Large caps in small spaces

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Abstract

We construct large caps in projective spaces of small dimension (up to 11) defined over fields of order at most 9. The constructions are both theoretical and computer-supported. Some more computer-generated 4-dimensional caps over larger fields are also mentioned.

Key words

Caps, ovoids, codes.

AMS classification

51E22,94B05.

1 Introduction

A **cap** in $PG(k,q)$ is a set of points no three of which are collinear. If we write the n points as columns of a matrix we obtain a $(k+1,n)$-matrix such that every set of three columns is linearly independent, hence the check matrix of a linear code of minimum distance ≥ 4. It follows that a cap of n points in $PG(k,q)$ is equivalent with a q-ary linear code $[n, n-k-1, 4]_q$.

Denote by $m_2(k,q)$ the maximum cardinality of a cap in $PG(k,q)$. We want to construct large caps in the cases of dimension $k \leq 11$, over fields $\mathbb{F}_q$, where $q \leq 9$, thus establishing lower bounds on $m_2(k,q)$ in these cases. We collect our results in Table 4.

As $m_2(k,2) = 2^k$ we can and will assume $q > 2$ in the sequel. Trivially $m_2(1,q) = 2$. In projective dimensions 2 and 3 we have

$$m_2(2,q) = \left[\begin{array}{ll} q+1 & \text{if } q \text{ is odd} \\ q+2 & \text{if } q \text{ is even} \end{array} \right.$$

$$m_2(3,q) = q^2 + 1.$$

Caps of size $q+1$ in $PG(2,q)$ and caps of size q^2+1 in $PG(3,q)$ may be constructed as quadrics. The $(q^2+1)-$caps in $PG(3,q)$ are known as **ovoids.** They have the property that they meet each hyperplane in either $q+1$ points (these are the **intersecting hyperplanes**) or in one point (these are the **tangent hyperplanes**). Each point of the ovoid is on precisely one tangent hyperplane.

For projective dimension $k > 3$ quadrics cannot be caps. No canonical model for large caps is known in dimension > 3. In fact, only three values $m_2(k,q)$ are known when $q > 2, k > 3$:

$$m_2(4,3) = 20,\ m_2(5,3) = 56 \text{ and } m_2(4,4) = 41$$

(see [13, 10, 4]). Direct constructions in dimension 4 were given in [1, 2]. For small q these are superseded by constructions obtained with different means.

The following result from [3] seems to be the most general recursive construction known:

Theorem 1. *Assume the following exist:*

1. *An n-cap $K_1 \subset PG(k,q)$ and a hyperplane H of $PG(k,q)$ such that $|K_1 \setminus H| = w$, and*

2. *an m-cap in $PG(l,q)$.*

Then there is an $\{wm + (n-w)\}$-cap in $PG(k+l,q)$.

The special case $w = n$ of Theorem 1 yields a product construction due to Mukhopadhyay ([12]) (an n-cap in $AG(k,q)$ and an m-cap in $PG(l,q)$ allow the construction of an mn-cap in $PG(k+l,q)$).

The following slight generalization of the product construction, which was first stated in [3], is useful for recursive constructions. It is easy to see that the following somewhat more general statement is true.

Theorem 2. *Assume the following exist:*

1. *An (affine) n-cap $K_1 \subset \mathcal{P}_k(q)$, which is avoided by some $i \geq 1$ hyperplanes in general position, and*

2. *an m-cap $K_2 \subset \mathcal{P}_l(q)$, which is avoided by some $j \geq 0$ hyperplanes in general position.*

Then the product cap is avoided by some $i + j - 1$ hyperplanes in general position.

The specialization $n = k = 2$ of Theorem 2 yields the **doubling construction:** an m-cap in $PG(l,q)$ yields a $2m-$cap in $AG(l+1,q)$.

Another specialization of Theorem 1 yields a recursive construction, which is originally due to Segre [14]:

Theorem 3. *An $n-$cap in $PG(k,q)$ implies a $\{q^2n+1\}-$cap in $PG(k+3,q)$.*

Theorem 3 is obtained by applying Theorem 1 in the case when the first cap is an ovoid.

Another useful recursive construction from [3] is the following:

Theorem 4. *Assume the following exist:*

1. *An n-cap $K_1 \subset PG(k,q)$ possessing a tangent hyperplane, and*

2. *an m-cap $K_2 \subset PG(l,q)$ possessing a tangent hyperplane.*

Then there is an $\{nm-1\}$-cap in $PG(k+l,q)$.

An application of Theorem 4 to ovoids yields $\{q^4+2q^2\}-$caps in $PG(6,q)$. It was shows in [3] that another application of Theorem 1 yields $q^2(q^2+1)^2-$caps in $PG(9,q)$.

Another construction from [3] led us to a family of $\{(q+1)(q^2+3)\}-$caps in $PG(5,q)$.

Table 1: The blocks of $\mathcal{D} : 2 - (6, 3, 2)$.

123	236
124	245
135	256
146	345
156	346

We start our description of improvements in small dimensions over small fields with the ternary case. It plays a particular role. Whereas most constructions over larger fields use ovoids as main ingredients, in the ternary case the Hill cap [10] is the principal ingredient.

2 The ternary case

We start with a brief description of the Hill cap, which follows [8],p.191f. One ingredient is the design $2 - (6, 3, 2)$, which is uniquely determined. Its ten blocks are given in Table 1.

The blocks of this design $\mathcal{D}$ may be described as one orbit of 3-sets under the action of the group $PSL_2(5) \cong A_5$. As $\binom{6}{3} = 20$ and as complements of blocks are not blocks we see that we obtain a partition of all 3-subsets into two designs $2 - (3, 6, 2)$, in other words: the 3-subsets, which are not blocks of our design $\mathcal{D}$, form a design $\overline{\mathcal{D}}$ with the same parameters. The cap will be described as a family of 56 one-dimensional subspaces of $\mathbb{F}_3^6$ (equivalently 112 nonzero vectors), where our designs are defined on the coordinates. The vectors of **type R** are defined as the vectors of weight 6 with an even number of entries 2. We observe the relationship with the binary all even-code. The number of 1-dimensional subspaces of type R is 16 (there are 32 vectors of type R, each such 1-subspace contains two of them). The points of **type** $\mathcal{D}$ are those represented by tuples of weight 3, whose support forms a block of the design. The number of such points is $10 \cdot 4 = 40$. The Hill cap $H \subset PG(5, 3)$ is defined as the union of the 1-dimensional subspaces of type R and of type $\mathcal{D}$, hence a set of $16 + 40 = 56$ points in $PG(5, 3)$. We omit the proof that H is indeed a cap. The proof makes use of the following property of design $\mathcal{D}$: whenever two blocks have two points in common, there is no

Table 2: A 236-cap in $AG(7,3)$

(0,$\mathcal{D}$)
(0,R)
(1,$\overline{\mathcal{D}}$)
(1,R)
(2,U)

third block contained in their union. Clearly an isomorphic copy $\overline{H}$ of the Hill cap is generated by the points of types R and $\overline{\mathcal{D}}$, that is we replace the design $\mathcal{D}$ by its complement.

The following construction from [7] is closely related to the Hill cap. The doubling construction as mentioned in Section 1 shows that the 112=32+80 vectors of types R or $\mathcal{D}$ form a cap in $AG(6,3)$. Likewise the vectors of types R or $\overline{\mathcal{D}}$ form a cap in $AG(6,3)$. Let U denote the family of 12 vectors of weight 1 in $\mathbb{F}_3^6$. We claim that the 236 vectors given in Table 2 form a 236-cap in $AG(7,3)$. This is the Calderbank-Fishburn cap.

Theorem 5. *The 236 vectors in $AG(7,3)$ as given in Table 2 form a cap.*

Proof. We have seen that the two first types, with prefix 0, form a 112-cap, likewise the two types with prefix 1. The last type consists of the 12 vectors with prefix 2 followed by a vector of weight 1. In order to see that we have an affine cap we have to show that no three of our vectors sum to 0. Assume x, y, z are from our collection and $x+y+z=0$. The prefixes cannot be all 0 or all 1. It is also clear that the prefixes cannot all be 2, as the sum of two vectors of weight 1 can never have weight 1. We conclude that the prefixes of x, y, z are 0,1,2. The sum of two vectors of type R never has weight 1. The sum of a vector of type R and a vector of type $\mathcal{D}$ or $\overline{\mathcal{D}}$ has weight ≥ 3. As it is also clear that the sum of a vector of type $\mathcal{D}$ and a vector of type $\overline{\mathcal{D}}$ has weight >1 we conclude that we have a 236-cap in $AG(7,3)$. ■

2.1 A 248-cap in $PG(7,3)$

The Calderbank-Fishburn cap is in $AG(7,3)$ and has 236 points. In projective notation it consists of the points generated by the vectors in $\mathbb{F}_3^8$ of the types given in Table 3.

Table 3:

(1,0,$\mathcal{D}$)
(1,0,R)
(1,1,$\overline{\mathcal{D}}$)
(1,1,R)
(1,2,U)

Theorem 6. *The 236 vectors in $AG(7,3)$ as given in Table 3 together with the 12 points generated by vectors of type $(0,1,U)$ form a complete 248-cap in $PG(7,3)$.*

Proof. It is clear that the 12 new points contained in the hyperplane $x_1 = 0$ form a cap. It remains to show that the difference between two points of the Calderbank-Fishburn cap never has the form $(0,1,U)$. Assume this is the case. The presence of the entry 1 in the second coordinate restricts the possibilities. Consider the last coordinate section. It is obvious that a vector of weight 1 cannot be involved in the difference. Also, the difference between type R and type $\mathcal{D}$ or $\overline{\mathcal{D}}$ has weight ≥ 3. We have already used the fact that the sum or difference of two vectors of type R cannot have weight 1. The only case remaining is the difference between type $\mathcal{D}$ and type $\overline{\mathcal{D}}$. Obviously, this results in weight ≥ 2. We leave it as an exercise to show this is a complete cap. It is indeed true that the points of type $(0,1,U)$ are the only ones which complete the Calderbank-Fishburn cap to a 237-cap in $PG(7,3)$. ■

2.2 Caps in $PG(8,3)$

It follows from Segre's construction (see Section 1) that $m_2(8,3) \geq 9 \cdot m_2(5,3)+1 = 505$. We will obtain larger caps by starting from a 504-cap obtained as an application of the product construction (the special case $w = n$ of Theorem 1) to the Hill cap in $PG(5,3)$ and the affine part of the ovoid. Our new caps will be obtained as extensions of this 504-cap. We use a description of the affine part of the ternary ovoid as given in [1]. It consists of the pairs (a,b), where $a \in \mathbb{F}_3, b \in \mathbb{F}_9, b^4 = a$. Observe that this is a set of vectors in $\mathbb{F}_3^3$. The elements of the product cap are the points (1-dimensional subspaces) generated by the vectors of the form (a,b,h), where $a \in \mathbb{F}_3, b \in \mathbb{F}_9, h \in \mathbb{F}_3^6, b^4 = a$ and h varies over representatives of the points of the Hill cap H. Different choices of representatives lead to essentially different product caps. Our choice of representatives is based on computer experiments, which produced large caps. As representatives of vectors of type R we choose $(1,1,1,1,1,1)$ and all vectors with four entries $= 2$. For every block $g \in \mathcal{D}$ the four vectors representing the elements of H with support g will be chosen such that the nonzero entries are the rows of the following array:

222
211
121
112

One reason for this choice is that the sum of any two different such vectors has weight 1. This will turn out to be profitable in the construction of extensions. Our representatives of the Hill cap are the columns of the following matrix:

2211221122110000000000002211000022110000	1112122212222221
2121212100002211000022110000000000002211	1121212221222212
2112000021210000221121210000221100000000	1211221222122122
0000211200002121212100002121212100000000	1222111222211222
0000000021122112211200000000000021212121	1222222111121222
0000000000000000000021122112211221122112	1222222122212111

Denote the set of these vectors by H_0. With this choice our product cap is generated by the vectors in

$$\mathcal{C}_0 = \{(a,b,h), h \in H_0, a \in \mathbb{F}_3, b \in \mathbb{F}_9, b^4 = a\}$$

We denote it by $\langle \mathcal{C}_0 \rangle$.

Lemma 1. *There are precisely 111 points in $PG(8,3)$, which complement $\langle \mathcal{C}_0 \rangle$ to a 505-cap. These extension points are generated by the vectors $(1,0,x), x \in \mathbb{F}_3^6$, where $wt(x)$ is even, the entries of x sum to 0 and in case $wt(x) = 4$ the following additional condition is satisfied:*

- *Let g_1, g_2 be the blocks of $\mathcal{D}$ such that $supp(x) = g_1 \cup g_2$. Then x is not constant on the intersection $g_1 \cap g_2$.*

Denote by $\mathcal{E}$ the set of these extension points. Observe that the second segment in $(1, 0, x)$ represents an element in $\mathbb{F}_9$. The entry 0 therefore stands for $(0, 0)$. Aside of the 0-vector we have 30 choices for x of weight 2. As vectors of weight 6 with vanishing sum have 3 entries 1 and 3 entries 2 this gives us $\binom{6}{3} = 20$ choices. Every 4-set is the union of two uniquely determined blocks of $\mathcal{D}$. The number of extension points $(1, 0, x)$, where $wt(x) = 4$, is therefore $\binom{6}{4} \cdot 4 = 60$.

Proof. We show at first that vectors yielding extensions must have the form $\pm(1, 0, x)$. In order to prove our claim it has to be shown equivalently that every vector $e \in \mathbb{F}_3^9$ not of this form can be written as a linear combination of vectors from $\mathcal{C}_0$ with ternary coefficients. Observe that due to the completeness of the Hill cap we have

$$\pm H_0 \pm H_0 = \mathbb{F}_3^6.$$

We will also need the following property of $\mathbb{F}_9$, which is easily checked to be true:

Lemma 2. *Let $U = \{u \mid u^4 = 1\} \subset \mathbb{F}_9$ and $\overline{U} = \{u \mid u^4 = 2\} \subset \mathbb{F}_9$. Whenever $V_1, V_2 \in \{U, \overline{U}\}$ and $\lambda, \mu = \pm 1$ we have*

$$\lambda V_1 + \mu V_2 \supseteq \mathbb{F}_9 \setminus \{0\}.$$

Let now $e = (z, y, x)$ be a vector generating an extension point. The fact that $\pm H_0 \pm H_0 = \mathbb{F}_3^6$ excludes case $(z, y) = (0, 0)$. Likewise, we can write $(0, y, 0) = (1, b_1, h) - (1, b_2, h)$ where $b_1^4 = b_2^4 = 1$ and $h \in H_0$ arbitrary, because of Lemma 2. Assume $e = (0, y, x)$, where $x, y \neq 0$. Because of the completeness of the Hill cap we can find $\lambda, \mu = \pm 1$ and $h_1, h_2 \in H_0$ such that $\lambda h_1 + \mu h_2 = x$. We find $a_1, a_2 = \pm 1$ such that $\lambda a_1 + \mu a_2 = 0$. Because of Lemma 2 there exist b_1, b_2 such that $b_i^4 = a_i$ and $\lambda b_1 + \mu b_2 = y$. We have shown that vectors starting with an entry 0 do not produce extension points. We can choose without restriction $e = (1, y, x)$. As $(1, y, 0) = (2, b_1, h) - (1, b_2, h)$ for suitable b_1, b_2 such that $b_1^4 = 2, b_2^4 = 1$, we have $x \neq 0$. The case $e = (1, y, x)$, where y and x are nonzero, is excluded by the same argument as before.

We have seen that every extension point is generated by a vector $e = (1, 0, x)$.

Lemma 3. *Let $e = (1, 0, x)$. Then e completes $\langle \mathcal{C}_0 \rangle$ to a 505-cap if and only if x cannot be written in the form $\pm(h_1 + h_2)$, where $h_1, h_2 \in H_0$.*

Proof. It is clear that e is not a multiple of an element of $\mathcal{C}_0$. Assume $e = (1,0,x) = \lambda(a_1,b_1,h_1)+\mu(a_2,b_2,h_2)$, where $\lambda,\mu = \pm 1$. The middle coordinate section shows $\lambda b_1 = -\mu b_2$, after taking fourth powers $a_1 = a_2$, which we denote by a. The first coordinate shows $(\lambda+\mu)a = 1$. It follows $\lambda = \mu = -a \neq 0$. The last coordinate section shows $x = \lambda(h_1+h_2)$ as claimed. It is clear that we have equivalence. ■

Lemma 3 shows directly that $(1,0,0)$ generates an extension point. Also, $(1,0,x)$ has this property if and only if $(1,0,-x)$ has.

Lemma 4. *If x has odd weight, then $e = (1,0,x)$ does not generate an extension point.*

Proof. Clearly we use Lemma 3. As
$(1,0,0,0,0,0) = (2,2,2,0,0,0) + (2,1,1,0,0,0)$ we see that x cannot have weight 1. Consider the case $wt(x) = 3$. Choosing h_1, h_2 of type $\mathcal{D}$ with identical support kills the vectors x of weight 3, whose support forms a block of $\mathcal{D}$. Assume now $supp(x)$ forms the complement of a block, without restriction $supp(x) = \{4,5,6\}$. The choices $h_1 = (a,b,c,0,0,0), h_2 = (-a,-b,-c,d,e,f)$ show that we do not obtain extension points. Similar arguments exclude the case $wt(x) = 5$. ■

Lemma 5. *Let $x \in \mathbb{F}_3^6$ such that $wt(x) = 2$. Then $(1,0,x)$ generates an extension point if and only if the coordinates of x sum to 0.*

Proof. As $(2,2,0,0,0,0) = (1,1,1,1,1,1) + (1,1,2,2,2,2)$ vector x cannot be constant on its support. Let now $x = (1,2,0,0,0,0)$. We have to show that $(1,0,x)$ generates an extension point. Clearly we use Lemma 3. Assume $x = \pm(h_1+h_2)$. In particular the entries of h_1+h_2 must sum to 0. This shows that without restriction h_1 is of type R and h_2 is of type $\mathcal{D}$. This however is impossible as h_1+h_2 has weight ≥ 3. ■

Lemma 6. *If $wt(x) = 6$, then $(1,0,x)$ generates an extension point if and only if x has three entries 1 and three entries 2.*

Proof. Consider $x = \pm(h_1+h_2)$ again, where $wt(x) = 6$. It is impossible that the h_i both have type $\mathcal{D}$. If both have type R, then necessarily $h_1 = h_2$. This excludes all vectors x of weight 6 with an even number of entries 2. Let now h_1 of type $\mathcal{D}$ and h_2 of type R, without restriction $h_1 = (a,b,c,0,0,0)$. It follows $h_2 = (a,b,c,*)$. Up to permutations two cases have to be considered. If $(a,b,c) = (2,2,2)$, then $h_2 = (2,2,2,2,1,1)$ and $x = \pm(1,1,1,2,1,1)$. If

$(a, b, c) = (2, 1, 1)$, then $h_2 = (2, 1, 1, 2, 2, 2)$ and $x = (1, 2, 2, 2, 2, 2)$. The lemma follows. ∎

Finally assume $wt(x) = 4$. If h_i are both of type R, then their sum has weight 4 if and only if up to permutation $h_1 = (2, 2, 2, 2, 1, 1), h_2 = (1, 2, 2, 2, 2, 1)$. This shows that x cannot have an odd number of entries 2. Let h_1 be of type $\mathcal{D}$ and h_2 of type R. Choose $h_1 = (2, 1, 1, 0, 0, 0), h_2 = (1, 2, 1, 2, 2, 2)$. Then $h_1 + h_2 = (0, 0, 2, 2, 2, 2)$. As for every 4-subset we can find a block intersecting it in precisely one point it follows that constant x of weight 4 are impossible. Other choices of h_i in this subcase do not yield further restrictions. Let the h_i be of type $\mathcal{D}$, with supports g_1, g_2, respectively. Clearly $g_1 \neq g_2$. If $|g_1 \cap g_2| = 1$, then $x = \pm(h_1 + h_2)$ of weight 4 has an odd number of entries 2. This case has been excluded already. Remains the case $|g_1 \cap g_2| = 2$. Without restriction $g_1 = \{1, 2, 3\}, g_2 = \{1, 2, 4\}$. We must have $h_1 = (a, b, c, 0, 0, 0), h_2 = (a, b, 0, d, 0, 0)$. and $x = \pm(-a, -b, c, d, 0, 0)$. No new conditions are obtained unless $(a, b, c) = (2, 2, 2)$. Necessarily $d = 2$ and $x = \pm(1, 1, 2, 2, 0, 0)$. The lemma follows. ∎

The following lemma will facilitate the construction of extensions of our product cap.

Lemma 7. *Let $e_i = (1, 0, x_i), i = 1, 2$. The following are equivalent:*

- *There is a vector of $\mathcal{C}_0$, which can be expressed as a linear combination of e_1 and e_2.*
- *There is $h \in H_0$ such that $h = \pm(x_1 - x_2)$.*

Proof. Assume $(a, b, h) = \lambda(1, 0, x_1) + \mu(1, 0, x_2)$. The first and second coordinate sections show $a = b = 0, \mu = -\lambda$. The result follows. ∎

Theorem 7. *The 20 extension points generated by $(1, 0, x)$, where $wt(x) = 6$, together with $\langle \mathcal{C}_0 \rangle$ generate a complete 524-cap in $PG(8, 3)$.*

Proof. Assume g is a line containing more than two points of our set. It is impossible that g contains only one extension point. Let g be the line connecting two extension points $P_i, i = 1, 2$ generated by $e_i = (1, 0, x_i)$, where the x_i have weight 6 and three entries of each nonzero kind. Assume g contains a point from $\langle \mathcal{C}_0 \rangle$. By Lemma 7 this means that $h = \pm(x_1 - x_2) \in H_0$. It follows that the entries of h must have sum 0. This shows that h is either constant or plus minus the characteristic function of a block. It is clear that this does not happen. Assume finally that g contains a third of our extension points of weight 6. By the usual argument based on the first entry it follows $x_1+x_2+x_3=0$. It is however clear that the sum of two vectors of type $1^3 2^3$ cannot have this type again.

Finally we show, that our 524-cap is complete. It suffices to show that no element from $\mathcal{E}$ extends it to a cap. This is obvious in case $x=0$. In the cases when $wt(x)=2$ and $wt(x)=4$ this is shown by the following identities:

$$(1,2,0,0,0,0)+(1,2,1,1,2,2)+(1,2,2,2,1,1)=0$$

$$(1,2,1,2,0,0)+(1,2,1,2,1,2)+(1,2,1,2,2,1)=0.$$

■

Let us construct extensions of $\langle\mathcal{C}_0\rangle$ starting from extension points generated by $(1,0,x)$, where $wt(x)=2$.

Lemma 8. *If $wt(x_1)=wt(x_2)=2$ and $P_i=\langle(1,0,x_i)\rangle\in\mathcal{E}$, then $\langle\mathcal{C}_0\rangle\cup\{P_1,P_2\}$ is a 506-cap if and only if either $supp(x_1)\cup supp(x_2)$ is not a block or if the entries of x_1 and x_2 at $supp(x_1)\cap supp(x_2)$ are the same.*

Proof. We apply Lemma 7. It is clear that $h=\pm(x_1-x_2)\in H_0$ can happen only when h has weight 3. This implies that $supp(x_1)\cup supp(x_2)$ must be a block and the entries of x_1 and x_2 at $supp(x_1)\cap supp(x_2)$ are different. It is clear that this is an equivalent description. ■

The usual argument based on the first entry shows that three different vectors $e_i=(1,0,x_i), i=1,2,3$ are linearly dependent if and only if $\sum_{i=1}^{3} x_i=0$.

Let us encode the information at hand in a graph. Consider the complete directed graph on vertices $1,2,3,4,5,6$. Identify the arc from i to j with the vector x with entry 1 in coordinate i, entry 2 in coordinate j. The arcs of the graph are in canonical bijection with the vectors x of weight 2 such that $(1,0,x)\in\mathcal{E}$. By the preceding lemmas a collection of k arcs will give us a $(504+k)$-cap extending $\langle\mathcal{C}_0\rangle$ if and only if the following are satisfied:

1. No three arcs form a directed triangle.

2. If two arcs form a directed path, then their endpoints do not form a block.

We present a solution with 10 arcs, where the arcs consist of all directed edges of the pentagon $(1,5,2,3,4)$ (in this order, observe that 1 and 5 are neighbours).

It is a solution as there are no triangles in the pentagon, and the union of three consecutive vertices never is a block of $\mathcal{D}$. Here is the matrix whose columns generate the 10 points of the extension.

$$\boxed{\begin{array}{cccccccccc}
1&1&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&2&1&2&1&0&0\\
0&0&2&1&0&0&0&0&2&1\\
2&1&0&0&0&0&0&0&1&2\\
1&2&0&0&0&0&1&2&0&0\\
0&0&1&2&1&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0
\end{array}}$$

The resulting 514-cap is not complete. An exhaustive computer search produced as largest extension a (complete) 532-cap. The 18 extension points are generated by the columns of the following array.

$$\boxed{\begin{array}{cccccccccccccccccc}
1&1&1&2&1&2&2&2&1&2&1&2&2&1&1&2&2&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&1&2&2&0&2&2&1&1&2&0&2&1&0&0&2&1&1\\
2&2&0&2&2&2&1&2&0&2&0&1&2&1&0&1&2&1\\
0&0&2&2&0&1&2&2&2&1&2&2&2&0&1&1&1&2\\
1&2&0&1&2&2&2&2&0&1&1&1&1&2&2&2&2&2\\
0&0&1&1&1&1&1&1&2&2&2&2&2&2&2&2&2&2\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}}$$

2.3 Larger dimensions

We have seen a complete 532-cap and a complete 524-cap in $PG(8,3)$. In dimension 9 we were not able to improve on the bound $m_2(9,3) \geq 20 \cdot 56 = 1120$, which results from an application of the product construction to the Pellegrino cap in $AG(4,3)$ and the Hill cap. There is a complete such cap. A complete $2744-$cap in $PG(10,3)$ was found by a computer construction. An application of the product construction to the Hill cap and the (affine) double of the Hill cap yields a complete cap of size $112 \cdot 56 = 6272$ in $PG(11,3)$.

3 The Glynn cap and generalizations

The original description of the Glynn cap is in [9]. We construct it from a slightly different point of view. Let $B = PG(2,q) \subset PG(2,q^2)$ be a Baer subplane, whose points we write in homogeneous coordinates as $(a:b:c)$, where $a,b,c \in \mathbb{F}_q$. The number of exterior points (points outside B) is $q^4+q^2+1-(q^2+q+1) = q^4-q$. It is a matter of elementary counting that every exterior point is on precisely one line of B and that every line has either 1 or $q+1$ points in common with B (the lines of the latter type are the lines of B). Clearly the group $G = G_0 \times \langle\phi\rangle = PGL_3(q) \times \langle\phi\rangle$, where $\phi(x) = x^q$ permutes B and the points outside B. We have

$$\mid PGL_3(q) \mid = q^3(q^2-1)(q^3-1).$$

We will use the following lemma, which is easy to prove:

Lemma 9. *The group $G_0 = PGL_3(q) \subset PGL_3(q^2)$ acts transitively on the points outside the Baer subplane B. It acts regularly (in particular transitively) on pairs (P,Q), where $P \notin B, Q \notin B, PQ \notin B$.*

The orbits of $\langle\phi\rangle$ on exterior points form pairs of **conjugate points.** These pairs are permuted by the action of G_0.

Definition 1. *Consider the following mapping $\gamma : (\mathbb{F}_{q^2})^3 \longrightarrow \mathbb{F}_q^6$*

$$\gamma(a,b,c) = (N(a), N(b), N(c), tr(ab^q), tr(ac^q), tr(bc^q)).$$

Here $N : \mathbb{F}_{q^2} \longrightarrow \mathbb{F}_q$ is the norm and tr is the trace. The norm part shows that $\gamma(a,b,c) = 0$ can only happen if $(a,b,c) = 0$. Also, $\gamma(ua,ub,uc) = N(u)\gamma(a,b,c)$. This shows that γ induces a mapping $\gamma : PG(2,q^2) \longrightarrow PG(5,q)$. It is clear that conjugate points in $PG(2,q^2)$ have the same image under γ. Let $P \in B$. We can choose notation such that $a,b,c \in \mathbb{F}_q$. It follows $\gamma(P) = (a^2 : b^2 : c^2 : 2ab : 2ac : 2bc)$.

Theorem 8. *There is an injective group homomorphism*

$$\iota : PGL(3,q) \longrightarrow PGL(6,q)$$

such that for every $(a:b:c) \in PG(2,q^2)$ and $g \in PGL(3,q)$ we have

$$\gamma((a:b:c)g) = \gamma(a:b:c)\iota(g).$$

If $g = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, *then*

$$\iota(g) = \begin{pmatrix} a_{11}^2 & a_{21}^2 & a_{31}^2 & a_{11}a_{21} & a_{11}a_{31} & a_{21}a_{31} \\ a_{12}^2 & a_{22}^2 & a_{32}^2 & a_{12}a_{22} & a_{12}a_{32} & a_{22}a_{32} \\ a_{13}^2 & a_{23}^2 & a_{33}^2 & a_{13}a_{23} & a_{13}a_{33} & a_{23}a_{33} \\ 2a_{11}a_{12} & 2a_{21}a_{22} & 2a_{31}a_{32} & a_{11}a_{22}+a_{21}a_{12} & a_{11}a_{32}+a_{31}a_{12} & a_{21}a_{32}+a_{31}a_{22} \\ 2a_{11}a_{13} & 2a_{21}a_{23} & 2a_{31}a_{33} & a_{11}a_{23}+a_{21}a_{13} & a_{11}a_{33}+a_{31}a_{13} & a_{21}a_{33}+a_{31}a_{23} \\ 2a_{12}a_{13} & 2a_{22}a_{23} & 2a_{32}a_{33} & a_{12}a_{23}+a_{22}a_{13} & a_{12}a_{33}+a_{32}a_{13} & a_{22}a_{33}+a_{32}a_{23} \end{pmatrix}.$$

Proof. The proof is straightforward. We use the obvious identity

$$N(a+b+c) = N(a) + N(b) + N(c) + tr(ab^q) + tr(ac^q) + tr(bc^q).$$

As an example, the first component of $(a:b:c)g$ has norm

$$a_{11}^2 N(a) + a_{21}^2 N(b) + a_{31}^2 N(c) + a_{11}a_{21}tr(ab^q) + a_{11}a_{31}tr(ac^q) + a_{21}a_{31}tr(bc^q).$$

This explains the first row of $\iota(g)$. ■

Apparently such a statement was first proved in [6].

Lemma 10. *If* $a, b \in \mathbb{F}_{q^2}$ *and* $N(a) = N(b), tr(a) = tr(b)$, *then* $b = a$ *or* $b = a^q$.

Proof. We can assume $ab \neq 0$. The trace equation shows $b = a + x^q - x$ for some x. The norm equation reads

$$N(a) = N(b) = (a^q + x - x^q)(a + x^q - x) = N(a) + (a^q - a)(x^q - x) - (x^q - x)^2,$$

hence $(a^q - a)(x^q - x) = (x^q - x)^2$. If $x^q = x$, then $b = a$, if not then $x^q - x = a^q - a$ and $b = a + a^q - a = a^q$. ■

Proposition 1. *Let* $P = (a:b:c)$ *and* $P' = (a':b':c')$ *be exterior points such that* $\gamma(P) = \gamma(P')$. *Then either* $P' = P$ *or* $P' = P^q$.

Proof. By Theorem 8 and the transitivity of $PGL(3,q)$ on exterior points we can assume $P = (a:1:0)$. The norm part of γ shows that the third coordinate of P' must vanish. We can choose $P' = (a':1:0)$. As $\gamma(P) = (N(a):1:0:tr(a):0:0)$ and analogously for $\gamma(P')$, we have $N(a) = N(a')$ and $tr(a) = tr(a')$ It follows from Lemma 10 that $a' = a$ or $a' = a^q$. ■

Definition 2. *Let $\Gamma_q \subset PG(5,q)$ be the image of γ when restricted to exterior points.*

We have seen that $\mid \Gamma_q \mid= q(q^3-1)/2$. Glynn's observation is equivalent with the claim that Γ_4 is a cap. The restriction of γ to the exterior points of the form $(a:1:0)$ gives us $(q^2-q)/2$ points of Γ_q, which are contained in a plane. It follows that Γ_q cannot be a cap when $q>4$.

Lemma 11. *The points $(N(a):tr(a):1)$, where a varies over representatives of orbits of the Galois group of $\mathbb{F}_{16}\setminus\mathbb{F}_4$, form a hyperoval in $PG(2,4)$.*

Proof. Let $a,b,c\in\mathbb{F}_{16}\setminus\mathbb{F}_4$ as above. We have to show that

$$M=\begin{pmatrix} 1 & tr(a) & N(a) \\ 1 & tr(b) & N(b) \\ 1 & tr(c) & N(c) \end{pmatrix}$$

is a regular matrix. Observe that all its entries are nonzero. We use the obvious facts that there are only two conjugate pairs of elements $\mathbb{F}_{16}\setminus\mathbb{F}_4$ with a given trace and also just two such conjugate pairs with a given nonzero norm. The second column cannot be constant. Assume $tr(a)=tr(b)$. It follows from Lemma 10 that $N(a)\neq N(b)$. The matrix is regular in this case. We conclude that the entries in the second row are pairwise different. As every linear combination of the first and the second column with nonzero coefficients has some vanishing entry, we conclude that M is regular unless the third column is a multiple of either the first or the second column. The first possibility is excluded because of our basic fact concerning norms. As $tr(x)/N(x)=tr(\frac{1}{x})$ the second possibility is excluded as well. ∎

Theorem 9 (Glynn cap). $\Gamma_4\subset PG(5,4)$ *is a cap.*

Proof. Let P_1,P_2,P_3 be any three exterior points, which are also pairwise non-conjugate. We have to prove that their images under γ are not collinear. Let $g\in PGL(3,q)$. It follows from Theorem 8 that $\gamma(P_1),\gamma(P_2),\gamma(P_3)$ are collinear if and only if $\gamma(P_1g),\gamma(P_2g),\gamma(P_3g)$ are collinear, in other words we can use the action of $PGL(3,q)$. Assume $\gamma(P_1),\gamma(P_2),\gamma(P_3)$ are collinear. Because of the transitivity of $PGL(3,q)$ on exterior points we can choose $P_1=(a_1:1:0)$. Assume at first P_1,P_2,P_3 are collinear on $[0,0,1]$. We have $P_i=(a_i:1:0)$, where a_1,a_2,a_3 are non-conjugate from $\mathbb{F}_{16}\setminus\mathbb{F}_4$. The corresponding matrix M is

$N(a_1)$	1	0	$tr(a_1)$	0	0
$N(a_2)$	1	0	$tr(a_2)$	0	0
$N(a_3)$	1	0	$tr(a_3)$	0	0

It follows from Lemma 11 that M has rank 3. We are reduced to the generic case that two of our points P_i are on a common exterior line. Because of Lemma 9 we can choose $P_1 = (a : 1 : 0), P_2 = (b : 0 : 1)$ and we can even fix a and b. Choose $b = a$ such that $N(a) = tr(a) = 1$. If P_3 vanishes in the last coordinate, then M clearly has rank 3. Let $P_3 = (x : y : 1)$. Matrix M is

1	1	0	1	0	0
1	0	1	0	1	0
$N(x)$	$N(y)$	1	$tr(xy^4)$	$tr(x)$	$tr(y)$

Assume M has rank 2. Clearly we must have $tr(y) = 0$, hence $y \in \mathbb{F}_4$. The fifth column must coincide with the third, so $tr(x) = 1$. Columns 4 and 2 are identical, hence $N(y) = y^2 = ytr(x) = y$. This shows $y \in \{0, 1\}$. Column 1 must be the sum of columns 2 and 3, hence $N(x) = y^2 + 1 = y + 1$. If $y = 1$, then $x = 0$ and P_3 is interior, contradiction. We have $y = 0$, hence $N(x) = tr(x) = 1$. It follows from Lemma 10 that P_3 and P_2 are either identical or conjugate, another contradiction. ■

3.1 A variation

Although Γ_q has no chance of being a cap for $q > 4$ it may contain large caps. We used a computer search and found a complete 434-cap in $\Gamma_7 \subset PG(5, 7)$, which has a hyperplane intersection of 7 points.

4 A recursive construction

Definition 3. *Let $\mathcal{C} \subset PG(k, q)$ be a $n-$cap, q odd and H the hyperplane $x_1 = 0$ intersecting $\mathcal{C}$ in w points. We call $\mathcal{C}$* **suitably generated** *with respect to H if there are sets $A, B \subset \mathbb{F}_q^n$ such that the following hold:*

1. *$\mathcal{C} \setminus H$ consists of the 1-dimensional subspaces of $\mathbb{F}_q^{k+1}$ generated by the $(1, a), a \in A$, and $\mathcal{C} \cap H$ consists of the 1-dimensional subspaces generated by the $(0, b), b \in B$.*
2. *$A = -A$*

3. $(B \pm B) \cap 2A = \emptyset$.

Lemma 12. *Consider the situation of Definition 3, where $n - w > 1$. The following hold:*

- $0 \notin A \cup B$.
- *If $a \in A$, then $\lambda a \in A$ if and only if $\lambda = \pm 1$.*
- $\mathbb{F}_q A \cap \mathbb{F}_q B = \{0\}$.
- $(\pm A \pm A) \cap B = \emptyset$

Proof. Clearly $0 \notin B$. If $0 \in A$, choose some $0 \neq a \in A$. The points generated by $(1,0), (1,a), (1,-a)$ are collinear, contradiction. Assume $\lambda a \in A$, where $\lambda \neq \pm 1$. Then the vectors $(1,a), (1,-a), (1, \lambda a)$ are linearly dependent, contradiction. Assume $\lambda a = \mu b$, where $0 \neq \lambda, \mu \in \mathbb{F}_q, a \in A, b \in B$. Then $(1,a), (1,-a)$ and $(0,b)$ generate three collinear points of $\mathcal{C}$, contradiction.
Assume $b = \pm a \pm a'$. We can choose notation such that $b = a - a'$. As $0 \notin B$ we have $a' \neq a$. We obtain the contradiction that $(1,a), (1,a')$ and $(0,b)$ generate three collinear points of $\mathcal{C}$. ■

Proposition 2. *Let $q \equiv 1 (\mathrm{mod}\ 4)$. There is an ovoid $\mathcal{O} \subset PG(3,q)$, which is suitably generated with respect to an (intersecting) hyperplane.*

Proof. We describe the ovoid by the equation $-cX_1^2 + X_2^2 + 2X_3X_4 = 0$, where $c \in \mathbb{F}_q$ is a non-square. Use homogeneous coordinates $(x_1 : x_2 : x_3 : x_4)$ for the description of points in $PG(3,q)$. Let $B = \{B_x \mid x \in \mathbb{F}_q\} \cup \{B_\infty\}$, where $B_x = (x, -x^2/2, 1), B_\infty = (0, 1/2, 0)$ and $A = \{(x_2, x_3, x_4) \mid x_2^2 + 2x_3x_4 = c\}$. Then $\mathcal{O}$ is generated by $(1,a), a \in A$ and $(0,b), b \in B$. Clearly we have $A = -A$. Remains to prove the last condition of Definition 3. Assume there are two elements of B whose sum or difference has the form $2a, a \in A$. If B_∞ is involved, then $a = (B_\infty \pm B_x)/2 = (\pm\frac{x}{2}, \frac{1}{4} \mp \frac{x^2}{4}, \pm\frac{1}{2})$. The definition of A yields $c = \pm\frac{1}{4}$, which is a contradiction as c is a non-square. Assume $a = (B_x \pm B_y)/2 \in A$. If the sign is -1 we obtain the contradiction $c = (x-y)^2/4$. If the sign is $+1$ we have $a = (\frac{x+y}{2}, -\frac{x^2+y^2}{4}, 1)$, and by definition of A finally $c = \frac{(x+y)^2}{4} - \frac{x^2+y^2}{2} = -(x-y)^2/4$, another contradiction. Observe that we have made substantial use of the fact that -1 is a square. ■

Theorem 10. *Let the $n-$cap $\mathcal{C} \subset PG(k,q)$ be suitably generated in the sense of Definition 3, where $|B| = w, |A| = n - w$. If $q > 3$ we can construct a cap $\mathcal{E} \subset PG(k+1,q)$ of size $2(n+w+1)$, which is affine if $q > 5$ and has a tangent hyperplane if $q = 5$.*

Proof. We define $\mathcal{E}$ as generated by the vectors from $U \cup V \cup W \cup Z$, where

$$U = (1,0,A), V = (0,1,A), W = (\pm 1, \pm 1, B), Z = \{(1,u,0),(1,-u,0)\}.$$

Here $u \in \mathbb{F}_q \setminus \{0,1,-1\}$. We have $\mid U \mid = \mid V \mid = n - w, \mid W \mid = 4w, \mid Z \mid = 2$. Clearly U and V are caps. It follows from [3] that $W \cup Z$ is a cap. Let g be a line. If g passes through two points of U, it is contained in the hyperplane $x_2 = 0$ and therefore does not contain a third point of $\mathcal{E}$. An analogous argument holds for V. As $0 \notin A \cup B$ by Lemma 12, the same argument works for the line connecting the two points of Z. Assume g passes through a point of Z and a point of W. A third point from $\mathcal{E}$ on g would have to be in $U \cup V$, without restriction in U. We obtain a linear relation $\lambda_1(1,0,a) + \lambda_2(1,u,0) = (\pm 1, \pm 1, b)$. The third component yields a contradiction to Lemma 12. Assume g passes through two points from W. A third point would have to be from $U \cup V$, without restriction from U. This leads to a linear relation $\lambda_1(1,0,a) + \lambda_2(\pm 1, \pm 1, b) = (\pm 1, \pm 1, b')$. The first two components show $\lambda_2 = \pm 1, \lambda_1 = \pm 2$. The third component yields a contradiction to the last axiom in Definition 3. We have seen that a line g through three points of $\mathcal{E}$ will have to contain a point from U and a point from V. Assume the third point comes from Z. We obtain $\lambda_1(1,0,a) + \lambda_2(0,1,a') = (1,u,0)$. It follows that $\lambda_1 = 1, \lambda_2 = u$, finally $a = -ua'$, contradiction to the second statement of Lemma 12. If the third point is from W we obtain a contradiction to the last statement of Lemma 12.
We have proved that $\mathcal{D}$ is a cap. Add λ times the first coordinate to the second coordinate, where $\lambda \notin \{0,1,-1\}$. The result is nonzero for all vectors generating $\mathcal{D}$ outside Z. If $q > 5$ we can choose $\lambda \neq \pm u$ and obtain that $\mathcal{D}$ is affine. In case $q = 5$ we obtain a tangent hyperplane. ∎

Theorem 10 improves upon the doubling construction (see Section 1). Application in case $q = 5$ yields a 66-cap in $PG(4,5)$ possessing a tangent hyperplane. We can apply Theorem 4 either with the ovoid or with our 66-cap as second ingredient. This results in a complete 1715-cap in $PG(7,5)$ and in a complete 4355-cap in $PG(8,5)$.

5 The known lower bounds

Table 4 contains the best lower bounds known to the authors on $m_2(k,q)$ in cases $2 \leq k \leq 11,\ 3 \leq q \leq 9$. More information on our caps is to be found in [16]. It has been noted earlier that these values are known to equal $m_2(k,q)$ only when $k \leq 3$ and in cases $m_2(4,3) = 20,\ m_2(5,3) = 56$ and $m_2(4,4) = 41$. We use label ^{c} to indicate that a complete cap is known. The following values follow from computer constructions based on symmetry groups:

$$m_2(4,7) \geq 132,\ m_2(4,8) \geq 208,\ m_2(4,9) \geq 212,$$

$$m_2(5,5) \geq 186,\ m_2(5,8) \geq 695.$$

For $m_2(5,7) \geq 434$ see Subsection 3.1. The remaining entries of Table 4 are based on caps constructed in the present paper by application of recursive constructions. In some cases the result is not a complete cap and we used a computer search to obtain completions. Some of the values for $k \geq 6,\ q > 3$ follow from [3]. The following 8-ary values are obtained by application of Theorem 2 to a 208-cap in $AG(4,8)$. The cap is avoided by $i = 3$ hyperplanes in general position. This yields

$$m_2(7,8) \geq 208 \cdot 65 = 13,520,\ m_2(8,8) \geq 208^2 = 43,264,$$

$$m_2(10,8) \geq 208 \cdot 65^2 = 878,800,\ m_2(11,8) \geq 208^2 65 = 2,812,160.$$

The $695-$cap in $PG(5,8)$ possesses a tangent hyperplane. Application of Theorem 4 produces a cap of size $695 \cdot 65 - 1 = 45,174$ in $PG(8,8)$.

A $2056-$cap in $PG(8,4)$ and a $21,399-$cap in $PG(8,7)$ are obtained via Theorem 4. In both cases the ovoid is one of the ingredients. In the quaternary case the second ingredient is a 121-cap in $PG(5,4)$ contained in the Glynn cap, in the other case we use a $428-$cap in $PG(5,7)$, which is contained in the $434-$cap from Subsection 3.1. It is in fact easy to see that there are hyperplanes intersecting the Glynn cap in 6 points, for example the hyperplane $x_1 = 0$ in the terminology of Section 3. The $21,399-$cap in $PG(8,7)$ was embedded in a computer-generated complete $21,555-$cap. The $2056-$cap in $PG(8,4)$ can be embedded in a complete $2110-$cap.

The sizes of quaternary caps derived from the theory in higher dimensions are 4926 in $PG(9,4)$, $15,126$ in $PG(10,4)$ and $34,566$ in $PG(11,4)$. The computer was used to generate complete extensions in the former two cases.

An application of Theorem 3 to the 186-cap in $PG(5,5)$ yields a 4651-cap in $PG(8,5)$, which can be completed to a $4700-$cap. A $16,900$-cap in

Table 4: Lower bounds on $m_2(k,q)$

$k\backslash q$	3	4	5	7	8	9
2	4^c	6^c	6^c	8^c	10^c	10^c
3	10^c	17^c	26^c	50^c	65^c	82^c
4	20^c	41^c	66^c	132^c	208^c	212^c
5	56^c	126^c	186^c	434^c	695^c	840^c
6	112^c	288^c	675^c	2499^c	4224^c	6723^c
7	248^c	756^c	1715^c	6472^c	13520^c	17220^c
8	532^c	2110^c	4700^c	21555^c	45174	68070
9	1120^c	4938^c	17124^c	122500	270400	544644
10	2744^c	15423^c	43876	323318	878800	1411830
11	6272^c	34566	120740	1067080	2812160	5580100

$PG(9,5)$ belongs to a family constructed in [3], see the last part of Section 1. We generated an extension by 224 points.

The product of the $132-$cap in $PG(4,7)$ and the affine part of the ovoid yields a $6468-$cap in $PG(7,7)$, which possesses a completion to a 6472-cap. The caps in $PG(7,9)$ and in $PG(10,9)$ are obtained from an application of the product construction, using a $210-$cap in $AG(4,9)$, which was found by a computer program. The complete 840-cap in $PG(5,9)$ (see [3]) has a hyperplane intersection of 10. Applications of Theorem 1 yield an $68,070-$cap in $PG(8,9)$ and a $5,580,100$-cap in $PG(11,9)$.

Finally we mention some more computer-generated caps in dimension 4 over larger fields:

$$m_2(4,11) \geq 316, \ m_2(4,13) \geq 388, \ m_2(4,16) \geq 545, \ m_2(4,32) \geq 3136.$$

References

[1] Y.Edel and J.Bierbrauer: *A family of caps in projective 4-space in characteristic 2, Congressus Numerantium,* to appear.

[2] J.Bierbrauer and Y.Edel: *A family of caps in projective 4-space in odd characteristic, Finite Fields and their Applications,* to appear.

[3] Y.Edel and J.Bierbrauer: *Recursive constructions for large caps, Bulletin of the Belgian Mathematical Society - Simon Stevin* **6**(1999), 249-258.

[4] Y.Edel and J.Bierbrauer: *41 is the largest size of a cap in* $PG(4,4)$, *Designs, Codes and Cryptography* **59** (1999),151-160.

[5] R.D.Baker, A.Bonisoli, A.Cossidente and G.L.Ebert: *Mixed partitions of* $PG(5,q)$*, Proceedings of Combinatorics 96, Assisi,* to appear.

[6] A.Bonisoli and A.Cossidente: *Inscribed bundles, Veronese surfaces and caps,* in *J.W.P.Hirschfeld et. al. (ed): Geometry, combinatorial designs and related structures, Proceedings of the First Pythagorean Conference, Island of Spetses, Greece, June 1-7,1996. Cambridge University Press, London Mathematical Society Lecture Notes Series* **245**(1997),27-32.

[7] A.R. Calderbank and P.C. Fishburn: *Maximal three-independent subsets of* $\{0,1,2\}^n$, *Designs, Codes and Cryptography* **4** (1994),203-211.

[8] P.J.Cameron and J.H. van Lint: *Designs, Graphs, Codes and their Links,* London Mathematical Society Student Texts 22, Cambridge 1991.

[9] David Glynn: *A 126-cap of* $PG(5,4)$ *and its corresponding [126,6,88]-code, Utilitas Mathematica,* to appear.

[10] R.Hill: *On the largest size of cap in* $S_{5,3}$, *Atti Accad. Naz. Lincei Rendiconti* **54**(1973),378-384.

[11] J.W.P. Hirschfeld and L.Storme: *The packing problem in statistics, coding theory and finite projective spaces, Journal of Statistical Planning and Inference* **72** (1998),355-380.

[12] A.C. Mukhopadhyay: *Lower bounds on* $m_t(r,s)$, *Journal of Combinatorial Theory A* **25**(1978),1-13.

[13] G.Pellegrino: *Sul massimo ordine delle calotte in* $S_{4,3}$, *Matematiche (Catania)***25**(1970),1-9.

[14] B. Segre: *Le geometrie di Galois, Ann.Mat.Pura Appl.***48** (1959),1-97.

[15] B. Segre: *On complete caps and ovaloids in three-dimensional Galois spaces of characteristic two, Acta Arithmetica* **5**(1959),315-332.

[16] Yves Edel's homepage: http://www.mathi.uni-heidelberg.de/∼yves