The largest cap in $AG(4,4)$ and its uniqueness

Yves Edel
Mathematisches Institut der Universität
Im Neuenheimer Feld 288
69120 Heidelberg (GERMANY),

Jürgen Bierbrauer
Department of Mathematical Sciences
Michigan Technological University
Houghton, Michigan 49931 (USA)

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Abstract

We show that 40 is the maximum number of points of a cap in $AG(4,4)$. Up to semi-linear transformations there is only one such 40-cap. Its group of automorphisms is a semidirect product of an elementary abelian group of order 16 and the alternating group A_5.

1 Introduction

A cap is a set of points no 3 of which are collinear. The maximum number of points of a cap in $PG(n,q)$ or $AG(n,q)$ for $n > 3, q > 2$ is known only in a few cases. In $PG(4,3)$ and $AG(4,3)$ the maximum is 20 (see Pellegrino [6]) and all these caps are known. In $PG(5,3)$ the maximum is 56 (Hill [5]), in $AG(5,3)$ the maximum is 45 [2]. In both cases the maximal caps are uniquely determined . The 45-cap in $AG(5,3)$ is an affine section of the Hill cap in $PG(5,3)$. Only one further value of the problem mentioned above is known: the maximum size of a cap in $PG(4,4)$ is 41 [1]. In this case the caps meeting the bound have not been classified. Two non-equivalent examples are known.

In the present paper we prove the following:

Theorem 1. *The maximum number of points of a cap in $AG(4,4)$ is 40. Call a cap in $PG(4,4)$ affine if it avoids a hyperplane. There is only one orbit of affine 40-caps in $PG(4,4)$ under the action of $P\Gamma L(5,4)$. This cap is complete in $PG(4,4)$. Its group of automorphisms has order 960 and is transitive on the points.*

In Section 2 we construct the 40-cap in $AG(4,4)$, starting from its automorphism group. The proof of maximality and uniqueness is described in the final section.

2 Description of the maximal cap in $AG(4,4)$

We start from a description of the group of automorphisms. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,4)$. The mapping

$$A \mapsto \iota(A) = \left(\begin{array}{cc|cc|c} a & b & 0 & 0 & (ab)^2 \\ c & d & 0 & 0 & (cd)^2 \\ \hline 0 & 0 & a^2 & b^2 & ab \\ 0 & 0 & c^2 & d^2 & cd \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

describes an embedding $\iota : SL(2,4) \to SL(5,4)$. Let $W(B) = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \in SL(5,4)$, where B is a $(2,3)$-matrix. Then $W = \{W(B)\}$ is an elementary abelian group of order 4^6 and $W(B_1)W(B_2) = W(B_1 + B_2)$. We have

$$\iota(A)^{-1} W(\begin{pmatrix} u & v & x \\ w & x & u \end{pmatrix}) \iota(A) = W(\begin{pmatrix} U & V & X \\ W & X & U \end{pmatrix})$$

where

$$X = ad^2x + b^2cu + cd^2v + ab^2w,\ U = bc^2x + a^2du + c^2dv + a^2bw,$$

$$V = bd^2x + b^2du + d^3v + b^3w,\ W = ac^2x + a^2cu + c^3v + a^3w$$

Lemma 1. *Consider the standard action of $SL(2,4)$ on a 2-dimensional $\mathbb{F}_4$-vector space S with basis v_1, v_2 :*

$$Av_1 = av_1 + cv_2,\ Av_2 = bv_1 + dv_2$$

and let $\phi(A)$ be the image of A under the Frobenius automorphism. The tensor product $S \otimes S$ is a 4-dimensional $\mathbb{F}_4$-vector space with basis $v_1 \otimes v_1, v_2 \otimes v_2, v_1 \otimes v_2, v_2 \otimes v_1$. Let $SL(2,4)$ act on $S \otimes S$ such that A acts on the first component and $\phi(A)$ acts on the second component ($v \otimes w \mapsto (Av) \otimes (\phi(A)w)$).

This action of $SL(2,4)$ is similar to the permutation action of $\iota(SL(2,4))$ on the $W(\begin{pmatrix} u & v & x \\ w & x & u \end{pmatrix}$. The $SL(2,4)$-equivariant isomorphism is given by

$$w(v_1 \otimes v_1) + v(v_2 \otimes v_2) + x(v_1 \otimes v_2) + u(v_2 \otimes v_1) \mapsto W(\begin{pmatrix} u & v & x \\ w & x & u \end{pmatrix})$$

This follows directly by inspection. Because of Lemma 1 each additive subgroup of $S \otimes S$, which is invariant under the action of $SL(2,4)$, describes a semidirect product embedded in $SL(5,4)$.

Lemma 2. *The $\mathbb{F}_2$-submodule (additive subgroup) V generated by $\overline{\omega}(v_1 \otimes v_1), \overline{\omega}(v_2 \otimes v_2)$ and the $\overline{\omega}\delta(v_1 \otimes v_2) + \overline{\omega}\delta^2(v_2 \otimes v_1)$ is an $SL(2,4)$-module under the action of $SL(2,4)$ from Lemma 1.*

Corollary 1. *The group $\iota(SL(2,4))$ acts by conjugation on the elementary abelian subgroup V consisting of $W(\begin{pmatrix} u & v & x \\ w & x & u \end{pmatrix})$ where $v, w \in \{0, \overline{\omega}\}$ and $(x,u) = \overline{\omega}(\delta, \delta^2)$ for some $\delta \in \mathbb{F}_4$. Denote by G the semidirect product $V \dot{S}L(2,4) \subset SL(5,4)$.*

Definition 1. *Let K be the orbit of $P = (0,0,0,0,1)^T$ under G.*

Lemma 3. *We have $|K| = 40$, and K consists of the points $Q = (\overline{\omega}a\delta + \overline{\omega}b\delta^2 + (ab)^2, \overline{\omega}c\delta + \overline{\omega}d\delta^2 + (cd)^2, ab, cd, 1)$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,4)$ and $\delta \in \mathbb{F}_4$.*

Proof. Application of $W(B)$ to P yields $(\overline{\omega}\delta, \overline{\omega}\delta^2, 0, 0, 1)^T$. Its image under $\iota(A)$ is

$$Q = (\overline{\omega}a\delta + \overline{\omega}b\delta^2 + (ab)^2, \overline{\omega}c\delta + \overline{\omega}d\delta^2 + (cd)^2, ab, cd, 1).$$

Assume $Q = P$. Then $ab = cd = 0$, which means that A is in a subgroup $SL(2,2)$. The first coordinates show $\delta(a+b\delta) = \delta(c+d\delta) = 0$. If $\delta \neq 0$ we obtain the contradiction $det(A) = 0$. It follows that the stabilizer of P in G consists of those elements $\iota(A)W(B)$, where $\delta = 0$ and $ab = cd = 0$. This group has order $4 \cdot 6$. The length of the orbit of P under G is therefore 40. ■

Lemma 4. *The intersection of K with the hyperplane $x_4 = 0$ consists of the affine ovoid $V(\omega X_2^2 + X_3^2 + X_1X_5 + X_2X_3) \setminus \{(1,0,0,0,0)\}$. The intersection of K with the hyperplane $x_3 = 0$ consists of the affine ovoid $V(\omega X_1^2 + X_4^2 + X_2X_5 + X_1X_4) \setminus \{(0,1,0,0,0)\}$.*

Proof. Consider point Q in Lemma 3, the generic image of P under an element of G. We have $Q \in (x_4 = 0)$ if and only if $cd = 0$. There are $16 \cdot 24$ elements of G having this property. As the stabilizer of P has order 24 it follows $|C \cap (x_4 = 0)| = 16$. The points $Q \in K \cap (x_4 = 0)$ have the form $Q = (\overline{\omega}a\delta + \overline{\omega}b\delta^2 + (ab)^2, \overline{\omega}c\delta + \overline{\omega}d\delta^2, ab, 0, 1)$. Its coordinates satisfy

$$\omega x_2^2 = \overline{\omega}c^2\delta^2 + \overline{\omega}d^2\delta^4 = \overline{\omega}c^2\delta^2 + \overline{\omega}d^2\delta$$

(because $\delta^4 = \delta$) and

$$x_3^2 + x_1x_5 = \overline{\omega}a\delta + \overline{\omega}b\delta^2, \ x_2x_3 = \overline{\omega}abc\delta + \overline{\omega}abd\delta^2.$$

Collecting terms we obtain

$$\omega(\omega x_2^2 + x_3^2 + x_1x_5 + x_2x_3) = \delta(a + abc + d^2) + \delta^2(b + abd + c^2).$$

Recall $cd = 0$. Assume $c = 0$. Then $ad = 1$ and the coefficient of δ^2 vanishes. The coefficient of δ is $a + d^2 = (1+d^3)/d = 0$. In case $d = 0$ a symmetric argument applies. This shows that the points $Q \in C \cap (x_4 = 0)$ are on the quadric as claimed. Case $x_3 = 0$ follows by symmetry. ■

Theorem 2. *The points of K form a cap.*

Proof. Recall that the 40 points of K form an orbit under the action of G and $P \in K$. Assume three points of K are collinear. Then there is a line through P containing two further points Q_1, Q_2 of K. The affine parts of these two points (the first four coordinates) must be scalar multiples of each other. Lemma 4 shows that this does not happen when these points satisfy $x_3 = 0$ or $x_4 = 0$. Consider a point $Q \in K$ such that $ab \neq 0, cd \neq 0$. We must have $ad \in \{\omega, \overline{\omega}\}$ and therefore $abcd = 1$. It follows that such points satisfy $x_4 = 1/x_3$. For any two such points the pair (x_3, x_4) is one of $(1,1), (\omega, \overline{\omega}), (\overline{\omega}, \omega)$. Any two such pairs which are scalar multiples of each other must be identical. ■

Consider the hyperplanes

$$H_1 = (x_3 = 0),\ H_2 = (x_4 = 0),\ H_3 = (x_3 + x_4 + x_5 = 0),$$

$$H_4 = (\omega x_3 + \overline{\omega} x_4 + x_5 = 0),\ H_5 = (\overline{\omega} x_3 + \omega x_4 + x_5 = 0).$$

Then $\{H_1, H_2, H_3, H_4, H_5\}$ form an orbit under G. Clearly $\cap_{i=1}^{5} H_i$ is the line $x_3 = x_4 = x_5 = 0$, and V acts on each H_i. The kernel of the permutation action of G on these hyperplanes is of course precisely V, and $\iota(SL(2,4))$ acts as A_5.

The intersection of K with hyperplane H_1 is an affine ovoid:

$$K \cap (x_3 = 0) = (x_3 = 0) \cap (x_5 = 1) \cap V(\omega X_1^2 + X_4^2 + X_2 X_5 + X_1 X_4).$$

The action of G shows that $K \cap H_i$ is an affine ovoid for all $i = 1, \ldots, 5$. In fact $K = \cup_{i=1}^{5}(K \cap H_i)$, and each point of K is in precisely two of the hyperplanes H_i. Further $H_i \cap H_j \cap K$ has precisely 4 points whenever $i \neq j$, and K is the disjoint union of $H \cap H' \cap K$, where $\{H, H'\}$ varies over the pairs of our hyperplanes.

3 Maximality and uniqueness

We show that the affine 40-cap K described in Section 2 is up to the action of the group $P\Gamma L(5,4)$ of semi-linear transformations the only affine cap in $PG(4,4)$. Also, K is complete in $PG(4,4)$ and the group G from Section 2 is the full stabilizer of K in $P\Gamma L(5,4)$. This suffices to prove all claims of Theorem 1. As G does not have a subgroup of index 2 it follows that there are precisely two orbits of affine 40-caps under the action of $PGL(5,4)$.

Let $A \subset PG(4,4)$ be an affine 40-cap. Consider a $(5,40)$-matrix M whose columns are representatives of the points of A. Consider M as generator matrix of a code $\mathcal{C} = \mathcal{C}(A)$. Then $\mathcal{C}$ is a linear $[40,5]_4$-code, and w is the weight of a codeword from $\mathcal{C}$ if and only if there is a hyperplane of $PG(4,4)$ intersecting A in precisely $40 - w$ points.

Let d be the minimum distance of $\mathcal{C}$. By the Griesmer bound of coding theory [3] we have $d \leq 28$. This means that A meets some hyperplane in at least 12 points.

Assume $d = 28$, equivalently that all hyperplane sections of A are ≤ 12. Denote by n_i the number of hyperplanes intersecting A in i points and by H_0 a hyperplane avoiding A. We use a generalization of the construction of residual codes, which can be found in [4]:

Theorem 3. *If there is a linear $[n, k, d]_q$-code, which contains a codeword of weight w, where $w < dq/(q-1)$, then we can construct an $[n-w, k-1]_q$-code of minimum distance $\geq d - \lfloor w(q-1)/q \rfloor$.*

Note that in the situation of Theorem 3 the $n-w$ points in the hyperplane yield the columns of the generator matrix of a code $[n-w, k-1, d-\lfloor w(q-1)/q \rfloor$.

Assume A intersects a hyperplane in 11 points. Then Theorem 3 produces a code $[11, 4, 7]_4$. As such a code does not exist we obtain a contradiction. By the same argument the non-existence of $[7, 4, 4]_4$ and $[6, 4, 3]_4$ show that A has no hyperplane section of 7 or 6 points. Let H_0 be the empty hyperplane at infinity. In homogeneous coordinates we write $H_0 = (x_0 = 0)$ and represent points not in H_0 as $(1 : x_1 : x_2 : x_3 : x_4)$. Call two hyperplanes $\neq H_0$ parallel if they intersect H_0 in the same plane. The 340 hyperplanes $\neq H_0$ come in 85 parallel classes of four each. Such a parallel class has type (s_1, s_2, s_3, s_4), where $s_1 \geq s_2 \geq s_3 \geq s_4$, if A intersects the hyperplanes of this parallel class in s_1, s_2, s_3 and s_4 points. As none of the s_i exceeds 12 and none equals 11, 7 or 6 the only possible types of parallel classes of hyperplanes are

$$(12, 12, 12, 4),\ (12, 12, 8, 8),\ (12, 10, 10, 8),\ (12, 10, 9, 9),\ (10, 10, 10, 10).$$

Let $a_1, \ldots, a_5$ be the number of parallel classes of the respective type. Assume $a_3 = a_5 = 0$. The standard equations on the hyperplane intersection numbers

$$\sum_{i \geq 0} \binom{i}{s} n_i = \binom{40}{s} \frac{4^{5-s} - 1}{3} \qquad s = 0 \ldots 3$$

(equivalent to $\mathcal{C}$ having dual distance > 3) yield equations on the a_i :

$$\begin{aligned} a_1 + a_2 + a_4 &= 85 \\ 204a_1 + 188a_2 + 183a_4 &= 16380 \\ 664a_1 + 552a_2 + 508a_4 &= 49400 \end{aligned}$$

The unique solution has $a_2 < 0$, contradiction.

Consequently parallel classes of type $(12, 10, 10, 8)$ or $(10, 10, 10, 10)$ must occur. We can assume that $H_1 = (x_1 = 0)$ is one of the hyperplanes intersecting A in 10 points. Theorem 3 shows in fact that the $(4, 10)$-matrix with columns $(1, x_2, x_3, x_4)^T$, where $(1 : 0 : x_2 : x_3 : x_4)$ varies over $A \cap H_1$, generates a code $[10, 4, 6]_4$. Such codes (containing the 1-word, of dual distance 4) do exist. Fortunately they can be classified. An exhaustive computer search was performed. Under the action of the stabilizer of H_0 and of H_1 in $P\Gamma L(5, 4)$ there are 3 orbits of such codes (equivalently, from the dual perspective, orbits of 10-caps in $H_1 \setminus H_0$, which generate a code of dual distance 6). Using a similar computer search as in [1] we see that none of these 10-caps in H_1 can be completed to an affine 40-cap intersecting the parallels of H_1 in $\{12, 10, 8\}$ or $\{10, 10, 10\}$ points.

This shows that $d < 28$, equivalently A must intersect some hyperplane in more than 12 points. Assume the largest hyperplane intersection is $13, 14$ or 15. It is possible to classify the caps of these sizes in $H_1 \setminus H_0$. The group induced by $P\Gamma L(5, 5)$ on H_1, mapping H_0 to itself, is a semidirect product of an elementary abelian group of order 4^3 and $P\Gamma L(3, 4)$. There are 4 orbits of 13-caps, 2 orbits of 14-caps and one orbit of 15-caps (of course). None of these can be completed to an affine 40-cap.

This shows that the maximal hyperplane intersection must be 16. The 16-cap in H_1 is uniquely determined. Another exhaustive search produced all the affine 40-caps containing this start cap. It turns out that they all are in one orbit under $P\Gamma L(5, 4)$. Moreover K is complete as a cap in $PG(4, 4)$. Another computer search shows that the stabilizer of K in $P\Gamma L(5, 4)$ has order 960. This completes the proof of Theorem 1. The hyperplane intersection numbers are

$$n_{16} = 5,\ n_{12} = 120,\ n_{10} = 160,\ n_8 = 15,\ n_4 = 40, n_0 = 1.$$

References

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