of automorphisms, h maps $(L_1, L_3)(L_4, L_5)$ and $g : (L_1, L_3, L_5, L_4)$. There are three orbits of line pairs, with representatives $\{L_2, L_3\}, \{L_4, L_5\}, \{L_3, L_4\}$. By inspection we see that the V_5-subspaces containing two lines and at least one point of the remaining 3 lines meet the point set in either 7 or 9 points in case of $\{L_3, L_4\}$, in 7 or 8 points in case of the remaining orbits of line pairs.

Proposition 3. *There is a unique additive* $[5, 3.5]$*-code of strength* 3. *Its automorphism group is dihedral of order* 8. *There is exactly one orbit of pairs of codelines which are contained in some* V_5 *that has* 9 *points in common with the union of the codelines.*

No $9 - V_5$ with two codelines

Proposition 4. *There is no* $9 - V_5$ *with two codelines. Any* 5 *codelines generate either the whole space or a hyperplane. There is no* $11 - V_6$ *with* 3 *codelines.*

Proof. Let U be a $9 - V_5$ containing 2 codelines. Let $V \supset U$ be the space generated by U and the 3 codelines that intersect U in isolated points. Then V is not the whole space. It follows from Proposition 2 that V cannot be a hyperplane. It follows that V must be a secundum (hyperplane of a hyperplane) V_7. By Proposition 3 we can choose the two lines contained in U as L_3, L_4 and $U = \langle v_2, v_3, v_4, v_5, v_7 \rangle$. Clearly V does not contain isolated points. The factor space $\Pi(U)$ is obtained by projection onto v_1, v_6, v_8, v_9 and V determines a line $g_0 = \{1100, 1000, 0100\}$ of $\Pi(U)$ (corresponding to L_1, L_2, L_5). Call the points of g_0 **special.** We use the notion of heaviness of lines of $\Pi(U)$ as in the proof of Lemma 5 and extend it by defining $h(P)$ to be the sum of the heavinesses of the lines containing point P and $h(E)$ as the sum of the heavinesses of the lines contained in plane E. Clearly each special point has weight 2 and heaviness 0. Recall from Lemma 4 that the points of weight 1 form a line of $\Pi(U)$. This line R_0 is skew to g_0. We can choose $R_0 = \{0010, 0001, 0011\}$. Each of the remaining 7 codelines determines a line of Π. It follows $\sum_g h(g) = 7$. This leads to the problem of determining the heaviness distributions on the lines of $PG(3,2)$ such that the following hold:

- The sum of all heavinesses of lines is 7.
- Each line intersecting g_0 has heaviness 0.
- $h(E) = 1$ if E contains R_0, and $h(E) = 2$ if E does not contain neither R_0 nor g_0.
- $h(P) = 1$ if $P \in R_0$, and $h(P) = 2$ if $w(P) = 2$ but P is not special.

For each solution of the heaviness problem we know our 5 codelines above and the last 4 rows of the generator matrix. Assume at first $h(R_0) = 1$. The remaining heavy lines (of positive heaviness) are parallel to R_0 and to g_0. Observe that these are 6 lines forming a grid. If one of them has heaviness 2 then the whole parallel class of the grid must have heaviness 2. Choose

$$g_1 = \langle 1010, 0101 \rangle,\ g_2 = \langle 0111, 1110 \rangle,\ g_3 = \langle 1101, 1011 \rangle$$

as lines with heaviness 2. This yields the following situation:

$$\left(\begin{array}{c|c|c|c|c||c|c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} \\
10 & 00 & 10 & 00 & 00 & 00 & & & & & & \\
01 & 00 & 00 & 10 & 00 & 00 & & & & & & \\
00 & 10 & 10 & 00 & 00 & 00 & & & & & & \\
00 & 01 & 01 & 10 & 00 & 00 & & & & & & \\
00 & 00 & 10 & 10 & 10 & 00 & & & & & & \\
\hline
00 & 00 & 01 & 01 & 00 & 00 & 10 & 10 & 01 & 01 & 11 & 11 \\
00 & 00 & 01 & 00 & 01 & 00 & 01 & 01 & 11 & 11 & 10 & 10 \\
00 & 00 & 00 & 00 & 00 & 10 & 10 & 10 & 11 & 11 & 01 & 01 \\
00 & 00 & 00 & 00 & 00 & 01 & 01 & 01 & 10 & 10 & 11 & 11
\end{array}\right).$$

A computer search produced no solutions. In the next case all the lines of the grid have heaviness 1.

$$
\left(\begin{array}{c|c|c|c|c||c|c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} \\
10 & 00 & 10 & 00 & 00 & 00 & & & & & & \\
01 & 00 & 00 & 10 & 00 & 00 & & & & & & \\
00 & 10 & 10 & 00 & 00 & 00 & & & & & & \\
00 & 01 & 01 & 10 & 00 & 00 & & & & & & \\
00 & 00 & 10 & 10 & 10 & 00 & & & & & & \\
\hline
00 & 00 & 01 & 01 & 00 & 00 & 10 & 01 & 11 & 10 & 01 & 11 \\
00 & 00 & 01 & 00 & 01 & 00 & 01 & 11 & 10 & 01 & 11 & 10 \\
00 & 00 & 00 & 00 & 00 & 10 & 10 & 11 & 01 & 11 & 01 & 10 \\
00 & 00 & 00 & 00 & 00 & 01 & 01 & 10 & 11 & 01 & 10 & 11
\end{array}\right)
$$

This is excluded by a computer search as well.

Let now $h(R_0) = 0$. Each point of R_0 is then on a different line of heaviness 1. These lines a_1, a_2, a_3 are pairwise skew as otherwise the plane generated by an intersecting pair would contain R_0 and have too many codelines. We have that a_1, a_2, a_3, g_0 form a partial spread. Let m be the line completing it. All remaining heavy lines are contained in the grid of 6 lines defined by the partial spread R_0, g_0. Consider $h(m)$. If $h(m) = 0$, then each point of m is on 2 codelines and those are all different, contradiction. Let $h(m) = 2$. Then the remaining two heavy lines must be the lines completing R_0, g_0, m to a spread, each with heaviness 1. Using the same R_0, g_0 as above and $m = \langle 1010, 0101\rangle$, we have

$$a_1 = \langle 0010, 1001\rangle,\ a_2 = \langle 0001, 0111\rangle,\ a_3 = \langle 0011, 1110\rangle$$

and the two complementing lines (here of heaviness 1) are

$$l_1 = \langle 1001, 0111\rangle \text{ and } l_2 = \langle 1011, 0110\rangle.$$

This leads to

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
10	00	10	00	00	00						
01	00	00	10	00	00						
00	10	10	00	00	00						
00	01	01	10	00	00						
00	00	10	10	10	00						
00	00	01	01	00	10	10	01	00	01	10	10
00	00	01	00	01	01	01	00	01	01	01	01
00	00	00	00	00	10	10	10	01	11	01	11
00	00	00	00	00	01	01	01	11	10	11	10

.

The final situation is when $h(m) = 1$. The remaining 3 heavy lines partition the points off R_0 and g_0. They form the lines dual to m, l_1, l_2. These are the lines

$$\langle 1010, 0111\rangle,\ \langle 0101, 1110\rangle,\ \langle 1111, 1001\rangle.$$

We have

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}	L_{12}
10	00	10	00	00	00						
01	00	00	10	00	00						
00	10	10	00	00	00						
00	01	01	10	00	00						
00	00	10	10	10	00						
00	00	01	01	00	10	01	00	01	10	01	11
00	00	01	00	01	01	00	01	01	01	11	10
00	00	00	00	00	10	10	01	11	11	01	10
00	00	00	00	00	01	01	11	10	01	10	11

.

Both cases yielded no code, after exhaustive search.

This shows that there is no $9 - V_5$ with 2 codelines. As a V_7 with 5 codelines contains a $9 - V_5$ with 2 codelines, such a space is out as well. Finally let W be an $11 - V_6$ with 3 codelines and T the space generated by W and the lines meeting W in isolated points. As T is generated by 5 codelines it follows that T is a hyperplane. By Proposition 2 this is impossible. □

The finishing touch

Proposition 5. *Any two codelines are contained in an* $8 - V_5$.

Proof. Let U be the V_4 generated by two codelines. Assume U is not contained in an $8 - V_5$. Then each point in $\Pi(U)$ (a $PG(4,2)$) has weight ≤ 1. As the sum of all weights is $36 - 6 = 30$, it follows that all points but one of $\Pi(U)$ have weight 1. Choose a hyperplane H all of whose points have weight 1. The corresponding preimage in $PG(8,2)$ has $6 + 15 = 21$ codepoints, contradiction. □

It remains to show that an $8 - V_5$ containing two codelines cannot exist. Let U be such a space and $V \supset U$ the V_7 generated by U and the two codelines meeting U in isolated points. Consider $\Pi(U)$, a $PG(3,2)$, and the line g_0 corresponding to V. As there are no $9 - V_5$ with two codelines all point weights are ≤ 2. It follows that the two special points on g_0 have weight 2. As V cannot contain isolated points the preimage of V contains 12 codepoints. This shows that the non-special point on g_0 has weight 0. As all planes containing g_0 have weight ≤ 8 (recall that the preimage of V is not contained in a hyperplane with 5 codelines), the point weights are uniquely determined: all points not on g_0 have weight 2.

We conclude that there is precisely one point P_0 of weight 0, and all other weights are 2. Each plane containing P_0 describes a hyperplane with 4 codelines, the remaining planes describe hyperplanes with 5 codelines. Let S_1, S_2 be the special points corresponding to the isolated points in U. The sum of all heavinesses of lines is 8. Let g_0 be the line whose points are P_0, S_1, S_2. Lines of positive heaviness must be parallel to g_0. We have $\sum_{P \in g} h(g) = 2$ for all $P \notin g_0$ and also $h(E) = 2$ for all planes E not containing g_0. In particular three lines forming a triangle cannot all have positive heaviness. Assume there is a line g_1 of heaviness 1. Each point of g_1 is then on a second line of heaviness 1. The absence of triangles shows that these three lines l_1, l_2, l_3 of heaviness 1 form a partial spread. They are all parallel to g_0. Let m be the line completing this to a spread (g_0, m, l_1, l_2, l_3). Starting with l_1 we obtain a partial spread g_1, g_2, g_3 of lines which all intersect the l_i. Clearly we have the two transversal completions of g_0, m to a spread. Each of the six completing lines $g_1, g_2, g_3, l_1, l_2, l_3$ has heaviness 1. We must have $h(m) = 2$. Use the same spreads as earlier:

$$g_0 = \{0010, 0001, 0011\},\ m = \{1000, 0100, 1100\},$$

first completion

$$l_1 = \{1010, 0101, 1111\},\ l_2 = \{0111, 1110, 1001\},\ l_3 = \{1101, 1011, 0110\},$$

second completion

$$g_1 = \{1010, 0111, 1101\}, g_2 = \{0101, 1110, 1011\},\ g_3 = \{1111, 1001, 0110\}.$$

Let $P_0 = 0010, S_1 = 0001, S_2 = 0011$. We have

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} \\
10 & 00 & 00 & 10 & 00 & & & & & & & \\
01 & 00 & 00 & 00 & 00 & & & & & & & \\
00 & 10 & 00 & 10 & 00 & & & & & & & \\
00 & 01 & 00 & 00 & 00 & & & & & & & \\
00 & 00 & 10 & 10 & 00 & & & & & & & \\
\hline
00 & 00 & 00 & 00 & 10 & 10 & 10 & 01 & 11 & 10 & 01 & 11 \\
00 & 00 & 00 & 00 & 01 & 01 & 01 & 11 & 10 & 01 & 11 & 10 \\
00 & 00 & 00 & 01 & 00 & 00 & 11 & 01 & 10 & 10 & 11 & 01 \\
00 & 00 & 01 & 01 & 00 & 00 & 01 & 10 & 11 & 01 & 10 & 11
\end{array}\right)$$

The second case is when all positive line heavinesses are 2. Those lines complete g_0 to a spread. Use m, g_1, g_2, g_3 above as the lines with heaviness 2.

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c}
L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} \\
10 & 00 & 00 & 10 & 00 & & & & & & & \\
01 & 00 & 00 & 00 & 00 & & & & & & & \\
00 & 10 & 00 & 10 & 00 & & & & & & & \\
00 & 01 & 00 & 00 & 00 & & & & & & & \\
00 & 00 & 10 & 10 & 00 & & & & & & & \\
\hline
00 & 00 & 00 & 00 & 10 & 10 & 10 & 10 & 01 & 01 & 11 & 11 \\
00 & 00 & 00 & 00 & 01 & 01 & 01 & 01 & 11 & 11 & 10 & 10 \\
00 & 00 & 00 & 01 & 00 & 00 & 11 & 11 & 01 & 01 & 10 & 10 \\
00 & 00 & 01 & 01 & 00 & 00 & 01 & 01 & 10 & 10 & 11 & 11
\end{array}\right).$$

Exhaustive searches were carried out and produced no solution, for both situations in this subsection. This completes the proof of nonexistence.

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