Short additive quaternary codes

Jürgen Bierbrauer, Yves Edel, Giorgio Faina, Stefano Marcugini and Fernanda Pambianco

***Abstract*— The best parameters of quaternary additive codes of small length are determined using the geometric description. Only one open question remains for length ≤ 13. Among the results obtained in this work are the non-existence of $[12, 7, 5]$-codes and $[12, 4.5, 7]$-codes as well as the existence of a $[13, 7.5, 5]-$code.**

***Index Terms*— Linear codes, quaternary additive codes, binary projective spaces.**

I. Introduction

Additive codes are generalizations of linear codes, see for example Chapter 17 of [2] for a general introduction and a theory of cyclic additive codes. Here we concentrate on the quaternary case.

Definition 1. *Let k be such that $2k$ is a positive integer. An additive quaternary $[n, k]$-code $\mathcal{C}$ (length n, dimension k) is a $2k$-dimensional subspace of $\mathbb{F}_2^{2n}$, where the coordinates come in pairs of two. We view the codewords as n-tuples where the coordinate entries are elements of $\mathbb{F}_2^2$.*

A generator matrix G of $\mathcal{C}$ is a binary $(2k, 2n)$-matrix whose rows form a basis of the binary vector space $\mathcal{C}$.

Definition 2. *Let $\mathcal{C}$ be an additive quaternary $[n, k]$-code. The weight of a codeword is the number of its n coordinates where the entry is different from 00. The minimum weight (equal to minimum distance) d of $\mathcal{C}$ is the smallest weight of its nonzero codewords. The parameters are then also written $[n, k, d]$.*

The strength of $\mathcal{C}$ is the largest number t such that all $(2k, 2t)$-submatrices of a generator matrix whose columns correspond to some t quaternary coordinates have full rank $2t$.

Notation for length and dimension has been chosen to facilitate comparison with quaternary *linear* codes. In fact it is clear that each linear $[n, k]$-code is also an additive $[n, k]$-code (where k of course is an integer) and the notations of minimum distance and strength of the linear code coincide with the additive notions introduced above.

The geometric description of an additive $[n, k]$-code is based on lines in $PG(2k-1, 2)$. In fact, consider a generator matrix G. For each quaternary coordinate $i \in \{1, 2, \ldots, n\}$ we are given points $P_i, Q_i \in PG(2k-1, 2)$. Let L_i be the line determined by P_i, Q_i. The geometric description of code $\mathcal{C}$ as in Definition 2 is based on this multiset of lines (the *codelines*) $\{L_1, L_2, \ldots, L_n\}$. Code $\mathcal{C}$ has minimum distance $\geq d$ if and only if for each hyperplane H of $PG(2k-1, 2)$ we find at least d codelines (in the multiset sense), which are not contained in H. Strength t means that any set of t codelines is in general position. Duality is based on the Euclidean bilinear form, the dot product for binary spaces. The dual of an additive $[n, k]$-code $\mathcal{C}$ is an $[n, n-k]$-code, and $\mathcal{C}$ has strength t if and only if $\mathcal{C}^\perp$ has minimum distance $> t$.

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As an example consider the following analogue of the Simplex codes:

Definition 3. *Let $\mathcal{S}_l$ be the additive quaternary code described by the set of all lines in $PG(l-1, 2)$, $l \geq 3$.*

As the number of lines in $PG(l-1, 2)$ is $(2^l - 1)(2^{l-1} - 1)/3$ it follows that $\mathcal{S}_l$ is an additive $[(2^l - 1)(2^{l-1} - 1)/3, l/2, 2^{l-2}(2^{l-1} - 1)]$-code. This code is optimal. In fact, concatenation yields a binary linear $[(2^l - 1)(2^{l-1} - 1), l, 2^{l-1}(2^{l-1} - 1)]_2$-code, which meets the Griesmer bound with equality. The smallest codes of independent interest in this family are the $[7, 1.5, 6]$-code $\mathcal{S}_3$ (geometrically the 7 lines of the Fano plane) and the $[155, 2.5, 120]$-code $\mathcal{S}_5$.

Recall that the geometric description of linear codes is based on multisets of points, whereas the geometric description of additive quaternary codes uses lines. A codeline not contained in hyperplane H meets it in one point. This motivates to consider mixed quaternary-binary codes.

Definition 4. *An $[(l, r), k]_{(4,2)}$-code is a $2k$-dimensional vector space of binary $(2l + r)$-tuples, where the coordinates are divided into l pairs (written on the left) and r single coordinates. We view each codeword as an $(l+r)$-tuple, where the left coordinates are quaternary, the right ones are binary.*

A code $[(l, r), k]_{(4,2)}$ is described geometrically by a multiset of l lines and r points (codelines and codepoints) in $PG(2k - 1, 2)$. The code has strength $\geq t$ if any set of t objects (codepoints or codelines) are in general position. The definition of minimum distance (equal to the minimum weight) is obvious. A generator matrix is a binary $(2k, 2l+r)$-matrix whose rows form a binary basis of the code. The dual of an additive $[(l, r), k]_{(4,2)}$-code of strength t is an additive $[(l, r), l + r/2 - k, t + 1]_{(4,2)}$-code.

Blokhuis-Brouwer [1] determine the optimal code parameters for additive quaternary codes of length ≤ 12, with two exceptions. We fill those gaps proving the following:

Theorem 1. *There is no additive $[12, 7, 5]$-code.*
There is no additive $[12, 4.5, 7]$-code.

On the constructive side we produce a $[13, 7.5, 5]$-code. A check matrix, described by 13 lines in $PG(10, 2)$, of strength 4 (the convention is $1 = 10, 2 = 01, 3 = 11$) is given in Figure 1.

INSERT FIGURE 1 HERE

Table I contains the list of the largest minimum distance d for additive quaternary $[n, k, d]$-codes of length $n \leq 13$. The

only question remaining open is the existence of a $[13, 6.5, 6]$-code.

TABLE I
LARGEST MINIMUN DISTANCE FOR
ADDITIVE QUATERNARY CODES OF LENGTH $n \leq 13$

$k\backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	2	3	4	5	6	7	8	9	10	11	12	13
1.5		1	2	3	4	5	6	6	7	8	9	10	11
2		1	2	3	4	4	5	6	7	8	8	9	10
2.5			1	2	3	4	5	6	6	7	8	8	9
3			1	2	3	4	4	5	6	6	7	8	9
3.5				1	2	3	4	4	5	6	7	8	8
4				1	2	2	3	4	5	6	6	7	8
4.5					1	2	3	3	4	5	6	6	7
5					1	2	2	3	4	5	6	6	7
5.5						1	2	3	3	4	5	6	6
6						1	2	2	3	4	5	6	6
6.5							1	2	3	3	4	5	$5-6$
7							1	2	2	3	4	4	5
7.5								1	2	2	3	4	5
8								1	2	2	3	4	4
8.5, 9									1	2	2	3	4
9.5, 10										1	2	2	3
10.5, 11											1	2	2

The geometric work happens in binary projective spaces. As we find it often more convenient to work with vector space dimensions we denote i-dimensional vector subspaces by V_i $(= PG(i-1, 2))$. The following obvious observation is often useful:

Proposition 1. *Let C be an additive $[n, k, d]$-code. Assume some i codelines generate a subspace V_{2i-j}. Then the subcode of C consisting of the codewords with vanishing entry in those i coordinates is an $[n-i, k-i+j/2, d]$-code.*

The non-existence of a $[12, 7, 5]$-code is proved in Section III. In Section II the non-existence proof for $[12, 4.5, 7]$ is outlined. A preliminary version of parts of the present paper appeared in [3].

II. NONEXISTENCE OF AN ADDITIVE $[12, 7, 5]$-CODE

It is easier to consider the dual, a $[12, 5]$-code of strength 4. What is the maximum hyperplane intersection of this code $\mathcal{C}$? It is impossible that there are at most 5 lines on each hyperplane as this would produce an additive $[12, 5, 7]$-code, which does not exist. It follows that there must be a hyperplane with at least 6 codelines. There can be no 8 codelines on any hyperplane as this would yield a $[8, 4.5]$ code of strength 4 whose dual would be a $[8, 3.5, 5]$-code. Such a code does not exist.

Lemma 1. *The maximum number of lines of a $[12, 5]$-code of strength 4 on a hyperplane is either 6 or 7.*

In particular we find a hyperplane that contains 6 codelines. This defines an additive $[6, 4.5]$-code. Its dual, a $[6, 1.5, 5]$-code, corresponds to using all lines but one of the Fano plane and is therefore uniquely determined. The following codelines can be used to describe our $[6, 4.5]$-code of strength 4 :

$$L_1 = \langle v_1, v_2 \rangle,\ L_2 = \langle v_3, v_4 \rangle,\ L_3 = \langle v_5, v_6 \rangle,\ L_4 = \langle v_7, v_8 \rangle,$$

$$L_5 = \langle v_1 + v_3 + v_5 + v_7, v_9 \rangle,$$

$$L_6 = \langle v_2 + v_4 + v_6 + v_8, v_9 + v_1 + v_4 + v_5 + v_6 \rangle.$$

We ran a computer program that determined the points completing those lines to a $(6, 1)$-code of strength 4. There are 45 such points. Exactly 24 of those points are distributed on lines that complete the $[6, 4.5]$-code to a $[7, 4.5]$-code of strength 4. There are thus 8 such lines.

Assume at first there is a hyperplane H containing 7 codelines of $\mathcal{C}$. We can choose $L_1, \ldots, L_6$ above and L_7 is one of the 8 lines that our computer search produced. The intersection with the codelines shows that this code must be embeddable in a mixed $[(7, 5), 4.5]_{(4,2)}$-code of strength 4. A computer search showed that not even a single point can be appended:

Proposition 2. *There is no $[(7, 1), 4.5]_{(4,2)}4$-code of strength 4.*

We conclude that the maximum number of codelines on a hyperplane is 6. Choose $L_1, \ldots, L_6$ as above. The intersection with the remaining codelines shows that this can be extended to a $[(6, 6), 4.5]_{(4,2)}$-mixed code of strength 4. The points forming the sextuple must be from the set of 45 extension points mentioned above. A computer search showed that there are exactly six such sextuples. In particular $[(6, 6), 4.5]_{(4,2)}$-mixed codes of strength 4 and their duals, $[(6, 6), 4.5, 5]_{(4,2)}$-codes do exist.

Another computer program showed that none of those six codes can be embedded in a $[12, 5]$-code of strength 4.

III. NONEXISTENCE OF AN ADDITIVE $[12, 4.5, 7]$-CODE

The proof is geometric in nature and much more involved than in the case of $[12, 7, 5]$. We work in $PG(8, 2)$. Geometric reasoning and information on optimal codes of shorter length shows the following:

Lemma 2. *There are no repeated codelines. Each V_6 contains at most 3 codelines and any three codelines generate V_5 or V_6. Any two codelines are mutually skew.*

Let M be the union of the points on the codelines. Then M is a set of 36 points, at most 22 on each hyperplane. This describes a binary code $[36, 9, 14]_2$, obtained from the hypothetical $[12, 4.5, 7]$ by concatenation. We study the distribution of the points of M (*codepoints*) on subspaces as well as the structure induced on corresponding factor spaces. In particular any hyperplane contains at most 22 codepoints and any $PG(4, 2)$ has at most 9 codepoints. The proof that any three codelines must be in general position already involves a computer search. Next we study subspaces S generated by 5 codelines. A computer-proof shows that S must be either the ambient space or a hyperplane and that the maximum number of codepoints on a subspace $PG(4, 2)$ is 8. A final computer search shows that this configuration in $PG(4, 2)$ cannot be completed to a $[12, 4.5, 7]$-code.

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$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c} L_1 & L_2 & L_3 & L_4 & L_5 & L_6 & L_7 & L_8 & L_9 & L_{10} & L_{11} & L_{12} & L_{13} \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 3 & 1 & 2 & 3 & 1 \\ 2 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 3 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 & 0 & 2 & 2 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 3 & 2 & 3 & 3 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 2 & 2 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{array}\right)$$

Fig. 1. Check matrix of a $[13, 7.5, 5]$-code.