Partition Identities

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Chapter 1

Introduction

Each representation of a natural number n as a sum of natural numbers is a **partition** of n :

$$n = a_1 + a_2 + \cdots + a_m(\text{ where} a_i \neq 0)$$

Here the order of the **summands** (or **parts**) a_i is irrelevant. Usually we will write the a_i in non-increasing order. The partition can be written in the form

$$\pi = (a_1, a_2, \ldots, a_m).$$

Let $Par(n)$ be the set of all partitions of n.

We want to study the number of partitions of n of certain types, which are defined either by restrictions on number or size of the summands or by restrictions on the relative position of the parts. Let $p(n) = |Par(n)|$ be the number of partitions of n. This defines the **partition function.** Further let $p_m(n)$ be the number of all $P(n)$ all of whose summands are $\leq m$. Here we use the expression $P(n)$ as a short form for the phrase *partition of* n.

The method of **generating functions** comes in handy. Whenever one is interested in a sequence of numbers $a_1, a_2, \ldots$, it is possible and often helpful to consider the corresponding power series

$$P(x) = 1 + \sum_{i=1}^{\infty} a_i x^i,$$

which is known as the generating function of the sequence (a_i). This power series has the some information as the sequence (a_i), but it is just one object, not an infinity of objects. We can use the **geometric series**

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

and obtain the generating functions for $p(n)$ and for $p_m(n)$:

$$1 + \sum_{n=1}^{\infty} p(n)x^n = \frac{1}{\prod_{i=1}^{\infty}(1-x^i)}$$

$$1 + \sum_{n=1}^{\infty} p_m(n)x^n = \frac{1}{\prod_{i=1}^{m}(1-x^i)}$$

In fact, consider the first of these equations and write each factor on the right side as an infinite sum by the geometric series: $(\sum x^j)(\sum x^{2j})(\sum x^{3j})\ldots$ and imagine multiplying out. Each partition of n will contribute a summand of 1 to the coefficient of x^n. For the second expression the procedure is analogous. To give an example, the choice of the summand x^{2j} from the second factor above corresponds to using the part 2 exactly j times in the partition. This also shows why i should be bounded by m when counting partitions with no summands $> m$.

Once this has become clear it is easy to vary this pattern:

$$\frac{1}{\prod_{i=1}^{\infty}(1-x^{2i})}$$

counts the partitions of n all of whose parts are even,

$$\frac{1}{\prod_{i=0}^{\infty}(1-x^{2i+1})}$$

counts the partitions of n all of whose parts are odd,

$$\frac{1}{\prod_{i=0}^{\infty}\{(1-x^{5i+1})(1-x^{5i+4})\}}$$

counts the partitions of n all of whose parts are $\pm 1 \pmod 5$, and

$$\frac{x^N}{\prod_{i=1}^{\infty}(1-x^{2i})}$$

counts the partitions of $n-N$. Here we have used the expressent *counts* ... short for *is the generating function of* ...

Things look somewhat different when for example we want all parts of the partition to be different:

$$\prod_{i=1}^{\infty}(1+x^i)$$

counts the partitions of n with different parts. This is clear from multiplying out the power series. It is the same procedure as earlier. Only the first two terms of each geometric series have survived. The choice of summand 1 from the factor $(1+x^i)$ means that part i does not occur, choice of x^i means that i is a part. Again this is easy to modify:

$$\prod_{i=0}^{\infty}(1+x^{2i+1})$$

counts partitions of n, whose parts are different and odd. If each part is used at most 3 times we obtain the generating function

$$\prod_{i\geq 1}(1+x^i+x^{2i}+x^{3i})$$

1.1 Proposition. *For every n, the number of partitions of n into odd parts, where each part is used at most 3 times, equals the number of partitions in different parts, which are not divisible by 4.*

Proof. This is easiest using generating functions. The generating function for the first type of partitions is

$$G(x)=\prod_{i\geq 1}(1+x^i+x^{2i}+x^{3i})$$

where the product is over all odd i. As $1+x^i+x^{2i}+x^{3i}=(1+x^i)(1+x^{2i})$ and the doubles of odd numbers are the evens, which are not divisible by 4, we have $G(x)=\prod_j(1+x^j)$, where j varies over all natural numbers, which are not multiples of 4. $\square$

The following statement is known as **Euler's partition theorem.** A proof is given in Chapter 12 of [1].

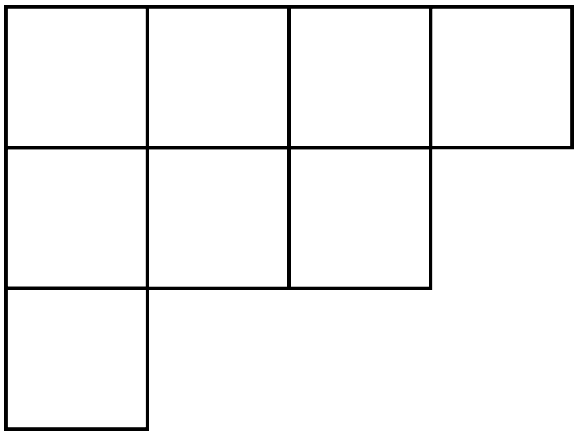

Figure 1.1: Partition $(4, 3, 1)$

1.2 Theorem. *For each n the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.*

Proof. A partition of n into distinct parts is described by numbers f_i such that $f_1 \cdot 1 + f_2 \cdot 3 + f_3 \cdot 5 \cdots = n$ (f_i counts how often part $2i-1$ is used). Write each f_i as a sum of different powers of 2 and multiply out. The result is a partition of n into different parts. As each natural number can be written as a product of an odd number and a power of 2 in precisely one way this mapping describes a bijection. $\square$

Another more combinatorial angle is obtained if we represent each partition by its **Ferrer diagram.** The Ferrer diagram of $8 = 4 + 3 + 1$ is Figure 1.1

and the diagram describing $16 = 4 + 4 + 4 + 3 + 1$ is Figure 1.2.

The number n is the number of little squares in the diagram (the **weight** of the diagram). Consider the operation σ, which reflects each Ferrer diagram across the main diagonal. The operation σ is called **conjugation.** It preserves the weight, is involutorial and induces a bijection between partitions with highest part $\leq m$ and partitions with $\leq m$ parts. We have seen the following elementary fact:

1.3 Lemma. *The number $p_m(n)$ of partitions of n into parts of size $\leq m$ is the same as the number of partitions of n into $\leq m$ parts.*

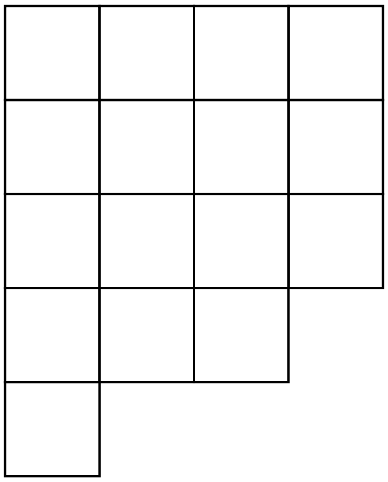

Figure 1.2: Partition $(4, 4, 4, 3, 1)$

Partitions, which are mapped to themselves by σ, are called **self-conjugate.** The corresponding Ferrer diagrams are symmetric about the main diagonal. Figure 1.3 gives an example.

We can derive partitions from self-conjugate diagrams in a different way: The highest part of the new partition will be obtained from the union of the elements in the first row and first column. Because of symmetry this will be an odd number. After erasing these symbols the same method produces the second part. Again, this will be odd, and it will be smaller than the first part.

In our example we obtain $(5, 4, 4, 3, 1) \mapsto (9, 5, 3)$. As promised, the result is a partition into different odd parts. The number of parts of the new partition is the length of the main diagonal in the old partition. It is clear that this yields a bijection. We have a first theorem:

1.4 Theorem. *The number of self-conjugate partitions of n is the same as the number of partitions of n into different odd parts.*

Observe that we know the corresponding generating function.

Let us count the symmetric Ferrer diagrams in a different way: we partition each such diagram into three pieces: the maximum (m, m)-subsquare in the top left corner (m is the length of the main diagonal), the diagram in the top right (it has $\frac{n-m^2}{2}$ cells) and the conjugate diagram in the bottom left. When m is given, the number of possible diagrams in the top right is the

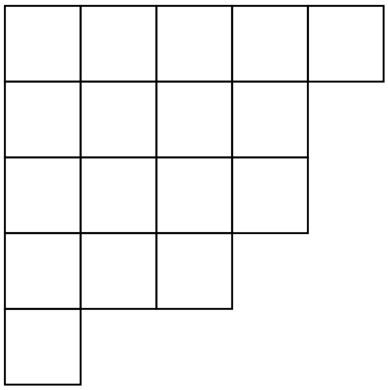

Figure 1.3: A self-conjugate partition

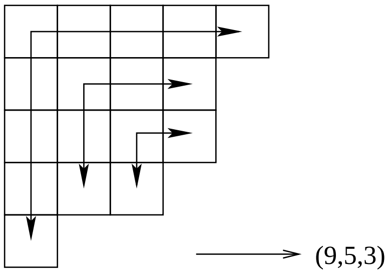

Figure 1.4: The corresponding partition into different odd parts

number of $P((n-m^2)/2)$ with $\leq m$ parts, which is the same as the number of $P(n-m^2)$ into $\leq m$ even parts. As we know the corresponding generating functions we obtain an equality of power series:

1.5 Proposition.

$$\prod_{i=0}^{\infty}(1+x^{2i+1}) = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{\prod_{i=1}^{m}(1-x^{2i})}$$

In fact, both sides in Proposition 1.5 are the generating function for the number of self-conjugate partitions of n.

Chapter 2

Euler's pentagonal number theorem

2.1 Theorem.

$$\prod_{i=1}^{\infty}(1-x^i) = \sum_{m\in\mathbb{Z}}(-1)^m x^{m(3m+1)/2}.$$

Proof. The first combinatorial proof of Theorem 2.1 is due to Franklin (1881) [6]. Let $d(n)$ be the coefficient of x^n on the left side. We can interpret this number combinatorially. Each partition of n with different parts contributes a summand of ± 1 to $d(n)$. The contribution is $+1$ if the partition has an even number of parts, it is -1 if it has an odd number of parts. This motivates to split the partitions of n with different parts into two disjoint parts: $G(n)$, the $P(n)$ with an even number of distinct parts, and $U(n)$, the $P(n)$ with an odd number of distinct parts. We have

$$d(n) = |G(n)| - |U(n)|.$$

We have to show that $d(n) = 0$ if n does not have one of the exceptional forms, which occur as exponents on the right side. Let $a(j) = j(3j+1)/2$ and observe that these numbers are pairwise different and $a(j) > 0$ if $j \neq 0$. We have to show $d(n) = 1$ if $n = a(j), j$ even and $d(n) = -1$ if $n = a(j), j$ odd. This will then prove the identity.

It was Franklin's idea to construct a mapping ϕ_F, which induces a bijection between $G(n)$ and $U(n)$ when n is not of the form $a(j)$. So let π be a partition of n into different parts. We can identify π with the set of its parts. Consider the following parameters: The largest part $t(\pi)$ is the **top**, the smallest part $b(\pi)$ is the **bottom**, $g(\pi)$ (the **gap**) is the largest number $< t$, which is not a part of π, and $i(\pi) = t - g$ (the **interval**). Observe that i is the length of the largest interval containing t, which belongs intirely to π.

Here is the bijection idea when $i < b$: Take away the top ($t \notin \pi'$), use these t elements to fill the gap ($g \in \pi'$) and to create a new bottom ($b' = t - g = i < b$). The resulting partition $\phi_F(\pi) = \pi'$ is still a partition of the same number. It has one part less than π and satisfies

$$t' = t - 1, b' = i, g' < g, i' \geq i = b'.$$

When $i \geq b$ we define $\phi_F(\pi) = \pi'$ in such a way that we can expect the desired bijection property to be satisfied. It is clear what we have to do: when $i \geq b$, then

$$\pi' = \pi \cup \{t+1\} \setminus \{b, t-b+1\}$$

This time π' has one part less than π (as expected). As $t - b + 1 \leq t - g$, we have that $g' = t - b + 1$. Clearly $t' = t + 1, b' > b$. It follows $i' = b < b'$. Also, whenever $\phi_F(\pi) = \pi'$ is defined, we have that π and π' are of different types (one in $G(n)$, the other in $U(n)$) and $\phi_F(\phi_F(\pi)) = \pi$.

When are there problems with the definition of ϕ_F? Consider at first case $i < b$. The only conceivable difficulty is when $i = g$. This forces $\pi = [i+1, 2i]$, an interval with i parts. The partitioned number is $a(i)$.

Analogously in case $i \geq b$ our mapping is not defined when $b = t - b + 1$. This forces $\pi = [b, 2b - 1]$, a partition into b parts. The partitioned number is $b(3b-1)/2 = a(-b)$. We have seen that for every number n, which is not of the form $a(j)$ for some integer j, the mapping ϕ_F yields a bijection between $G(n)$ and $U(n)$. When n has one of these forms, there is either one more element in $G(n)$ than in $U(n)$ or vise versa. These special partitions are those where ϕ_F is not defined. $\square$

We mention that the pentagonal number theorem is needed for the Riemann-Dedekind η-function, whose expansion at infinity is

$$q^{1/24} \prod_{k=1}^{\infty} (1 - q^k),$$

where $q = exp(2\pi i\tau)$ and τ varies in the upper halfplane.

The exceptional exponents or rather half of them, the numbers $a(-j) = j(3j-1)/2, j = 1, 2, 3, 4 \ldots$ on the right side in Theorem 2.1 are known as

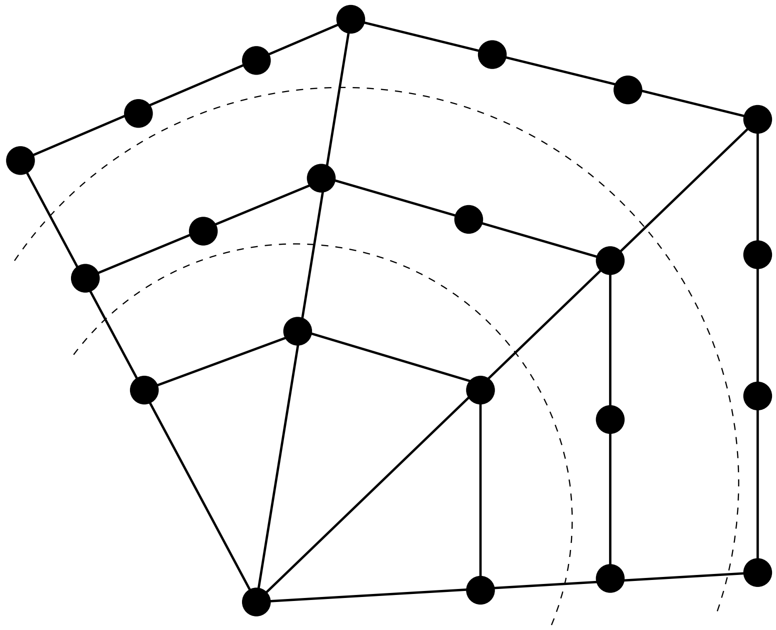

Figure 2.1: Pentagonal numbers

pentagonal numbers. These are the numbers $1, 5, 12, 22, \ldots$ The reason is given in Figure 2.1.

The infinite product in the pentagonal number theorem and the generating function of the partition function are reciprocals. We can therefore use the pentagonal number theorem to obtain a recursion formula for $p(n)$. Euler used it already for the calculation of $p(n)$. In fact, as the product of these reciprocals has a vanishing coefficient at x^n when $n > 0$ we obtain the identity

$$\sum_{j \in \mathbb{Z} \text{ even}} p(n - a(j)) = \sum_{j \in \mathbb{Z} \text{ odd}} p(n - a(j)),$$

where $a(j) = j(3j+1)/2$ as before. The first values of the partition are easily calculated:

$$p(0) = 1,\ p(1) = 1,\ p(2) = 2,\ p(3) = 3,\ p(4) = 5,\ p(5) = 7,\ p(6) = 11.$$

The identity which determines $p(5)$ recursively is

$$p(5) + p(0) = p(3) + p(4).$$

The value of $p(6)$ can then be determined from

$$p(6) + p(1) = p(4) + p(5).$$

It is clear that a proof for this identity is equivalent to a proof of the pentagonal number theorem. A simple combinatorial proof if given in [4], as follows:

2.2 Theorem. *For every natural number there is a bijection*

$$\phi_{BZ} : \bigcup_{j\in\mathbb{Z}\ even} Par(n - a(j)) \longleftrightarrow \bigcup_{j\in\mathbb{Z}\ odd} Par(n - a(j)).$$

Proof. The idea is the following: let $\pi = (a_1, a_2, \ldots, a_m) \in Par(n - a(j))$. Observe that j is integer, $a(j)$ is always non-negative and $a(j) - a(j-1) = 3j - 1$. Recall that n is given. If possible we construct $\phi_{BZ}(\pi) \in Par(n - a(j-1))$ by taking away one element from each part of ϕ and, together with $3j - 1$ additional elements, form a new top (if $3j - 1 < 0$ the wording is not appropriate). The new top has $m + 3j - 1$ elements. In order to be the new top, we must have $m + 3j \geq a_1$. Here is the formal description:

Assume $m + 3j \geq a_1$. Then $\phi_{BZ}(\pi) = (m + 3j - 1, a_1 - 1, \ldots, a_m - 1) \in Par(n - a(j-1))$. Here some of the smaller parts may be empty (if $a_m = 1$).

Let us take a look at an example: $n = 21, \pi = (5, 5, 4, 3, 2) \in Par(19) = Par(21 - a(1))$. We have $j = 1, m = 5, 3j - 1 = 2$. As $m + 3j = 8 \geq a_1 = 5$, we have $\phi_{BZ}(\pi) = (7, 4, 4, 3, 2, 1) \in Par(21 - a(0)) = Par(21)$. The image $\pi' = \phi_{BZ}(\pi)$ in case $m + 3j \geq a_1$ satisfies $m' + 3j' < a_1$.

So let π be given such that $m + 3j < a_1$. We want to define ϕ_{BZ} in this case such that ϕ_{BZ} becomes involutorial. It is clear what we have to do in order to undo the effect of ϕ_{BZ} from the previous case: the image has to be in in $Par(n - a(j+1))$. We destroy the top, add one cell to each remaining part and add the appropriate number of $1s$ at the end, formally

$$\phi_{BZ}(\pi) = (a_2 + 1, a_3 + 1, \ldots, a_m + 1, 1, \ldots, 1),$$

where the number of 1 parts is $a_1 - 3j - m - 1 \geq 0$.

Consider the example $n = 21, \pi = (7, 4, 4, 3, 2, 1) \in Par(21 - a(0)) = Par(21)$. We have $j = 0, m = 6$ and $6 = m + 3j < a_1 = 7$. We obtain $\phi_{BZ}(\pi) = (5, 5, 4, 3, 2)$. It is easy to check that ϕ_{BZ} is indeed involutorial.

The completes the proof of the Bressoud-Zeilberger bijection and with it of the recursion formula for the partition function and of our second proof for the pentagonal number theorem. ∎

Chapter 3

The triple product identity

The following famous formal identity for Laurent series in 2 variables apparently was found by Euler. It makes its appearence in a natural way in the theory of elliptic Theta functions. Rademacher [10] proves it in this context.

3.1 Theorem (triple product identity).

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+yx^{2n-1})(1+y^{-1}x^{2n-1}) = \sum_{n\in\mathbb{Z}} y^n x^{n^2}$$

Proof. The following combinatorial proof is from [9]. At first we use the generating function of $p(n)$ to move the first factor to the right side. This gives us the following equivalent statement:

$$\prod_{n=1}^{\infty}(1+yx^{2n-1})(1+y^{-1}x^{2n-1}) = (\sum_{k\in\mathbb{Z}} y^k x^{k^2})(\sum_{n=0}^{\infty} p(n)x^{2n}).$$

Compare coefficients at $x^N y^k$, where k is an integer and N a natural number. On the right side this coefficient is $p(\frac{N-k^2}{2})$. On the left side we count how often can be written in the form $N = (a_1+a_2+dots+a_{k_1})+(b_1+b_2+\cdots+b_{k_2})$, where the a_j, b_j are odd natural numbers, $k_1 - k_2 = k$ and $a_1 > a_2 > \cdots > a_{k_1}, b_1 > b_2 > \cdots > b_{k_2}$. This combinatorial description seems to go back to Sylvester. Clearly both these numbers are 0 if N and k have different parity. We can therefore assume that their parity is the same.

Observe that k, N are fixed, k_1, k_2 are natural numbers such that $k_1 - k_2 = k$. The number $p(\frac{N-k^2}{2})$ has made its appearance in the proof of

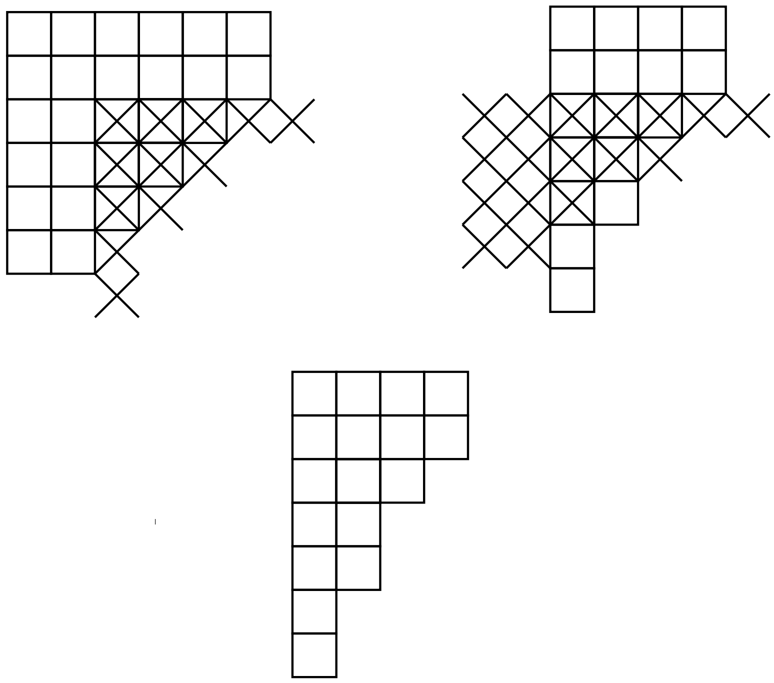

Figure 3.1: triple product identity

Proposition 1.5. In that instance we summed over all k. Here we fix k, and the analysis is somewhat more refined.

Here is the desired bijection: let N be represented in the prescribed way. Consider the self-conjugate diagrams corresponding to $a_1+a_2+\cdots+a_{k_1}$ and to $b_1+b_2+\cdots+b_{k_2}$. Write these symmetric diagrams with different symbols, say squares and x's, with the b-diagram moved k spaces along the main diagonal with respect to the a-diagram (to the southeast if $k>0$, to the northwest if $k<0$). Erase the (k,k)-diagram in the northwest corner. Here comes the decisive step: flip the symbols below the main diagonal (nothing changes if a cell belongs to both diagrams). The resulting scheme has $(N-k^2)/2$ squares and equally many entries x. The image of our mapping is the Ferrer diagram of $(N-k^2)/2$, which is formed by the squares. It is immediately clear that we have a bijection. $\square$

As an example consider the case $N=38, k=2, N=(11+9+5+1)+(9+3)$.

In Figure 3.1 we see the three stages of the construction. In the second stage we see two conjugate diagrams appear. The diagram of squares is the Ferrer diagram of $(4, 4, 3, 2, 2, 1, 1)$.

More interesting identities can be derived from the triple product identity by specialization. In particular the substitution $x \mapsto x^{3/2}, y \mapsto -x^{1/2}$ yields the pentagonal number theorem again. This shows that the combinatorial argument above contains our third proof of the pentagonal number theorem.

Chapter 4

The Rogers-Ramanujan identities

The most famous members of the family of identities we consider here are the **Rogers-Ramanujan equations.** Here are the statements in combinatorial form:

4.1 Theorem (Rogers-Ramanujan). *The number of partitions of n into parts with a pairwise difference of at least 2 equals the number of partitions of n into parts, which are ± 1 mod 5.*
The number of partitions of n into parts ≥ 2 with a pairwise difference of at least 2 equals the number of partitions of n into parts, which are ± 2 mod 5.

Ramanujan had discovered this empirically but was unable to find a proof. They had the status of unsolved problems until Ramanujan found them formulated and proved in an 1894 paper by L.J.Rogers. The problems had thus been solved 20 years before they had been formulated.

Let us express the identities in terms of generating functions. The right sides are obvious. Consider the left side of the first equation. Let m be the number of parts, $\pi = (a_1, a_2, \ldots, a_m)$. If the conditions are satisfied, we can subtract 1 from a_m, 3 from a_{m-1}, and so on, finally $2m-1$ from a_1. This yields the partition $(a_1 - (2m-1), a_2 - (2m-3), \ldots, a_m - 1)$, where some parts may be 0. As $1+3+\cdots+(2m-1) = m^2$, our partitions with precisely m parts are in bijection with the partitions of $n - m^2$ with $\leq m$ parts. We know the generating function of these partitions. Summation over m yields the desired expression. The left side of the secind equation is obtained in an analogous way. Observe $2+4+\cdots+2m = m(m+1)$.

4.2 Theorem (Rogers-Ramanujan).

$$1+\sum_{m=1}^{\infty}\frac{x^{m^2}}{\prod_{i=1}^{m}(1-x^i)}=\prod_{m=1}^{\infty}(1-x^{5m-1})(1-x^{5m-4})$$

$$1+\sum_{m=1}^{\infty}\frac{x^{m(m+1)}}{\prod_{i=1}^{m}(1-x^i)}=\prod_{m=1}^{\infty}(1-x^{5m-2})(1-x^{5m-3})$$

I.Schur gave a proof [11] whose essential part is of a combinatorial nature. Here is the algebraic preparation: Let $R_0(x), R_1(x)$ be the right sides above, and $P(x)=\prod(1-x^i)$. We have

$$P(x)R_0(x)=\prod(1-x^{5m})(1-x^{5m-2})(1-x^{5m-3})$$

$$P(x)R_1(x)=\prod(1-x^{5m})(1-x^{5m-1})(1-x^{5m-4})$$

These expressions have the form of a triple product after the substitutions

$$x\mapsto -x^{5/2},\ y\mapsto x^{1/2}\text{ resp. }x\mapsto -x^{5/2},\ y\mapsto x^{3/2}.$$

Because of the triple product we can write these expressions as sums:

$$P(x)R_0(x)=\sum_{m\in\mathbb{Z}}(-1)^m x^{m(5m+1)/2}$$

$$P(x)R_1(x)=\sum_{m\in\mathbb{Z}}(-1)^m x^{m(5m+3)/2}$$

It suffices to prove these identities.

In order to do this we have to determine the combinatorial meaning of the left sides. In Franklin's proof of the pentagonal number theorem we saw that $P(x)=1+\sum_{m=1}^{\infty}d(m)x^m$, where $d(m)=|G(m)|-|U(m)|$. Let $R_i(x)=1+\sum_{m=1}^{\infty}z_i(m)x^m$ $(i=0,1)$. Let further $P(x)R_i(x)=1+\sum_{m=1}^{\infty}w_i(m)x^m$. We have

$$w_i(n)=\sum_{k=0}^{n}(|G(k)|-|U(k)|)z_i(n-k)=\sum_{k=0}^{n}|G(k)|z_i(n-k)-\sum_{k=0}^{n}|U(k)|z_i(n-k).$$

We see that it suffices to prove the following:

$$w_i(n)=(-1)^m\text{ if }n=m(5m+1+2i)/2,\ w_i(n)=0\text{ otherwise}$$

Let $V = G \cup U$ the set of partitions with different summands, and $V(k) = G(k) \cup U(k)$ with obvious terminology. Denote by Z_i the set of partitions with difference ≥ 2 between parts and minimal part $\geq i+1$ (clearly $Z_i \subset V$).

Schur constructs permutations $\beta_i (i = 0, 1)$ of $V \times Z_i$ with the following properties:

1. β_i preserves the weight (the weight of a pair of partitions is the sum of the weights),
2. For each number $n = m(5m+1+2i)/2$ the permutation β_i has precisely one fixed point of this weight. This fixed point is in $G \times Z_i$ if m is even, it is in $U \times Z_i$ if m is odd. There are no other fixed points.
3. β_i is involutorial and affords a bijection

$$G \times Z_i \setminus Fix(\beta_i) \longleftrightarrow U \times Z_i \setminus Fix(\beta_i).$$

This suffices to prove the identity above, and hence the Rogers-Ramanujan identities. In fact we need only β_0. The permutation β_1 is then defined as the restriction of β_0 to $V \times Z_1$. Nonetheless we treat both cases simultaneously. We have to make sure that

$$\beta_i(\pi, \pi') \in U \times Z_i \iff (\pi, \pi') \in G \times Z_i.$$

Let $(\sigma, \sigma') = \beta_i(\pi, \pi')$ and

$$\pi = (a_1, \ldots, a_k),\ \pi' = (b_1, \ldots, b_l).$$

The cases $a_1 \geq b_1 + 2$ and $a_1 < b_1$ are easy to handle. In these cases β_i is defined as follows:

1. If $a_1 \geq b_1 + 2$, then $\sigma = (a_2, \ldots, a_k), \sigma' = (a_1, b_1, \ldots, b_l)$.
2. If $a_1 < b_1$, then $\sigma = (b_1, a_1, \ldots, a_k), \sigma' = (b_2, \ldots, b_l)$.

This is a simple involutorial operation, which maps cases $a_1 \geq b_1 + 2$ and $a_1 < b_1$ into each other.

In the remaining cases $a_1 = b_1$ and $a_1 = b_1 + 1$ we have to take a closer look. We consider the following three parameters for pairs (π, π') :

$$p = a_k \text{ (the smallest part of } \pi).$$

$$q = Max\{j : a_1 - a_2 = \cdots = a_{j-1} - a_j = 1\} \; (q = 0 \text{ if } a_1 - a_2 > 1)$$

$$r = Max\{j : b_1 - b_2 = \cdots = b_{j-1} - b_j = 2\} \; (r = 0 \text{ if } b_1 - b_2 > 2)$$

Consider a rough subdivision into A (if $a_1 = b_1$) and B (if $a_1 = b_1 + 1$) and a finer subdivision $A = A_1 \cup A_2 \cup A_3$ and $B = B_1 \cup B_2 \cup B_3$, where the A_j, B_j are defined by the relative position of $m = Min\{p, q, r\}$ as follows:

$$A_1 : m = p \qquad A_2 : m = r \neq p \qquad A_3 : m = q \neq p, r$$

$$B_1 : m = q \neq p \qquad B_2 : m = p \qquad B_3 : m = r \neq p, q$$

We will attempt to find bijections : $A_j \longrightarrow B_j$. The description of the mappings (which define β_i in the corresponding cases) will be undefined at certain exceptional pairs. These are then the fixed points.

Consider at first $(\pi, \pi') \in A_1$. Recall that means the minimum m of our three parameters is $m = p$. We define

$$\sigma = (a_1 + 1, a_2 + 1, \ldots, a_m + 1, a_{m+1}, \ldots, a_{k-1}), \sigma' = \pi'.$$

The idea is to destroy the smallest part $a_k = p = m$ of π and to add these elements one by one to the m largest parts. The result will have $p' > p, q' = m, r' = r$, and will therefore be in B_1. Also, the parity of the number of parts of the first partition has changed, as required. When are there problems? If $m = p = q = k$, hence $\pi = (2m - 1, \ldots, m)$. As $b_1 = a_1$ and $r \geq p = m$ we must have $\pi' = (2m - 1, 2m - 3, \ldots, 3, 1)$. As π' has a part $= 1$ this happens only when $i = 0$. The weight is $n = m(5m - 1)/2$. We have defined β_i on all elements of A_1, with image in B_1, with the exception of fixed points occurring for $i = 0, n = m(5m - 1)/2$. It is clear how to define $\beta : B_1 \longrightarrow A_1$ such that the inverse of the former case is obtained. This time fixed points are obtained when $i = 0, \pi = (2m, 2m - 1, \ldots, m + 1), \pi' = (2m - 1, 2m - 3, \ldots, 3, 1), n = m(5m + 1)/2$.

Next let $(\pi, \pi') \in A_2$. This means the minimum is $m = r < p$. The idea is: take one away from each of the leading $m = r$ parts on the right, constitute a new smallest part on the left:

$$\sigma = (\pi, m), \; \sigma' = (b_1 - 1, \ldots, b_m - 1, b_{m+1}, \ldots, b_l).$$

We have $p' = m, q' \geq q, r' \geq r = m$. It follows that we land in B_2. When are there problems? The case $b_l = 1, l = r = m$ cannot happen as we would have $b_1 = a_1 = 2m - 1$, consequently $q < m$, contradiction. Trouble can occur only when $b_l = 2, l = r = m$. This yields $\pi' = (2m, 2m-2, \ldots, 4, 2), \pi = (2m, 2m-1, \ldots, m+1), i = 1, n = m(5m+3)/2$. Again the definition is clear in the other direction. This time there are no fixed points.

Finally consider $(\pi, \pi') \in A_3$. This means the minimum is $m = q < p, r$. We want the image to have its unique minimum in r'. The idea is: copy the top part from right to left, remove one from each of the m top elements left (all in all we add $a_1 = b_1$ elements and remove $m = q$ elements on the left), on the right we remove the top part and add 1 to each of the m following parts, algebraically:

$$\sigma = (b_1, a_1-1, \ldots, a_m-1, a_{m+1}, \ldots, a_k),\ \sigma' = (b_2+1, \ldots, b_{m+1}+1, b_{m+2}, \ldots, b_l).$$

The image has $r' = q, q' > q, p' = p$. It follows that we are in B_3. There are no fixed points. The inverse has a fixed point in case $i = 1, \pi = (2m+1, 2m, \ldots, m+1), \pi' = (2m, 2m-2, \ldots, 4, 2), n = (m+1)((m+1)-3)/2$. This ends the proof of the Rogers-Ramanujan identities.

Chapter 5

Weyl's denominator formula

This formula says the following: let S be half the sum of the positive roots of a root system, Q the lattice of fundamental weights (the dual of the lattice of coroots), and e an isomorphism between Q and its multiplicative variant (e can be seen as exponentiation). Finally, let Θ be the projection from the group algebra $\mathbb{R}[e(Q)]$ to the subring, which is relatively invariant under W with respect to det ($\Theta(a) = \sum_{w \in W} det(w)w(a)$). Then Weyl's formula claims

$$\Theta(e(S)) = e(-S) \prod_{r \in \Phi^+} (e(r) - 1).$$

The proof ([5], pp.149-151) uses the geometry of the Coxeter complex. In the special case of the root system A_n the denominator formula is the **Vandermonde determinant.**

5.1 Theorem.

$$\prod_{1 \leq i < j \leq n} (x_i - x_j) = \sum_{\sigma \in S_n} sgn(\sigma) \prod_i x_{\sigma(i)}^{n-i}.$$

On the right side we have the definition of the determinant of matrix (x_i^j). This is easy to prove as a polynomial equation (each term $x_i - x_j$ divides the determinant, compare degrees and highest coefficient).

We want to see this in a combinatorial fashion: multiply out on the left. Each pair $i < j$ yields a factor, x_i or $-x_j$. We encode this in the following way: an arrow (i, j) from i to j means factor x_i if $i < j$, factor $-x_i$ if $i > j$. It follows that each **tournament graph** T on $\{1, 2, \ldots, n\}$ yields contribution

$w(T)$, where $w(T) = \prod w(e)$ and

$$w(e) = w(i,j) = (-1)^{\chi(i>j)} x_i,\ \chi(true) = 1, \chi(false) = 0.$$

The claim is that we need to sum only over **transitive** tournament graphs. Here a tournament graph is transitive if one of the following equivalent conditions are met:

5.2 Proposition. *Let T be a tournament graph on $\{1, 2, \ldots, n\}$. Then the following are equivalent:*

- *There is a total ordering on the symbols such that $(i, j) \in T$ if and only if i precedes j with respect to this ordering.*
- *The outdegrees of the vertices are all different.*
- *The outdegrees of the vertices are $0, 1, \ldots, n - 1$.*

In order to show that transitive tournament graphs suffice we construct a fixpointfree involution on the non-transitive tournament graphs whose contributions $w(T)$ cancel. This can be done as follows:

Impose an arbitrary total ordering on the pairs of vertices. Let T be a non-transitive tournament graph. Let a, b be the first pair such that the outdegree of a and of b are the same. The image T' is defined as follows:

- Predecessors of a (arrows ending in a) become predecessors of b and vice versa,
- followers of a (arrows originating in a) become followers of b.
- all others arrows remain unchanged

Observe in particular that the arrow between a and b changes direction. Clearly the mapping $T \mapsto T'$ is involutorial and fixed-point-free on the non-transitive tournament graphs. It is undefined on transitive tournament graphs. The first important observation is that $|w(T')| = |w(T)|$. This follows from the fact that a and b have the same outdegree. Only the $\pm$ sign is in question. Count the parity of the sign changes in the contributions to $w(T)$ and to $w(T')$. Let $V(a)$ be the set of predecessors of a (excluding b), analogously $N(a), V(b), N(b)$. If $i \in V(a) \cap V(b)$, or $i \in N(a) \cap N(b)$, there is no change. Each $i \in V(a) \cap N(b)$ and each $i \in V(b) \cap N(a)$ yields two minus signs, so no change. So fgar we excluded a and b. The fact that pair a, b does lead to a negative sign shows $w(T') = -w(T)$.

This proves our claim. The transitive tournament graphs are in bijection with permutations. Let $\sigma \in S_n$ and $T(\sigma)$ the corresponding tournament graph. Then $w(T(\sigma))$ is precisely the expression on the right side of the equation.

A similar proof is given in [7]. Stanton asked if this type of combinatorial proof could be found for the other root systems as well. The answer is affirmative, but the proof given in [3] is case by case.

Bibliography

[1] George E. Andrews: *Number Theory,* Saunders 1971, Dover 1994.

[2] George E. Andrews, *Combinatorics and Ramanujan's Lost Notebook,* 1985.

[3] Bressoud, *Colored tournaments and Weyl's denominator formula, European Journal of Combinatorics* **8**(1987),245-255.

[4] Bressoud-Zeilberger, *Bijecting Euler's partitions-recurrence,* Americal *Mathematical Monthly, January 1985,* 54-55.

[5] Roger W.Carter: *Simple groups of Lie type,* Wiley 1972.

[6] 1881.

[7] I.Gessel, *Tournaments and Vandermonde's determinant,* Journal of *Graph Theory* **3**(1979),303-307.

[8] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers.* Oxford University Press 1939, First Indian Edition 2004.

[9] R.P.Lewis, *A combinatorial proof of the triple product identity,* American *Mathematical Monthly 1984,*420-423.

[10] Rademacher, *Topics in Analytic Number Theory.*

[11] I.Schur: *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche,* Sitzungsberichte der Preussischen Akademie der Wis- *senschaften,* 1917.