An infinite family of 7-designs

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Abstract

We study designs in the binary affine space invariant under the affine group in its 3-transitive action. The main result is a family $7-(2^n, 8, 45), n \geq 6$ of non-simple designs. We also obtain $5-(2^n, 6, 3)$ for every $n \geq 3$ and $5-(2^n, 7, 7(v-16)/2)$ for every even $n \geq 6$.

1 Introduction

Consider the affine group $G = E \cdot L$ over the field with two elements, in its 3-transitive action on the 2^n points of the underlying vector space $V = \mathbb{F}_2^n$. Here E is an elementary abelian group of order 2^n and L is the n-dimensional general linear group $GL_n(2)$. The elements of E have the form τ_w, where $w \in V$ and act as translations on V :

$$\tau_w(v) = v + w.$$

We see that L is the stabilizer in G of the vector 0. As $GL_n(2)$ is doubly transitive on the nonzero vectors of the vector space V, it follows that G acts triply transitively. It was Alltop [1] who first used this group to construct t-designs with $t > 3$. As G is 3-transitive it is clear that for every cardinality k every orbit of G on k-subsets of V is a 3-design. Alltop uses single orbits. He proves that such an orbit is a 5-design if and only if it is a 4-design and produces an explicit example of a 5-design with $k = 24$ in case $n = 8$.

We want to study the situation in more detail in order to construct infinite families of non-simple t-designs whose blocks are unions of orbits of G. We restrict attention to block sizes at most 8. In the next section we collect general information and describe some invariants. In the final sections we describe the orbits and derive families of designs. I wish to thank Vladimir Tonchev for bringing this subject to my attention.

2 Orbits and invariants

Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of the vector space V and $v = 2^n$.

Lemma 1. *Let $B \subseteq V$ be a subset of even cardinality k. We have $\sum_{x \in B} x = 0$ if and only if this is true for all members of the orbit containing B.*

This is trivial. Observe that the statement of Lemma 1 definitely is not true for odd cardinalities k.

Definition 1. *Let $B = \{x_1, x_2, \ldots, x_k\} \subseteq V$. The* **affine span** *of B is defined as*

$$\langle B \rangle = \{\sum_{i=1}^{k} \lambda_i x_i | \sum_i \lambda_i = 0\}.$$

The affine span is a subspace of the binary vector space V. We define the dimension of B by $dim(B) = dim(\langle B \rangle)$.

It is in fact obvious that $\langle B \rangle$ is closed under addition. Also $\langle B+u \rangle = \langle B \rangle$. We can therefore assume $0 \in B$. In that case $\langle B \rangle$ is the subspace of V generated by B. The dimension is constant on orbits. We define two more invariants. These will suffice to characterize all orbits of subsets with at most 8 elements.

Definition 2. *Let $B \subseteq V$. Denote by $\alpha(B)$ the number of 4-subsets of sum 0, which are contained in B, and by $\beta(B)$ the number of 6-subsets with sum 0 contained in B.*

It follows from Lemma 1 that α and β are constant on orbits.

3 Description of orbits

Observe that the orbits of subsets B to be described in the sequel are present only when $n \geq dim(B)$. It follows from Lemma 1 that the 4-sets $\{0, v_1, v_2, v_3\}$ (of sum $\neq 0$) and $\{0, v_1, v_2, v_1 + v_2\}$ (of sum $= 0$) are in different orbits. Denote these orbits by V_1 and V_2, respectively. V_1 may also be described as the family of 4-sets in general position (or of dimension 3), whereas the elements of V_2 are precisely the subspaces of dimension 2. The lengths of these orbits are obviously

$$\boxed{\begin{array}{lcl} a(V_1) & = & v(v-1)(v-2)(v-4)/24 \\ a(V_2) & = & v(v-1)(v-2)/24 \end{array}}$$

As these numbers add up to $\binom{v}{4}$, group G has just two orbits on 4-sets.

Two orbits on 5-sets immediately come to mind: the orbit F_1 of 5-sets in general position (equivalently: of dimension 4), with representative $\{0, v_1, v_2, v_3, v_4\}$, and the 5-sets obtained by adding an independent vector to a 4-set in orbit V_2. These latter 5-sets form an orbit F_2 with representative $\{0, v_1, v_2, v_1+v_2, v_3\}$. Elements of F_2 have dimension 3. Obviously the lengths of these orbits are

$$\boxed{\begin{array}{lcl} a(F_1) & = & v(v-1)(v-2)(v-4)(v-8)/120 \\ a(F_2) & = & v(v-1)(v-2)(v-4)/24 \end{array}}$$

These numbers add up to $\binom{v}{5}$. We conclude that G has only two orbits of 5-subsets. Consider the 6-subsets next.

Some notation comes in handy. Recall $v = 2^n$. Moreover let

$$p_i = p_i(v) = v(v-1)(v-2)(v-4)\dots(v-2^i).$$

These expressions occur regularly in the formulas for the orbit lengths. The following is a list of four representatives B of different orbits of 6-subsets. For each representative we give the parameters dim and α. These two parameters suffice to determine the orbits.

name B	representative	dim	α	length $a(B)$ of orbit
S_1	$0, v_1, v_2, v_3, v_4, v_5$	5	0	$p_4/6!$
S_2	$0, v_1, v_2, v_3, v_4, v_1 + v_2 + v_3 + v_4$	4	0	$p_3/6!$
S_3	$0, v_1, v_2, v_1 + v_2, v_3, v_4$	4	1	$p_3/48$
S_4	$0, v_1, v_2, v_3, v_1 + v_2, v_1 + v_3$	3	3	$p_2/48$

Again, the lengths of the orbits add up to the right number $\binom{v}{6}$. It follows that G has precisely four orbits on 6-subsets.

Definition 3. *The* $(t, k)-$ **incidence matrix** *is the matrix whose rows are indexed by the orbits on* $t-$*sets, columns indexed by the orbits on* k*-sets, where the entries denote, for a fixed* t*-set* T *from that orbit, in how many* $k-$*sets of the given orbit* T *is contained.*

Here is the $(5, 6)-$incidence matrix.

representative in	S_1	S_2	S_3	S_4
F_1	$v-16$	1	10	0
F_2	0	0	$v-8$	3

We see that we may use the blocks of S_4 and those of S_2, where the latter are to be used with multiplicity 3, and get a design with parameters $5-(2^n, 6, 3)$.

Theorem 1. *Choose as blocks*

- *the 6-sets with sum 0, each with multiplicity 3, and*
- *the 6-sets of dimension 3.*

This defines a design

$$5-(2^n, 6, 3).$$

The following table contains names, representatives and parameter values for six orbits of 7-sets. The parameters show that these orbits are different.

name X	representative	*dim*	α	β
X_1	$0, v_1, v_2, v_3, v_4, v_5, v_6$	6	0	0
X_2	$0, v_1, v_2, v_3, v_4, v_5, v_1+v_2$	5	1	0
X_3	$0, v_1, v_2, v_3, v_4, v_5, v_1+v_2+v_3+v_4$	5	0	1
X_4	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_3+v_4$	4	2	1
X_5	$0, v_1, v_2, v_3, v_4, v_1+v_2, v_1+v_3$	4	3	0
X_6	$0, v_1, v_2, v_3, v_1+v_2, v_1+v_3, v_2+v_3$	3	7	0

Sets from X_2 consist of a V_2-set and three independent elements. Sets from X_3 are obtained by adding an independent point to a set from S_2. The unions of two sets from V_2 intersecting in precisely one point yield orbit X_4. Sets from X_5 arise by adjoining an independent point to a set from S_4. Finally, sets X_6 arise by deleting a point from a 3-dimensional subspace. These descriptions yield the formulae for the lengths of the orbits, as follows:

name	orbit length
X_1	$p_5/7!$
X_2	$p_4/(6 \cdot 24)$
X_3	$p_4/6!$
X_4	$p_3/72$
X_5	$p_3/48$
X_6	$p_2/(7 \cdot 24)$

The orbit lengths sum to $\binom{v}{7}$. This shows that we have described all orbits on 7-sets.

Let $B \subseteq V, |B| = k$. Then $\alpha(B)$ is the number of 4-subsets of B from orbit V_2, and the number of 5-subsets of B from orbit F_2 is clearly $(k-4)\alpha(B)$. A trivial double-counting argument suffices to determine the $(5,7)-$incidence matrix:

	X_1	X_2	X_3	X_4	X_5	X_6
F_1	$\frac{(v-16)(v-32)}{2}$	$15(v-16)$	$\frac{7(v-16)}{2}$	25	30	0
F_2	0	$\frac{(v-8)(v-16)}{2}$	0	$2(v-8)$	$\frac{9(v-8)}{2}$	3

Theorem 2. *Let $n \geq 6$ be even. Choose as blocks*

- *the 7-sets X of dimension 5 not containing a 4-subset of dimension 2, each with multiplicity 1, and*
- *the 7-sets of dimension 3, with multiplicity $7(v-16)/6$.*

This defines a design

$$5-(2^n, 7, 7(2^{n-1}-8)).$$

Here is the $(6,7)-$incidence matrix:

	X_1	X_2	X_3	X_4	X_5	X_6
S_1	$v-32$	20	6	0	0	0
S_2	0	0	$v-16$	10	0	0
S_3	0	$v-16$	0	4	6	0
S_4	0	0	0	0	$v-8$	2

4 Sets of size $k = 8$.

Here is a list of some orbits of G on 8-subsets. We give names, representatives and the values of the basic parameters.

name	representative	*dim*	α	β
A_1	$0, v_1, v_2, v_3, v_4, v_5, v_6, v_7$	7	0	0
A_2	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_5, v_6$	6	1	0
A_3	$0, v_1, v_2, v_3, v_4, v_1+v_2+v_3+v_4, v_5, v_6$	6	0	1
A_4	$0, v_1, v_2, v_3, v_4, v_5, v_6, \sum_{i=1}^{6} v_i$	6	0	0
A_5	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_5, v_3+v_4+v_5$	5	2	0
A_6	$0, v_1, v_2, v_3, v_1+v_2, v_1+v_3, v_4, v_5$	5	3	0
A_7	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_3+v_4, v_5$	5	2	1
A_8	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_5, v_1+v_3+v_4+v_5$	5	1	2
A_9	$0, v_1, v_2, v_3, v_4, v_1+v_2, v_1+v_3, v_2+v_3$	4	7	0
A_{10}	$0, v_1, v_2, v_1+v_2, v_3, v_4, v_1+v_3, v_1+v_4$	4	6	0
A_{11}	$0, v_1, v_2, v_3, v_4, v_1+v_2+v_3+v_4, v_1+v_4, v_2+v_3+v_4$	4	5	2
A_{12}	$0, v_1, v_2, v_3, v_1+v_2, v_1+v_3, v_2+v_3, v_1+v_2+v_3$	3	14	0

A_1 has dimension 7,

A_2 arises from adding four independent vectors to an element of V_2. The orbit length is therefore $a(V_2)\frac{1}{24}(v-4)(v-8)(v-16)(v-32) = p_5/24^2$.

A_3 is an extension of S_2 by two independent vectors. The orbit length is obvious.

A_4 consists of the 8-tuples of sum zero generating a space of dimension 6. It is clear that $a(A_4) = p_5/8!$

A_5 are the disjoint unions of two V_2, generating a space of dimension 5. We get $a(A_5) = \frac{1}{2}a(V_2) \cdot (v-4)(v-8)(v-16)/24 = p_4/(24 \cdot 48)$.

A_6 are the extensions of S_4 by two independent vectors. The orbit length is therefore $a(S_4)(v-8)(v-16)/2 = p_4/96$.

A_7 are the independent 1-vector extensions of X_4. The orbit length is therefore $a(X_4)(v-16) = p_4/72$.

A_8 consists of unions of two S_2 that intersect in an independent 4-set (type V_1). We have $\alpha = 1, \beta = 2$, the V_2-subset being the complement of the intersection of the two S_2. This makes counting easy: $a(A_8) = a(V_1)\frac{1}{2}\frac{v-8}{2}\frac{v-16}{2} = p_4/192$.

A_9 consists of independent 1-vector extensions of X_6. The orbit length is therefore $a(X_6)(v-8)$.

A_{10} consists of orbits with a stabilizer of order 2 under the action of the translation group E. There are $v-1$ choices for such a group. Given the group, the set consists of four independent orbits. We get $a(A_{12}) = (v-1)\frac{1}{24}\frac{v}{2}\frac{v-2}{2}\frac{v-4}{2}\frac{v-8}{2} = p_3/(16 \cdot 24)$.

A_{11} consists of unions of two S_2, intersecting in a 4-set of type V_1. This orbit differs from A_8 in that a space of different dimension (4 instead of 5) is generated. We have $\alpha = 5, \beta = 2$. There are $a(V_1)$ choices for this intersection X. Let $\sigma \neq 0$ be the sum over its elements. Each of the two S_2 is uniquely determined by the choice of just one further element x as the remaining element must then be $x+\sigma$. The number of choices of the first S_2 is therefore $(v-8)/2$. It is clear that there are then exactly three choices for the second S_2, each one being determined by a complementary pair of pairs from X. Therefore $a(X_{11}) = a(V_1)\frac{1}{2}\frac{v-8}{2}\cdot 3 = p_3/32$.

A_{12} consists of the affine 3-spaces. Its length is therefore $v(v-1)(v-2)(v-4)/(8 \cdot 7 \cdot 6 \cdot 4) = 30p_2/8!$

These descriptions yield the parameter values and the orbit lengths. We see that these orbits are pairwise different.

name B	length $a(B)$ of orbit
A_1	$p_6/8!$
A_2	$p_5/24^2$
A_3	$p_5/(6!2)$
A_4	$p_5/8!$
A_5	$p_4/(24 \cdot 48)$
A_6	$p_4/96$
A_7	$p_4/72$
A_8	$p_4/192$
A_9	$p_3/(7 \cdot 24)$
A_{10}	$p_3/(16 \cdot 24)$
A_{11}	$p_3/32$
A_{12}	$p_2/(8 \cdot 7 \cdot 6 \cdot 4) = 30p_2/8!$

The orbit lengths sum to $\binom{v}{8}$, showing that we have all orbits on 8-sets. The number of $5-$sets from orbit F_2 contained in B is $(k-4)\alpha(B) = 4\alpha(B)$. Denote by $\lambda(F_i, A_j)$ the corresponding entry of the $(5,8)-$incidence matrix. The number of pairs (F, A), where $F \in F_2, A \in A_j$, is $4\alpha(A_j)a(A_j) = a(F_2)\lambda(F_2, A_j)$, hence

$$\lambda(F_2, A_j) = \frac{96\alpha(A_j)a(A_j)}{p_2}.$$

The corresponding formula for orbit F_1 is

$$\lambda(F_1, A_j) = \frac{480(14-\alpha(A_j))a(A_j)}{p_3}.$$

This yields the $(5,8)-$incidence matrix as follows:

	A_1	A_2	A_3
F_1	$\frac{(v-64)(v-32)(v-16)}{6}$	$\frac{65(v-32)(v-16)}{6}$	$\frac{14(v-32)(v-16)}{3}$
F_2	0	$\frac{(v-32)(v-16)(v-8)}{6}$	0

	A_4	A_5	A_6	A_7
F_1	$\frac{(v-32)(v-16)}{6}$	$5(v-16)$	$55(v-16)$	$80(v-16)$
F_2	0	$\frac{(v-16)(v-8)}{6}$	$3(v-16)(v-8)$	$\frac{8(v-16)(v-8)}{3}$

	A_8	A_9	A_{10}	A_{11}	A_{12}
F_1	$\frac{5(v-16)}{2}$	20	10	135	0
F_2	$\frac{(v-16)(v-8)}{2}$	$\frac{v-8}{4}$	$\frac{3(v-8)}{2}$	$15(v-8)$	1

Comparing the $(5,6)-$incidence matrix and the $(5,8)-$incidence matrix we see the following:

Theorem 3. *Choose as blocks*

- *the 6-sets of sum 0, and*
- *the 8-sets, which form 3-dimensional subspaces.*

This defines a $5-$*balanced design*

$$5-(2^n, \{6,8\}, 1).$$

This family was first obtained by R.M.Wilson, see the discussion in [2, 3].

In order to calculate the (6,8)-incidence matrix we have to determine how many 6-sets from a given orbit are contained in a given 8-set. This information is in the following table. Observe that the entries in column S_2 are the values of the $\beta-$parameter.

	S_1	S_2	S_3	S_4
A_1	28	0	0	0
A_2	22	0	6	0
A_3	27	1	0	0
A_4	28	0	0	0
A_5	16	0	12	0
A_6	12	0	15	1
A_7	15	1	12	0
A_8	20	2	6	0
A_9	0	0	21	7
A_{10}	0	0	24	4
A_{11}	0	2	24	2
A_{12}	0	0	0	28

Here is the $(6,8)$-incidence matrix:

	A_1	A_2	A_3	A_4
S_1	$\frac{1}{2}(v-64)(v-32)$	$\frac{55}{2}(v-32)$	$\frac{27}{2}(v-32)$	$\frac{1}{2}(v-32)$
S_2	0	0	$\frac{1}{2}(v-32)(v-16)$	0
S_3	0	$\frac{1}{2}(v-32)(v-16)$	0	0
S_4	0	0	0	0

	A_5	A_6	A_7	A_8
S_1	10	90	150	75
S_2	0	0	$10(v-16)$	$\frac{15}{2}(v-16)$
S_3	$\frac{1}{2}(v-16)$	$\frac{15}{2}(v-16)$	$8(v-16)$	$\frac{3}{2}(v-16)$
S_4	0	$\frac{1}{2}(v-16)(v-8)$	0	0

	A_9	A_{10}	A_{11}	A_{12}
S_1	0	0	0	0
S_2	0	0	45	0
S_3	6	3	36	0
S_4	$2(v-8)$	$\frac{1}{2}(v-8)$	$3(v-8)$	1

We concentrate on case $n = 5$. Use orbits A_5 and A_7, each with multiplicity 1, as well as orbit A_{10} with multiplicity 8, A_{12} with multiplicity 64. This yields a design

$$6-(32,8,160)$$

Here is the table giving the number of 7-sets contained in a given 8-set:

	X_1	X_2	X_3	X_4	X_5	X_6
A_1	8	0	0	0	0	0
A_2	4	4	0	0	0	0
A_3	6	0	2	0	0	0
A_4	8	0	0	0	0	0
A_5	0	8	0	0	0	0
A_6	0	6	0	0	2	0
A_7	0	6	1	1	0	0
A_8	0	4	4	0	0	0
A_9	0	0	0	0	7	1
A_{10}	0	0	0	0	8	0
A_{11}	0	0	0	4	4	0
A_{12}	0	0	0	0	0	8

This gives us the (7,8)-incidence matrix:

	A_1	A_2	A_3	A_4	A_5	A_6
X_1	$v-64$	35	21	1	0	0
X_2	0	$v-32$	0	0	1	9
X_3	0	0	$v-32$	0	0	0
X_4	0	0	0	0	0	0
X_5	0	0	0	0	0	$v-16$
X_6	0	0	0	0	0	0

	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}
X_1	0	0	0	0	0	0
X_2	12	3	0	0	0	0
X_3	10	15	0	0	0	0
X_4	$v-16$	0	0	0	9	0
X_5	0	0	2	1	6	0
X_6	0	0	$v-8$	0	0	1

Theorem 4. *Let $n \geq 6$. Choose as blocks the 8-sets from orbits $(A_4, A_5, A_8, A_{10}, A_{11}, A_{12})$, with respective multiplicities $(45, 36, 3, 15, 5, 45)$. This defines a design*

$$7-(2^n, 8, 45)$$

References

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[3] E.S.Kramer: *Some results on t-wise balanced designs,* *Ars Combinatoria* **15**(1983),179-192.