Projective planes, coverings and a network problem

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Abstract

A problem concerning packet switched networks leads to a natural combinatorial design problem, whose solution involves (k, n)-arcs in projective planes, 3-dimensional linear codes, the theory of fractional matchings and designs which approximate projective planes.

1 Introduction

The following problem arose in the context of designing scalable, congestion-free network topologies. The translation into combinatorial language is made in [5].

Definition 1. *A* **covering** $C(n, k, r)$ *is a family of subsets (called* **blocks***) of an n-set, such that the following are satisfied:*

- *Each block has size $\leq k$.*
- *Each point is on $\leq r$ blocks.*

- *Any pair of points is on at least one common block.*

Denote by $Cov(k,r)$ the maximum n such that $C(n,k,r)$ exists.

In the application we have n network sites, connected by links or busses, where each site has at most r communication ports, each link can connect at most k sites and any pair of sites appear on a common link. For more background see [5, 6].

Clearly, we have $Cov(k,r) \leq r(k-1)+1$, with equality if and only if there exists a Steiner 2-design on $r(k-1)+1$ points with block size k. In the case of equality we have $k \leq r$. From the point of view of applications the case $k > r$ is most interesting. We work under the assumption $k \geq r$. Let $r = q+1$. We will see that case $k = r$ is of fundamental importance for our problem. Our bound states that $Cov(q+1, q+1) \leq q^2 + q + 1$, with equality if and only if a projective plane of order q exists. A link from $C(n,k,q+1), k \geq q+1$ to projective planes has been observed in [14] and further explored in [5]. We start from a formal definition of the basic construction of coverings from projective planes.

Definition 2. *Let w be a function assigning integer weights $w(P)$ to the points P of a projective plane π_q of order q. The* **Radon transform** *$\hat{w}$ is defined by*

$$\hat{w}(g) = \sum_{P \in g} w(P).$$

It assigns weights to the lines of π_q. Let

$$mass(w) = \sum_{P} w(P),$$

where the sum is over all points of the projective plane. We call w a **weighted k-arc** *if $\hat{w}(g) \leq k$ for all lines g. The weight function w is* **non-negative** *if $w(P) \geq 0$ for all P.*

The approach from [1] may be described as follows:

Proposition 1. *A non-negative weighted k-arc in a projective plane of order q yields a covering $C(mass(w), k, q+1)$.*

Proof. Replace every point P of the projective plane by a set of $w(P)$ points. The blocks are the blown-up lines of the plane. Each such block has at most k points, by construction. It is clear that the remaining conditions of Definition 1 are also satisfied. □

A covering constructed via Proposition 1 will be called a **geometric covering,** if the underlying projective plane is desarguesian we speak of a **linear geometric covering.** Comparison with the fundamental description of linear codes in terms of projective geometry shows that linear geometric coverings can be described equivalently in terms of 3-dimensional linear codes.

Theorem 1. *Let q be a prime power. The following are equivalent:*

- *A linear geometric covering $C(n, k, q+1)$.*
- *A nonnegative weighted $k-$arc in $PG(2, q)$ of mass n.*
- *A linear $[n, 3, n-k]_q$ code.*

In the next section we will determine the maximum mass of weighted k-arcs in projective planes of order q when q is a prime power. As a corollary we obtain the largest geometric coverings $C(n, k, q+1)$ when q is a prime power and $k > q$.

2 Weighted k-arcs in projective planes

Lemma 1. *Let w be a weighted k-arc in a projective plane of order q. Then the following hold:*

1. *$mass(\hat{w}) = (q+1)mass(w)$.*
2. *For every point P we have*

$$(q+1)k \geq \hat{\hat{w}}(P) = \sum_{g:P\in g} \hat{w}(g) = mass(w) + qw(P).$$

Proof. Both statements follow from double counting. □

Proposition 2. *For every projective plane of order q the maximum mass of a weighted* 0*-arc is* 0*.*

Proof. Let w be a weighted 0-arc. By definition $mass(\hat{w}) \leq 0$. It follows from Lemma 1 that $mass(w) \leq 0$. Now, it is obvious how to construct a weighted 0-arc of weight 0. □

Proposition 3. *For $1 \leq \kappa \leq q$ the mass of a weighted κ-arc in a plane of order q is at most $(q+1)(\kappa-1)+1$.*

Proof. Assume the weighted κ-arc has mass $\geq (q+1)(\kappa-1)+2$. The second statement of Lemma 1 implies $w(P) \leq 0$ for all P, which yields the contradiction $mass(w) \leq 0$. □

Proposition 3 is also known as the **Barlotti bound.** In the case of the Desarguesian plane $PG(2,q)$ it coincides with the **Griesmer bound** of coding theory when applied to 3-dimensional linear codes (see [10] for binary codes, [12] for general linear codes).

Lemma 2. *The mapping : $w \mapsto w+c$ is a bijection from the weighted $k-$arcs of mass m to the weighted $\{k+(q+1)c\}-$arcs of mass $m+(q^2+q+1)c$.*

This is obvious. In particular it suffices to consider κ-arcs, where $0 < \kappa < q+1$.

Proposition 4. *For every $0 < \kappa < q+1$ the maximum mass of a weighted κ-arc in $PG(2,q)$ is $(q+1)(\kappa-1)+1$, and there is such a weighted arc with minimum weight ≥ -1.*

Proof. By Lemma 2 an equivalent statement is that there is a non-negative weighted $(q+1+\kappa)$-arc with mass $q^2+q+1+(q+1)(\kappa-1)+1 = 2(q^2+q+1)-(q+1)(q+1-\kappa) = q^2+1+(q+1)\kappa$. Another equivalent statement is the existence of a code

$$[q^2+1+(q+1)\kappa, 3, q(q+\kappa-1]_q.$$

Let $l_i, i=1,2,\dots,q+1-\kappa$ be a family of lines such that no more than two have a common point. Define the weighted arc as

$$w = 2 \cdot PG(2,q) - \sum_{i=1}^{q+1-\kappa} l_i,$$

in other words $w(P)=0$ if P is the intersection of two of the $l_i, w(P)=1$ if P is on exactly one of the l_i and $w(P)=2$ otherwise. It is clear that this defines a non-negative weighted arc as claimed. □

In the language of coding theory Proposition 4 says that for every $d > (q-1)^2$ there is a code $[n,3,d]_q$ meeting the Griesmer bound with equality. This was proved in [11].

Theorem 2. *Let $k = c(q+1)+\kappa$, where $0 \le \kappa \le q$ and $c \ge 1$. The largest mass of a non-negative weighted k-arc in $PG(2,q)$ is $c(q^2+q+1)$ if $\kappa = 0$, it is $c(q^2+q+1)+(q+1)(\kappa-1)+1$ if $\kappa > 0$. These are the largest masses among all projective planes of order q.*

Proof. This follows from the Barlotti bound, Proposition 4 and Lemma 2. $\square$

Only cases $c = 0$, equivalently $k = \kappa \le q$ remain open. In fact, this problem remains wide open as it includes the problem of finding the largest cardinality ν of $\{\nu, \kappa\}$-arcs in $PG(2,q)$, an old and famous problem which is nowhere near solution. Due to our assumption $k \ge r$ we do not have to worry about these cases here. We have determined the largest geometrical coverings when $q = r-1$ is a prime power and $k \ge q+1$.

Theorem 3. *Let q be a prime power, $k = c(q+1)+\kappa$, where $0 \le \kappa \le q$. The maximum value n for which a geometric covering $C(n,k,q+1)$ exists is*

1. $c(q^2+q+1)$ *if* $\kappa = 0$,
2. $c(q^2+q+1)+(q+1)(\kappa-1)+1$ *if* $\kappa > 0, c > 0$.

These maxima are achieved by linear geometric coverings.

We will show that the coverings in Theorem 3 are optimal in the generic case (when c is sufficiently large). The proof involves the real relaxation of the covering and matching problems in graph theory.

3 Intersecting hypergraphs

Let $\mathcal{H}$ be a hypergraph, in other words a family of subsets (hyperedges) of a finite set V. A **cover** of $\mathcal{H}$ is a set of vertices which meets all the hyperedges. A **fractional cover** is a mapping

$$t : V \longrightarrow \mathbb{R}^+$$

such that

$$\hat{t}(E) = \sum_{P \in E} t(P) \ge 1 \text{ for every hyperedge } E$$

Here, every vertex is assigned a non-negative weight, the sum of the weights of the points of any hyperedge is ≥ 1. Every cover is also a fractional cover.

The notion of fractional cover is the real relaxation of the notion of a cover. The dual notion is a **fractional matching,** defined as a mapping

$$w : E(\mathcal{H}) \longrightarrow I\!R^{+}$$

satisfying

$$\hat{w}(P) = \sum_{P \in E} w(E) \leq 1$$

for every vertex P Here, every hyperedge is assigned a non-negative weight, the sum of the weights of the hyperedges through any vertex is ≤ 1. This is the real relaxation of the notion of a matching. The main observation is the following: The determination of the minimum mass $\tau^*(\mathcal{H})$ of a fractional cover and of the maximum mass $\nu^*(\mathcal{H})$ of a fractional matching are dual problems of linear programming. These values are therefore equal. This leads to the following fundamental inequality:

Theorem 4. *For every hypergraph $\mathcal{H}$ the fractional matching number, the fractional covering number, the matching number (maximum size of a matching) $\nu(\mathcal{H})$ and the covering number (minimum size of a cover) $\tau(\mathcal{H})$ are related as follows:*

$$\nu(\mathcal{H}) \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq \tau(\mathcal{H})$$

In [8] the following is proved:

Theorem 5. *Let $\mathcal{H}$ be a hypergraph with hyperedge sizes $\leq q+1$ such that any two hyperedges have at least one vertex in common ($\mathcal{H}$ is* **intersecting***). Then one of the following holds:*

- *$\mathcal{H}$ is a projective plane of order q, or*
- *$\tau^*(\mathcal{H}) \leq q$.*

Our main result in this section is a slight generalization of Theorem 5. We introduce a class of partially balanced designs.

Definition 3. *Denote by $G(q, \delta)$ a design with the following properties:*

- *$G(q, \delta)$ has $q^2 + q + 1 - \delta$ points and equally many blocks.*
- *Each block has $q + 1$ points and each point is on $q + 1$ blocks.*

- *Any two points are on* 1 *or* 2 *common blocks.*
- *Any two blocks have* 1 *or* 2 *points in common.*

Clearly the notion of a $G(q, 0)$ is identical to the notion of a projective plane of order q. Let us call $G(q, \delta)$ an **almost projective plane** of defect δ. Almost projective planes of defect 1 will show up in the following theorem.

Theorem 6. *Let $\mathcal{H}$ be an intersecting hypergraph with hyperedge sizes $\leq q+1$ and $\tau^*(\mathcal{H}) = q$. Assume also that $\mathcal{H}$ is* **critical** *(meaning that removal of any edge would decrease the fractional matching number). Then $\mathcal{H}$ is one of the following:*

- *a dual affine plane of order q, or*
- *a $G(q, 1)$.*

Proof. Let n be the number of hyperedges and m the number of points. Let w be an optimal fractional matching, which is a vertex of the polytope. Every hyperedge yields a cover. It follows that the number of points on a hyperedge is either q or $q+1$. Let now $P \in E, |E| = k$ and sum the $\hat{w}(Q)$ for $Q \in E$. We obtain the inequality

$$q + (k-1)w(E) \leq \sum_{Q \in E} \hat{w}(Q) \leq \hat{w}(P) + k - 1.$$

Summing up this inequality over all E containing P we obtain

$$q deg(P) + (k-1)\hat{w}(P) \leq deg(P)(\hat{w}(P) + k - 1),$$

$$deg(P)(\hat{w}(P) + k - 1 - q) \geq (k-1)\hat{w}(P).$$

We see that $k = q$ is impossible. Every hyperedge has $q+1$ points. Our inequality simplifies to $deg(P) \geq q$. Assigning weight $\frac{1}{q+1}$ to each point yields a fractional cover. It follows that $m \geq q(q+1)$. Our original inequality is $q \cdot w(E) \leq \hat{w}(P) \leq 1$.

Assume $deg(P) = q$. We must have equality in all the arguments that lead to the inequality $deg(P) \geq q$. Let $E_1, \dots, E_q$ be the edges through P and $\mathcal{Q} = E_1 \cup \dots \cup E_q \setminus \{P\}$. We must have $\hat{w}(Q) = 1$ for every $Q \in \mathcal{Q}$ and $|E \cap E_i| = 1$ for all edges E not containing P. Moreover $w(E_i) = \hat{w}(P)/q$. We see that $|\mathcal{Q}| = q^2$. Let $\mathcal{R}$ be the union of the points not on one of the

E_i. Every edge E not containing P intersects $\mathcal{R}$ in precisely one point R. As $\{P\} \cup \mathcal{R}$ is a cover we must have $|\mathcal{R}| \geq q-1$. Counting along E_1 yields $q^2 \leq n \leq q^2+q$. It follows $|\mathcal{R}| \leq q$.

Assume at first $|\mathcal{R}| = q$. We have $m = q^2+q+1, n = q^2+q$ and $deg(R) = q, deg(Q) = q+1 (R \in \mathcal{R}, Q \in \mathcal{Q})$. Assigning weight $1/q$ to all edges E not containing P defines a fractional matching of mass q. We see that we can omit the E_i without decreasing the fractional matching number. This contradicts the criticality.

We have $|\mathcal{R}| = q-1$, hence $m = q^2+q$. Assume $deg(R) = q+1$. Then one of the edges through R would have to intersect E_1 in more than one point, contradiction. We have $deg(R) = q$ for all $R \in \mathcal{R}$. It follows $n = q^2$ and all points have degree q. We have a dual affine plane.

The last case is that every point has degree $q+1$. It follows $m = n$. Assigning weight $1/(q+1)$ to points and edges, respectively, shows that the fractional matching number is automatically $= q$ in this situation, and $n = m = q^2+q$. We see that the edges come in pairs: for each edge E there is a unique E' such that $|E \cap E'| > 1$. This intersection cardinality is $= 2$. Let P be a point and assume there is no $Q \neq P$ such that $\{P, Q\}$ is on more than one edge. Then the union of the edges through P would contain q^2+q+1 points, contradiction. It follows that the points come in pairs as well: for every P there is exactly one P' such that $\{P, P'\}$ is on more than 1 edge. The pair is then on precisely two edges. It is obvious now that we have a $G(q, 1)$. □

The following corollary is our main result in this section:

Corollary 1. *Let $q > 2$ and $\mathcal{H}$ an intersecting hypergraph with hyperedge sizes $\leq q+1$ and $\tau^*(\mathcal{H}) = q$. Then either $\mathcal{H}$ is contained in a projective plane of order q and contains a dual affine plane of order q or $\mathcal{H}$ is a $G(q, 1)$.*

Proof. Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a critical subhypergraph. By Theorem 6 $\mathcal{H}_0$ either is a dual affine plane or a $G(q, 1)$. In the first case it is clear that every block B of $\mathcal{H}$, which is not a block of $\mathcal{H}_0$, has a nontrivial intersection with every line of the projective plane containing $\mathcal{H}_0$. It follows that B is a line of this projective plane.

Let $\mathcal{H}_0$ be a $G(q, 1)$ and assume there is a block B in $\mathcal{H}$ which is not in $\mathcal{H}_0$. Denote by x_i the number of blocks of $\mathcal{H}_0$, which meet B in precisely i points, $i = 1, \ldots, q$. Counting (a) all blocks of $\mathcal{H}_0$, (b) the point-block incidences

(P, g), where $P \in B$ and g a block of $\mathcal{H}_0$, and (c) the incidences of pairs of points of B with blocks g yield two equations and an inequality:

$$\boxed{\begin{array}{ccc} \sum_{i=1}^{q} x_i & = & q^2+q \\ \sum_{i=1}^{q} ix_i & = & (q+1)^2 \\ \sum_{i=2}^{q} i(i-1)x_i & \geq & q(q+1) \end{array}}$$

Subtracting the first equation from the second yields $\sum_{i=2}^{q}(i-1)x_i = q+1$. Comparison of the q-fold of this equation with the last inequality shows that we must have $x_i = 0$ unless $i = 1$ or $i = q$. The inequality shows $x_q \geq 2$, the equation $q+1 = (q-1)x_q$ implies $q \leq 3$. Assume $q = 3$. Then $x_1 = 10, x_3 = 2$. The inequality is an equality. This shows that any two points of B are in precisely one block of $\mathcal{H}_0$. However, the two blocks intersecting B in 3 points each have two points of B in common, contradiction. □

4 Reduction to geometric coverings

The link between coverings $C(n, k, q+1)$ and hypergraphs is obtained simply by dualization: let a $C(n, k, q+1)$ be given. We construct a multihypergraph $\mathcal{H}$ (multiple hyperedges are allowed) as follows: every block of the covering is a vertex of $\mathcal{H}$, every point of the covering is a hyperedge of $\mathcal{H}$. Then $\mathcal{H}$ has hyperedges of size $\leq q+1$ and is intersecting. We obtain a matching w of $\mathcal{H}$ by assigning weight $\frac{1}{k}$ to each hyperedge. It follows that $\mathcal{H}$ has matching number $\nu(\mathcal{H}) \geq \frac{n}{k}$. Let $\mathcal{H}_0$ be the hypergraph obtained from $\mathcal{H}$ by counting each hyperedge of $\mathcal{H}$ only once. Comparison with the definition of a geometrical covering (see Proposition 1) shows that our $C(n, k, q+1)$ is geometric if and only if $\mathcal{H}_0$ is contained in a projective plane of order q.

Theorem 7. *Let a covering $C(n, k, q+1)$ in the sense of Definition 1 be given. The multihypergraph $\mathcal{H}$ constructed by dualization has hyperedges of size $\leq q+1$, is intersecting and has fractional matching number $\geq \frac{n}{k}$. The covering is geometric if and only if the hypergraph $\mathcal{H}_0$ underlying $\mathcal{H}$ is contained in a projective plane of order q.*

It is obvious that the fractional matching number is the same for a multiphypergaph and for the hypergraph underlying it. We conclude from Corollary 1 that $C(n, k, q+1)$ is geometric provided $n > qk$.

Theorem 8. *Every $C(n,k,q+1)$, where $n > qk$, is geometric. If there is no $G(q,1)$, the same is true when $n = qk$. In particular $n \leq qk$ if q is not the order of a projective plane, and $n < qk$ if also there is no $G(q,1)$.*

Corollary 2. *The values of optimal geometric coverings as given in Theorem 3 are indeed optimal among all coverings provided either $n \geq qk$, or $n = qk - 1$ and there is no $G(q,1)$.*

We are mostly interested in case $k > r$. If $r = q + 1$, where q is a prime-power, we know from Theorem 3 the optimal size n of a geometric covering. In these cases the only open question remaining is the following: if this optimal geometric value n satisfies $n < qk$, it is conceivable that the optimal geometric covering may be superseded by a non-geometric covering. This may happen when k (equivalently c) is relatively small.

5 $Cov(k,r)$ for $k \geq r \leq 6$.

Lemma 3. $Cov(k,2) = k + \lfloor k/2 \rfloor$.

Proof. Let $n > k$, let B_1 be a block containing P and $Q \notin B_1$. Let $B_2 \supset \{P,Q\}$ and $R \in B_1 \setminus B_2$. Let B_3 be a block containing $\{Q,R\}$. Blocks B_1, B_2, B_3 are different and pairwise intersecting. Their union is the whole ground set as a point outside their union could not be on a common block with P. For the same reason each point in B_1 must be either in B_2 or in B_3, symmetrically for the other blocks. This shows that there can be no other blocks. We can describe the block structure as follows: X is the disjoint union $X = X_1 \cup X_2 \cup X_3$ and $B_i = X_j \cup X_k$, where $\{i,j,k\} = \{1,2,3\}$. It is now obvious that for even k the optimum sizes of the X_i are $k/2$ each. It is also clear what to do when k is odd. $\square$

The structure underlying the blocks in case $r = 2$ is a triangle, which may be seen as the projective plane of order 1.

Theorem 9. *If q is a prime-power, then $Cov(c(q+1), q+1) = c(q^2+q+1)$ for all c and*

$$Cov(c(q+1)+\kappa, q+1) = c(q^2+q+1) + (q+1)(\kappa-1) + 1$$

if $1 \leq \kappa \leq q$ and either $c \geq Max\{1, q-\kappa\}$ or $c = q-1-\kappa \geq 1$ and there is no $G(q,1)$.

Proof. Consider case $\kappa > 0$. The value in the statement of the theorem is the maximal value for a geometric covering when $c \geq 1$. Assume there is a covering with n one more than that value. As the cover cannot be geometric, Theorem 8 implies

$$c(q^2+q+1)+(q+1)(\kappa-1)+2 \leq c(q^2+q)+q\kappa,$$

equivalently $c \leq q-1-\kappa$. Also, when $c = q-1-\kappa$, a $G(q,1)$ must exist. $\square$

Theorem 10. *For* $c \geq 1$

$$Cov(3c,3)=7c,\ Cov(3c+1,3)=7c+1,\ Cov(3c+2,3)=7c+4,$$

$$Cov(4c,4)=13c,\ Cov(4c+1,4)=13c+1,$$
$$Cov(4c+2,4)=13c+6,\ Cov(4c+3,4)=13c+9.$$

Theorem 11. *Let* $c \geq 1$.

$$Cov(5c,5)=21c,\ Cov(5c+1,5)=21c+1 \textit{ if } c \geq 2,$$

$$Cov(5c+2,5)=21c+6,\ Cov(5c+3,5)=21c+11,$$
$$Cov(5c+4,5)=21c+16.$$

Theorem 12. *Let* $c \geq 1$.

$$Cov(6c,6)=31c,\ Cov(6c+1,6)=31c+1 \textit{ if } c \geq 3,$$

$$Cov(6c+2,6)=31c+7 \textit{ if } c \geq 2, Cov(6c+3,6)=31c+13,$$
$$Cov(6c+4,6)=31c+19,\ Cov(6c+5,6)=31c+25.$$

Proof. Most of the statements of the preceding theorems follow directly from Theorem 9. We will see in Section 7 that $G(4,1)$ cannot exist. It remains to show the non-existence of $C(15,5,4),\ C(45,9,6),\ C(70,14,6),\ C(95,19,6)$. In each of these cases consider the dual hypergraph $\mathcal{H}$ and the simple hypergraph $\mathcal{H}_0$ underlying it. We have $\tau^* \geq q$ by Theorem 7, with equality as the covering is not geometric. Corollary 1 implies that $\mathcal{H}_0$ is a $G(q,1)$. The covering defines a non-negative weighted $\{kq,k\}$-arc in $G(q,1)$, in the sense of Definition 4. Corollary 3 yields a contradiction. $\square$

The first open problem concerns $Cov(6,5)$. The largest geometric such covering has $n=22$. Does $C(23,6,5)$ exist? The open existence questions in case $r=6 \leq k$ are

$$C(33,7,6),\ C(64,13,6),\ C(39,8,6).$$

6 The use of substitutes for projective planes

Assume $q = r - 1$ is not a prime-power. In order to obtain good coverings $C(n, k, q+1)$ for $k \geq q+1$ we proceed as follows: construct $C(n, q+1, q+1)$ where n is as close as possible to $q^2 + q + 1$ and derive $C(n, k, q + 1)$ by a generalization of the geometric construction from Section 1. In other words, in the absence of projective planes of order q we try to substitute them by coverings $C(n, q+1, q+1)$. Starting point is a generalization of Definition 2 to arbitrary block systems.

Definition 4. *Let $(\mathcal{P}, \mathcal{B})$ be a block system. This means that $\mathcal{P}$ is a finite set and $\mathcal{B}$ a family of subsets (blocks) of $\mathcal{P}$. Let w be a function assigning integer weights to the points. The* **Radon transform** *$\hat{w}$ is defined by*

$$\hat{w}(B) = \sum_{P \in B} w(P).$$

Let

$$mass(w) = \sum_{P \in \mathcal{P}} w(P).$$

Call w a **weighted k-arc** *if $\hat{w}(B) \leq k$ for all blocks B. The weight function w is* **non-negative** *if $w(P) \geq 0$ for all P. We also call k the* **species** *of the weighted arc. A weighted $\{m, k\}$-arc is a weighted k-arc of species k and mass m. It is an arc of species k if all the weights are 0 or 1.*

Clearly Proposition 1 generalizes:

Proposition 5. *A non-negative weighted k-arc in a $C(n, q+1, q+1)$ yields a covering $C(mass(w), k, q+1)$.*

Lemma 4. *Call a $C(n, q + 1, q + 1)$* **uniform** *if all its blocks have $q + 1$ points and all points are on $q+1$ blocks. For a uniform $C(n, q+1, q+1)$ the mapping : $w \mapsto w + c$ is a bijection from the weighted $k-$arcs of mass m to the weighted $\{k + c(q + 1)\}-$arcs of mass $m + cn$.*

This is obvious. It should be clear how the mechanism of Section 2 generalizes. We start from a uniform $C(n, k + 1, k + 1)$ and determine its weighted $\{m, \kappa\}$-arcs for $0 \leq \kappa \leq q$, with m as large as possible. Let w be such a weighted arc. If c_0 is the minimum value of w (possibly negative), then $w + c$ for $c + c_0 \geq 0$ is a non-negative weighted $\{m + cn, \kappa + c(q + 1)\}$-arc (see Lemma 4), which by Proposition 5 yields a $C(m + cn, \kappa + c(q + 1), q + 1)$.

Lemma 5. *Let w be a weighted k-arc in a uniform $C(n, q+1, q+1)$ Then*

$$mass(\hat{w}) = (q+1)mass(w).$$

This follows simply from double counting.

Proposition 6. *For every uniform $C(n, q+1, q+1)$ the maximum mass of a weighted 0-arc is 0.*

Proof. Let w be a weighted 0-arc. By definition $mass(\hat{w}) \leq 0$. It follows from Lemma 5 that $mass(w) \leq 0$. A weighted 0-arc with weight 0 clearly exists. $\square$

For points P, Q define $i(P, Q)$ to be the number of blocks through P, Q. By definition $i(P, Q) \geq 1$ and $i(P, P) = q + 1$. Let w be a weighted k-arc. Fix a point P.

Lemma 6. *Let w be a weighted k-arc in a uniform $C(n, q+1, q+1)$. For every point we have*

$$mass(w) \leq k(q+1) - qw(P) - \epsilon(P),$$

where $\epsilon(P) = \sum_{Q \neq P} (i(P, Q) - 1)w(Q)$.

Proof. We have $\sum_{g:P \in g} \hat{w}(g) \leq k(q+1)$. On the other hand $\sum_{g:P \in g} \hat{w}(g) = \sum_Q i(P, Q)w(Q) = mass(w) + qw(P) + \epsilon(P)$. $\square$

Observe that Lemma 6 is a generalization of the Barlotti bound Proposition 3 on the size of caps of species k in projective planes of order q.

Theorem 13. *Let w be a weighted k-arc in a uniform $C(n, q+1, q+1)$. Then*

$$mass(w) \leq \frac{kn}{q+1}.$$

Proof. Write $m = mass(w)$. By Lemma 5 we have $mass(\hat{\hat{w}}) = (q+1)^2 m$. As $\hat{\hat{w}}(P) \leq (q+1)k$ for all P we have $mass(\hat{\hat{w}}) \leq (q+1)kn$. Comparison yields the statement of the theorem.

Observe the following simple proof of Theorem 13 in case of $G(q, \delta)$: Let w be a weighted $\{m, k\}$-arc in C, where C is a $G(q, \delta)$. Then $\hat{w}$ is a weighted $\{(q+1)m, (q+1)k\}$-arc in C' with k as maximal value, and $\hat{w} - k$ is a $\{(q+1)m - kn, 0\}$-arc in C' with 0 as maximal value. This forces $(q+1)m - kn \leq 0$. $\square$

Corollary 3. *There is no weighted non-negative $\{kq, k\}$-arc in a $G(q, 1)$ when $k < q^2 - 1,\ k \equiv -1 \pmod q$.*

Proof. Assume w is such an arc. Let P be a point. As $\epsilon(P) \geq 0$, Lemma 6 implies $qw(P) \leq k$, hence $w(P) \leq (k - (q - 1))/q$. It follows that w is a weighted $\{kq, k'\}$-arc, where $k' = (q + 1)(k - (q - 1))/q < k$. Theorem 13 yields a contradiction. □

A direct construction is based on the κ-arcs of a uniform $C(n, q+1, q+1)$. In fact, let A be such a $\kappa-$arc of maximum cardinality. Identify A with its characteristic function. Let $k = c(q+1) + \kappa$. The weighted arc $cC + A$ shows the following:

Theorem 14. *Let C be a uniform $C(n, q+1, q+1)$. Write $k = c(q+1) + \kappa$, where $0 \leq \kappa \leq q$ and let $m_C(\kappa)$ be the maximal size of a κ-arc in C. We can construct*

$$C(cn + m_C(\kappa), k, q + 1).$$

When $C(n, q + 1, q + 1)$ satisfies additional properties certain weighted arcs can be constructed. The following generalizes Theorem 3.

Theorem 15. *Let C be a uniform $C(n, q + 1, q + 1)$. Assume further there exist blocks $l_1, \ldots, l_q$ such that the following hold:*

- *Each point of C is in at most two of the l_i (the l_i form a* **dual arc** *in C), and*

- *for every i and every block g we have that l_i intersects g nontrivially (this is the* **intersection property***).*

Then for every $0 < \kappa < q + 1$ there is a weighted κ-arc in C of mass $n - (q + 1)(q + 1 - \kappa)$, with minimum weight ≥ -1.

Proof. By Lemma 4 an equivalent statement is that there is a non-negative weighted $(q+1+\kappa)$-arc with mass $2n - (q+1)(q+1-\kappa)$. Define the weighted arc as

$$w = 2C - \sum_{i=1}^{q+1-\kappa} l_i.$$

Clearly $mass(w)$ is as promised. We have $w \geq 0$ as the l_i form a dual arc. If g is a block $\neq l_i, i = 1, \ldots, q + 1 - \kappa$, then the intersection hypothesis implies $\hat{w}(g) = 2(q + 1) - (q + 1 - \kappa) = q + 1 + \kappa$. If $g = l_i$, then clearly $\hat{w}(g) < q + 1$. □

Corollary 4. *Let C be a uniform $C(n, q+1, q+1)$ and $k = c(q+1)+\kappa$, where $0 \leq \kappa \leq q$ and $c \geq 1$. Assume C contains a dual $(q+1-\kappa)$-arc with the intersection property. We can construct*

$$C((c+1)n-(q+1)(q+1-\kappa), c(q+1)+\kappa, q+1).$$

7 Almost projective planes with small defect

Natural candidates for uniform $C(n, q+1, q+1)$ are almost projective planes $G(q, \delta)$ with small δ, see Definition 3. Recall that a $G(q, \delta)$ is also a uniform $C(q^2+q+1-\delta, q+1, q+1)$.

Definition 5. *Define a graph Γ on the points of a $G(q, \delta)$ such that two points form an edge if they are contained in two blocks. Call Γ the* **excess graph.** *The excess graph is regular of degree δ.*

7.1 $G(q, 1)$

It follows from our earlier results that a $C(q^2+q, q+1, q+1)$ is either geometric or a $G(q, 1)$. This makes $G(q, 1)$ particularly interesting. The excess graph is a 1-factor. Observe that $G(q, 1)$ are special cases of partially balanced regular group-divisible designs. The Bose-Connor theorem gives necessary conditions for the existence (see [3, 7]), as follows:

Theorem 16. *Assume $G(q, 1)$ exists. Then the following must be satisfied:*

- *If $\binom{q+1}{2}$ is even, then $q+1$ is a square. If $\binom{q+1}{2}$ is 2 (mod 4), then $q-1$ is a sum of two squares.*
- *If $\binom{q+1}{2}$ is odd, then $q-1$ is a square and the quadratic equation*

  $$(q+1)x^2 + (-1)^{t(t-1)/2} 2y^2 = z^2$$

  has a nontrivial solution in the integers.

We are chiefly interested in the question for which composite numbers q such designs exist. The smallest numbers q satisfying the conditions of Theorem 16 are $q = 2, 3, 5, 8, 10, 15$. Case $q = 10$ is of special interest in our context as there is no projective plane of order 10 and a $G(10, 1)$ is a $C(110, 11, 11)$.

$G(2,1)$ and $G(3,1)$ are easy to construct. Design $G(3,1)$ has been constructed by the method of differences (a cyclic group of automorphisms regular on the points) in [4]. An interesting question is the determination of the largest τ^* such that there exist intersecting hypergraphs with hyperedges of size $\geq q+1$ and fractional matching number $\geq \tau^*$, which are not embedded in a projective plane of order q. We have seen $\tau^* \leq q$, with equality if and only if a $G(q,1)$ exists.

Designs $G(q,1)$ have also been studied in [2]. These authors show in particular that a $G(q,1)$ for $q>2$ can admit a regular abelian group on its points only if $q+1$ is a square whose only prime divisors are 2 and 3. No $G(q,1)$ is known for $q>3$.

7.2 $G(q,2)$

Lemma 7. *The excess graph Γ of an $G(q,2)$ does not contain a triangle.*

Proof. Let P,Q,R be such a triangle. Let g_1,g_2 be the blocks through P,Q, g_3,g_4 the blocks through P,R and $g_5,\dots,g_n$ the remaining blocks through P. These blocks partition the points outside the triangle. Consider a block h through Q,R. The $q-1$ points of h outside Q,R are distributed on the $q-1$ blocks $g_5,\dots,g_{q+1}$. It follows that either h intersects one of the g_i in at least 3 points, contradiction, or h intersects at least two of the g_i in two points each. The latter case is impossible as h would contain more than two edges of the eccess graph. $\square$

There are two possible types of points. A point P is of type 1 if there is a block containing P and its two neighbours in the eccess graph, it is of type 2 otherwise. Likewise, a block has type 1 if the edges of Γ contained in the block form a path, it has type 2 otherwise.

Assume P has type 1. Let its neighbours be Q,R. We can choose notation such that the blocks through P are as follows:

$$\begin{array}{|ccc|}\hline g_1 & = & (R,P,Q,A_1,\dots,A_{q-2}) \\ g_2 & = & (P,Q,B_1,\dots,B_{q-1}) \\ g_3 & = & (P,R,C_1,\dots,C_{q-1}) \\ g_4 & = & (P,X_1,\dots,X_q) \\ \dots & = & \dots \\ g_{q+1} & = & (P,\dots,X_{q^2-2q}) \\ \hline\end{array}$$

The following lemma is rather trivial but important to keep in mind.

Lemma 8. *Let P be a point of type* 1*. Each block not through P, Q or R intersects each $g_i, i = 1, \ldots, q+1$ in precisely one point. There are $q^2 - 2q = (q-1)^2 - 1$ such* **exterior** *blocks.*

Proof. This follows from the axiom that any two blocks intersect non-trivially. The number of blocks containing P, Q or R is $(q+1) + 2(q-1) = 3q - 1$. $\square$

In the sequel we indicate the edges of G by writing $u - v$ if u, v form such an edge. In this terminology we would write $g_1 = (R - P - Q, A_1, \ldots, A_{q-2})$.

Theorem 17. *If two points of an $G(q, 2)$ are in the same cycle of the exxess graph, then they have the same type. If a block B contains two points of type* 1*, which form an edge in Γ, then B has type* 1*.*

Proof. It suffices to show that whenever point P has type 1 then also its neighbours Q, R in the excess graph have type 1. With terminology as before consider the $(q-1)^2$ pairs $\{B_i, C_j\}$. As the number of exterior blocks is $(q-1)^2 - 1$ and because of Lemma 8 one such pair must be on an interior block. Without restriction that block contains Q. It follows that g_2 has type 1. The $q-1$ incidences between Q and the C_j must appear on the $q-1$ blocks $\neq g_1, g_2$ through Q. Assume g_3 has type 2, without restriction $C_1 - C_2$. The second block g through $\{C_1, C_2\}$ contains Q. As g generates two of the above incidences it follows that some pair $\{Q, C_j\}$ must be on two blocks. This yields the contradiction that Q has degree > 2 in Γ.

The blocks containing an edge one of whose endpoints is P (of type 1) are g_1, g_2, g_3. As Q, R have type 1, blocks g_2, g_3 must be of type 1. $\square$

Theorem 18. *All points in an $G(q, 2), q \geq 6$ are of type* 1*.*

Proof. Assume P has type 2. We know that its neighbours Q, R in the excess graph have type 2 as well. This shows that all blocks containing either $\{P, Q\}$ or $\{P, R\}$ have type 2. We can choose notation such that the points through P are as follows:

g_1	$=$	$(P - Q, A_1 - A_2, A_3, \ldots, A_{q-1})$
g_2	$=$	$(P - Q, B_1 - B_2, B_3, \ldots, B_{q-1})$
g_3	$=$	$(P - R, C_1 - C_2, C_3, \ldots, C_{q-1})$
g_4	$=$	$(P - R, D_1 - D_2, D_3, \ldots, D_{q-1})$
g_5	$=$	$(P, X_1, X_2, \ldots, X_q)$
$\ldots$	$\ldots$	$\ldots$
g_{q+1}	$=$	$(P, \ldots)$

Call points $\neq P, Q, R$ **exterior points** and blocks not through P, Q or R **exterior blocks.** There are $(q-1)^2$ exterior blocks. Each exterior block intersects each $g_i, i = 1, \dots, q+1$ in precisely one (clearly exterior) point.

Consider the unique block h through $\{Q, R\}$. As h does intersect g_1, g_2 only in Q (otherwise Q would have another neighbour in g_1 or in g_2) and by symmetry h intersects g_3, g_4 only in R, we must have that h intersects two of the $g_i, i = 5, \dots, q+1$ in 2 points, all the others in one point each. We can choose notation such that

$$g_5 = (P, X_1 - X_2, X_3 \dots, X_q),\ g_6 = (P, Y_1 - Y_2, Y_3, \dots, Y_q)$$

and

$$h = (Q, R, X_1 - X_2, Y_1 - Y_2, \dots).$$

Blocks g_5, g_6 have type 2. In fact, assume $X_1 - X_2 - X_3$ on g_5. Without restriction we can assume that the new block through X_2, X_3 contains Q. However, this implies that Q, X_2 are on two common blocks (one being h), leading to the contradiction that X_2 has degree > 2 in Γ. As g_5, g_6 have type 2, we can choose notation such that

$$g_5 = (P, X_1 - X_2, X_3 - X_4, \dots),\ g_6 = (P, Y_1 - Y_2, Y_3 - Y_4, \dots)$$

such that the new blocks contain $\{Q, X_3, X_4\}$ and $\{R, Y_3, Y_4\}$, respectively. We see that the second neighbour of Q is in g_5, whereas the second neighbour (aside of P) of R is in g_6.

We count edges between exterior points. Consider a pair $\{g_i, g_j\}$ and count the edges of Γ whose end points are exterior points in g_i and g_j, respectively. Let us call these edges **exterior transversal edges** between g_i and g_j. We can count this number because we can count the incidences of such pairs of points on blocks. The exterior blocks always contribute precisely $(q-1)^2$ such incident pairs. The remaining blocks contributing such incidences pass through Q or R. As an example, consider the pair $\{g_1, g_3\}$. In this case no line through Q or R contains exterior points in both g_1 and g_3. We are left with the exterior blocks alone, hence with $(q-1)^2$ incidences. This shows that there are no exterior transversal edges between g_1 and g_3. By symmetry we see that there are no exterior transversal edges between $g_1 \cup g_2$ and $g_3 \cup g_4$. Consider $\{g_1, g_2\}$. Blocks through Q do not contribute such incidences. The number of incidences on blocks through R is $2 + 2 + 1 \cdots + 1 = q + 1$. This shows that there are $(q+1) + (q-1)^2 - (q-1)^2 = q + 1$ exterior transversal edges between g_1, g_2 likewise between g_3, g_4. The same type of counting shows that there are no exterior transversal edges between $\{g_1, g_5\}$, there is one such edge between $\{g_1, g_6\}$ and there is again one such edge between $\{g_1, g_i\}$ for $i \geq 7$.

It follows from Lemma 17 that $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2$ are points of type 2. Let l_1 be the second block (aside of g_1) through A_1, A_2. The blocks g_1, l_1 play a role symmetric to our pair $\{g_1, g_2\}$, where the pair of points $\{A_1, A_2\}$ replaces the pair $\{P, Q\}$. We have

$$l_1 = (A_1 - A_2, R, B, U_5, U_6, \ldots, U_{q+1}),$$

where $B \in g_2, U_i \in g_i, i = 5, \ldots, q + 1$. By what we proved on the pair g_1, g_2 the following must be true about the pair g_1, l_1 : there are precisely $q+1$ edges of Γ having one endpoint in $\{P, Q, A_3, \ldots, A_{q-1}\}$ and the second endpoint in $M = \{R, B, U_5, \ldots, U_{q+1}\}$. Let us count the number of incidences of the individual points from M with such edges. Point R is on one such edge $(R - P)$, each U_i is on at most one such edge. It follows that we must have equality. In particular, as there are no exterior transversal edges between g_5, g_1, it must be that U_5 is the second neighbour of Q. The situation is the same when we base ourselves on pair $B_1 - B_2$ instead of $A_1 - A_2$. Let l_2 be the second block through B_1, B_2. By symmetry we have that $U_5 \in l_2$. As $\{R, U_5\} \subset l_1 \cap l_2$, we have that U_5 is a neighbour of R as well as of Q. However, this is impossible as the second neighbour of R (aside of P) is in g_6, not in g_5. We have reached a contradiction. $\square$

Theorem 19. *If the excess graph of an $G(q, 2), q \geq 6$ consists of a single cycle, then the cyclic group Z of order $q^2 + q - 1$ is a group of automorphisms.*

Proof. Let $S - Q - P - R - T$ be a section of the excess graph. We can choose notation such that three of the blocks are as follows:

$$\boxed{\begin{array}{lcl} g_P & = & (Q - P - R, A_1 \ldots, A_{q-2}) \\ g_Q & = & (S - Q - P, B_1, \ldots, B_{q-2}) \\ g_R & = & (P - R - T, C_1 \ldots, C_{q-2}) \end{array}}$$

The A_i, B_j, C_k are pairwise different points. A point is **exterior** if it is not one of P, Q, R. A block is **exterior** if it contains only exterior points. There are $q^2 - 2q$ exterior blocks, and each picks up exactly one point from each of the $q + 1$ blocks through P. Consider the incidences between points $\{A_1, \ldots, A_{q-2}\}$ and $\{S, B_1, \ldots, B_{q-2}\}$. Each exterior block contributes one such incidence, and there are no other blocks giving a contribution. It follows that the number of edges of Γ with one endpoint in $\{A_1, \ldots, A_{q-2}\}$ and the second endpoint in $\{S, B_1, \ldots, B_{q-2}\}$ is $q^2-2q-(q-1)(q-2) = q-2$. Also, S is on no such edge and each B_j is on one of these edges. The same argument is true for the pair (g_P, g_R) instead of (g_P, g_Q). We can choose notation such that $B_i - A_i - C_i$ is a segment of the excess graph, for each $i = 1, \ldots, q-2$.

Let now the points be the elements of $\mathbb{Z}/(q^2+q-1)\mathbb{Z}$ and two points neighbours if and only if they differ by 1. Denote by g_i the block containing $i-1, i, i+1$. Let

$$g_0 = ((-1) - 0 - 1, A_1, \ldots, A_{q-2}.$$

By what we have just proved we can choose notation such that

$$g_1 = (0 - 1 - 2, B_1, \ldots, B_{q-2}) \text{ and}$$

$$g_{-1} = ((-2) - (-1) - 0, C_1, \ldots, C_{q-2}) \text{ ,where}$$

and $\{B_i, C_i\} = \{A_i-1, A_i+1\}$. Assume there is some i such that $B_i = A_i-1$. It follows by induction that g_j contains $A_i - j$, for all j. Choose j such that $j = A_i - j$, equivalently $2j = A_i$. This is possible as we work in an abelian group of odd order. It follows that g_j has less than $q+1$ elements, a contradiction. We conclude that $B_i = A_i + 1$ and $C_i = A_i - 1$ for all i. By induction it follows that $g_j = \{j-1, j, j+1, A_1+j, \ldots, A_{q-2}+j\}$, in other words g_j is in the same orbit as g_0 under the action of the cyclic group. $\square$

Theorem 20. *There is no* $G(6,2)$.

Proof. Let G be a $G(6,2)$. If Γ is connected, then by Theorem 19 G admits the action of a regular cyclic group. However, the largest $C(n, q+1, q+1)$ with a regular abelian group action on the points have been determined in [13]. The largest such cyclic difference cover in case $q = 6$ has $n = 39$. We conclude that Γ is not connected. Let $\Gamma_1, \ldots, \Gamma_m$ be the connected components. Then Γ_i is a cycle of length γ_i, and clearly $\sum \gamma_i = 41$. Denote by $g(P)$ the block with center P. Give an orientation to each Γ_i and calculate mod γ_i on Γ_i. The main argument in the proof of Theorem 19 shows the following: if $g(P) = \{P-1, P, P+1, A_1, A_2, A_3, A_4\}$, then $g(P+1) = \{P, P+1, P+2, B_1, B_2, B_3, B_4\}$, where $\{A_j, B_j\}$ form an edge in Γ. In particular the intersection size with Γ_j is the same for $g(P)$ and $g(P+1)$. Let $d_{ij} = |g(P) \cap \Gamma_j|$, where $P \in \Gamma_i$. Clearly $d_{ii} \geq 3$ and $\sum_j d_{ij} = 7$. Also, $d_{ii} < 7$ as otherwise blocks with center not in Γ_i would have empty intersection with the blocks contained in Γ_i.

Let $P \in \Gamma_i$ and $Q \in g(P) \cap \Gamma_j$ for some $j \neq i$. Assume γ_j does not divide γ_i. Then $g(P \pm \gamma_i) = g(P)$ contains $Q \pm \gamma_i \pmod{\gamma_j}$. By induction all points of the form $Q + c\gamma_i \pmod{\gamma_j}$ are contained in $g(P)$. There are $\frac{\gamma_j}{(\gamma_i,\gamma_j)}$ such points. This number is > 1 by assumption. All these points are contained in $g(P + (\gamma_i, \gamma_j))$. It follows that $g(P) = g(P + (\gamma_i, \gamma_j))$, hence $\gamma_i | \gamma_j$. We have seen the following:

Lemma 9. *If a block with center in Γ_i has a nontrivial intersection with Γ_j, then either γ_i is a multiple of γ_j or γ_j is a multiple of γ_i.*

Let γ_1 be the minimum of the γ_i and $\Gamma_i, i = 1, \ldots, u$ the components having a nontrivial intersection with the blocks that have their center in Γ_1. Clearly $\gamma_1 \geq 5$. By Lemma 9 we have $\gamma_i = a_i\gamma_1, i = 1, \ldots, u$. Moreover $d_{1i} \geq a_i$ for $i = 2, \ldots, u$. Assume there is γ_j coprime to γ_1. Clearly $j > u$. As any two blocks intersect we must have $d_{ji} > 0$ for some $i \leq u$. Lemma 9 implies that $a_i\gamma_i$ is a multiple of γ_j. It follows that a_i is a multiple of γ_j, hence the contradiction $a_i \geq 5$.

We conclude that all γ_i have a nontrivial common divisor with γ_1. As 41 is prime γ_1 cannot be a prime-power. Assume $\gamma_1 = 6$. Let γ_j be coprime to 3. Then γ_j is even and > 6, hence $\gamma_j \geq 8$. By Lemma 9 there is $i \leq u$ such that $\gamma_i = 6a_i$ is a multiple of γ_j. The only possibility is $\gamma_j = 8$ and $a_i = 4$. This shows $u = 2, \gamma_2 = 24, \gamma_j = 8$. This yields a contradiction as $41 - (6 + 24 + 8) = 3 < 6$. It follows $\gamma_1 \geq 10$. It is now easier to obtain a contradiction using the same arguments. □

8 $Cov(7, 7)$ and $G(6, \delta)$

We know that there is no projective plane of order 6, no $C(42, 7, 7)$ and no $G(6, 2)$. It is still possible that a $C(41, 7, 7)$ exists. As shown in [13] there is a $C(39, 7, 7)$ with a regular cyclic group action on the points, and 39 is maximal with this property. It can be shown that there is no $C(40, 7, 7)$ with a regular abelian group action on the points.

Theorem 21. *There is a $G(6, 3)$ admitting Z_{20} as a group of automorphisms.*

Proof. Let the points be $P(i, j)$, where $i = 1, 2$ and $j = 0, \ldots 19$, with Z_{20} acting on the second coordinate. We have two block orbits, with representatives $(1, A_1) \cup (2, B_1)$ and $(1, A_2) \cup (2, B_2)$, respectively. With obvious notation we have $a_1 + b_1 = a_2 + b_2 = 7$. The condition that pairs of points are covered translates as follows:

$$(A_1 - A_1) \cup (A_2 - A_2) = Z_{20}$$
$$(B_1 - B_1) \cup (B_2 - B_2) = Z_{20}$$
$$(A_1 - B_1) \cup (A_2 - B_2) = Z_{20}$$

The last condition shows that all the a_i, b_i must be ≥ 2. We can choose notation such that $a_1 \leq 3$. The first condition shows $a_2 \geq 5$, hence $a_2 = 5, b_2 = 2$. The second equation shows $b_1 = 5$, hence $a_1 = 2$. The following solution, due to Alan Ling, was found by computer. Its validity is now easy to check:

$$A_1 = B_2 = \{0, 10\}, B_1 = \{0, 1, 2, 5, 14\}, A_2 = \{1, 2, 3, 14, 17\}.$$

Consider the points of A_2 on the circle of length 20. All the possible differences occur with the exception of 10. This verifies the first equation. The same property is satisfied by B_1, verifying the second equation. The last equation is equivalent to the fact that $-B_1, 10 - B_1, A_2, A_2 - 10$ partition Z_{20}. We have found a $C(40, 7, 7)$, which admits Z_{20} as group of automorphisms. Any two blocks intersect nontrivially. This verifies that our $C(40, 7, 7)$ is a $G(6, 3)$. The neighbours of $P(i, j)$ in the excess graph Γ are $P(i, j-1), P(i, j+1)$ and $P(i, j+10)$. $\square$

The $G(6, 3)$ constructed in Theorem 21 is not isomorphic to its dual.

We have seen that $Cov(7, 7) \in \{40, 41\}$. In the next section we study the combinatorial properties of Alan Ling's $G(6, 3)$. This allows us to construct good coverings $C(n, k, 7)$ by the method of Section 6.

9 The combinatorics of $G(6, 3)$ and construction of $C(n, k, 7)$

Denote by C the $G(6, 3)$ constructed in Section 8 and by C' its dual. We determined the largest arcs of species κ for $\kappa \leq 4$, both for C and for C'. The sizes of these arcs are identical for C and for C'. The largest such arcs are $\{1, 1\}$-arcs (of course), $\{7, 2\}$-arcs, $\{13, 3\}$-arcs and $\{19, 4\}$-arcs. For $\kappa = 2, 3, 4$ these are computer results.

$$\{(1, 0), (1, 2), (1, 4), (1, 8), (2, 0), (2, 4), (2, 8)\}$$

is a $\{7,2\}$-arc. An example for a $\{13,3\}$-arc is

$$\{(1,0),(1,1),(1,2),(1,7),(1,9),(1,18),(2,0),(2,3),(2,8),(2,11),(2,13),(2,15),(2,17)\}$$

For $\kappa = 5$ and $\kappa = 6$ we apply Theorem 15. This yields a weighted $\{26,5\}$-arc of minimum -1 and an $\{33,6\}$-arc (the complement of a block).

Proposition 7. *Let $k = 7c + \kappa$, where $0 \leq \kappa \leq 6, c \geq 1$. There exist*

$$C(40c,7c,7), C(40c+1,7c+1,7), C(40c+7,7c+2,7), C(40c+13,7c+3,7),$$

$$C(40c+19,7c+4,7), C(40c+26,7c+5,7), C(40c+33,7c+6,7).$$

It is interesting to study how far the masses of these (weighted) arcs are from the maximum mass of a weighted arc of the corresponding species in C or C'. The upper bound on weighted arcs from Theorem 13 is close for large κ. In fact, it is possible to show that for $\kappa = 5$ and $\kappa = 6$ our weighted arcs are optimal. There are no weighted $\{33,6\}$-arcs or $\{27,5\}$-arcs in C or C'.

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