FINITE ELEMENT METHODS FOR MAXWELL'S TRANSMISSION EIGENVALUES *

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Abstract. The transmission eigenvalue problem plays a critical role in the theory of qualitative methods for inhomogeneous media in inverse scattering theory. Efficient computational tools for transmission eigenvalues are needed to motivate improvements to theory, and, more importantly as part of inverse algorithms for estimating material properties. In this paper, we propose two finite element methods to compute a few lowest Maxwell's transmission eigenvalues which are of interest in applications. Since the discrete matrix eigenvalue problem is large, sparse, and, in particular, non-Hermitian due to the fact that the problem is neither elliptic nor self-adjoint, we devise an adaptive method which combines the Arnoldi iteration and estimation of transmission eigenvalues. Exact transmission eigenvalues for balls are derived and used as a benchmark. Numerical examples are provided to show the viability of the proposed methods and to test the accuracy of recently derived inequalities for transmission eigenvalues.

1. Introduction. The interior transmission problem (ITP) has attracted a lot of attention recently in the inverse scattering community due to its importance in the study of direct and inverse scattering problem for non-absorbing inhomogeneous media [10, 13, 22, 6, 18, 7, 11]. The ITP is a boundary value problem in a bounded domain which is neither elliptic nor self-adjoint. It is known that the associated spectrum, the so called transmission eigenvalues, closely relates to the denseness of the far field operator and plays a critical role in qualitative methods in inverse scattering, e.g., the linear sampling method and factorization method [3, 17]. Furthermore, transmission eigenvalues can be obtained from either far field or near field data and used to estimate the index of refraction of an inhomogeneous medium [4, 25].

The transmission eigenvalue problem is not directly covered by standard partial differential equation theory. Until recently, little was known about the existence of transmission eigenvalues except in the case of a spherically stratified medium [10]. In [22], Päivärinta and Sylvester proved the existence of (real) transmission eigenvalues in the general case. Cakoni et al. [5] showed that transmission eigenvalues form a countable set with infinity as the only accumulation points. For more results on the existence of transmission eigenvalues, we refer the readers to [23, 5, 18, 7, 6] and the references therein. Note that the existence of complex transmission eigenvalues is suggested by numerical examples [11] and can be proved for spherically stratified media under certain conditions [2]. A region in the complex plane where complex eigenvalues must lie was derived in [2] for scalar problems and later in [27] for vector problems. However, for general cases, the existence of eigenvalues with non-zero imaginary part is an open problem.

The need for efficient computational tools is of great importance because theoretical results are still partial and numerical evidence might lead theorists in the right direction. Moreover, numerical methods for computing transmission eigenvalues are needed in some algorithms to estimate the index of refraction [4, 25]. There are only a few papers dealing with the numerical computation of transmission eigenvalues. In

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[11], Colton et al. proposed three finite element methods for the computation of the transmission eigenvalues for the Helmholtz equation. However, the mesh has to be kept rather coarse since the final matrix problem is non-Hermitian and the numerical eigenvalues are computed by an eigensolver that estimates all the eigenvalues, which is very expensive. Moreover, no convergence result is proved therein. Based on one of the existence proofs, in [26], Sun proposed two iterative methods by solving a series of fourth order problems. A convergence result was also provided. A coupled finite-element method and boundary element method was also proposed by Hsiao et al. [15] to treat inhomogeneous interior transmission problem for anisotropic media. Note that all the above works deal with the scalar case, i.e., the Helmholtz equation.

The goal of this paper is to develop efficient finite element methods for transmission eigenvalues for the vector case, i.e. Maxwell's equations. In particular, we propose two finite element methods. One is a curl conforming finite element method based on a formulation first given by Kirsch [18] (see also [11] for a similar derivation). The second is a mixed finite element method for the fourth order reformulation of the transmission eigenvalue problem. Both methods enjoy simplicity and easy implementation by using the edge elements. We show that the methods are strongly related. The resulting non-Hermitian matrix eigenvalue problem is then computed by an adaptive Arnoldi method.

The rest of the paper is organized as follows. In Section 2, we introduce the transmission eigenvalue problem for the Maxwell's equations and recall some existence results. Then we derive the Maxwell's transmission eigenvalues for balls with constant index of refraction in Section 3. This serves as a benchmark problem and is used to verify the computational results in Section 6. In Section 4, we propose two finite element methods, i.e., the curl conforming finite element method and the mixed finite element method. To solve the fully discretized matrix eigenvalue problem, we devise an adaptive Arnoldi method in Section 5. The method employs a Faber-Krahn type inequality for the Maxwell's transmission eigenvalues and adaptively updates search intervals. Various numerical examples are provided to show the effectiveness of the proposed methods and test the tightness of several inequalities for transmission eigenvalues in Section 6. Finally, in Section 7, we draw some conclusions and discuss future work.

2. Maxwell's transmission eigenvalue problem.

Let $D \subset \mathbb{R}^3$ be a bounded connected region with piece-wise smooth boundary ∂D. Let ν be the unit outward normal to ∂D. The Hilbert space $H(\text{curl}, D)$ is defined as

$$H(\text{curl}, D) = \{u \in (L^2(D))^3 : \text{curl } u \in (L^2(D))^3\}$$

equipped with the scalar product

$$(u, v)_{\text{curl}} = (u, v) + (\text{curl } u, \text{curl } v)$$

where $(\cdot, \cdot)$ is the L^2 inner product on D. We also define

$$\begin{aligned} H_0(\text{curl}, D) &= \{u \in H(\text{curl}, D) : \nu \times u = 0 \text{ on } \partial D\}, \\ \mathcal{U}(D) &= \{u \in H(\text{curl}, D) : \text{curl } u \in H(\text{curl}, D)\}, \\ \mathcal{U}_0(D) &= \{u \in H_0(\text{curl}, D) : \text{curl } u \in H_0(\text{curl}, D)\}. \end{aligned}$$

Let N be a 3×3 matrix valued function defined on D such that $N \in L^\infty(D, \mathbb{R}^{3\times3})$.

DEFINITION 2.1. *A real matrix field N is said to be bounded positive definite on D if $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ and there exists a constant $\gamma > 0$ such that*

$$\bar{\xi} \cdot N\xi \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{C}^3, \qquad a.e.\ in\ D.$$

We assume that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix fields on D as in [6].

Let the known time-harmonic electromagnetic incident plane wave be given by

$$E^i(x,d,p) = \frac{i}{k}\operatorname{curl}\operatorname{curl} p\, e^{ikx\cdot d}, \quad H^i(x,d,p) = \operatorname{curl} p\, e^{ikx\cdot d}$$

where $d \in \mathbb{R}^3$ is a unit vector giving the direction of propagation of the wave, and the vector p is called the polarization. Then the scattering by the anisotropic medium leads to the following problem for the interior electric and magnetic fields E, H and the scattered electric and magnetic field E^s, H^s satisfying [3]

$$\operatorname{curl} E^s - ikH^s = 0, \qquad \text{in } \mathbb{R}^3 \setminus D, \tag{2.1a}$$

$$\operatorname{curl} H^s + ikE^s = 0, \qquad \text{in } \mathbb{R}^3 \setminus D, \tag{2.1b}$$

$$\operatorname{curl} E - ikH = 0, \qquad \text{in } D, \tag{2.1c}$$

$$\operatorname{curl} H + ikN(x)H = 0, \qquad \text{in } D, \tag{2.1d}$$

$$\nu \times (E^s + E^i) - \nu \times E = 0, \qquad \text{on } \partial D, \tag{2.1e}$$

$$\nu \times (H^s + H^i) - \nu \times H = 0, \qquad \text{on } \partial D, \tag{2.1f}$$

and the Silver-Müller radiation condition

$$\lim_{r\to\infty} (H^s \times x - rE^s) = 0 \tag{2.2}$$

where $r = |x|$ and k is the wave number. Under suitable conditions on N and D, the well-posedness of the above problem is well-known (Theorem 4.2 of [3]) and the scattered fields have the following asymptotic behavior

$$E^s(x,d,p) = \frac{e^{ikr}}{r} E_\infty(\hat{x},d,p) + O\left(\frac{1}{r^2}\right), \qquad r \to \infty, \tag{2.3a}$$

$$H^s(x,d,p) = \frac{e^{ikr}}{r} \hat{x} \times E_\infty(\hat{x},d,p) + O\left(\frac{1}{r^2}\right), \qquad r \to \infty, \tag{2.3b}$$

where $\hat{x} = x/r$ and E_∞ is the electric far field pattern [9]. Given E_∞, one can define the far field operator $F : L^2_t(\Omega) \to L^2_t(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, \mathrm{d}s \tag{2.4}$$

where $\Omega = \{\hat{x} \in \mathbb{R}^3; |\hat{x}| = 1\}$ and $L^2_t(\Omega) := \left\{\mathbf{u} \in (L^2(\Omega))^3 : \nu \cdot \mathbf{u} = 0 \quad \text{on} \quad \Omega\right\}$. The far field operator F has fundamental importance in the study of qualitative methods, for example, the linear sampling method (see Section 3.3 of [3]). For the case of anisotropic media, F has dense range provided k is not a transmission eigenvalue which we define next. We refer the readers to [3, 10, 9] for the mathematical derivation and interpretation of the problem.

In terms of electric fields, the transmission eigenvalue problem for the anisotropic Maxwell's equations can be formulated as the following (see [18]).

DEFINITION 2.2. *A value of $k^2 \neq 0$ is called a transmission eigenvalue if there exist real-valued fields $E, E_0 \in (L^2(D))^3$ with $E - E_0 \in \mathcal{U}_0(D)$ such that*

$$
\begin{aligned}
&\textit{curl curl } E - k^2 N E = 0, && \textit{in } D, && (2.5a)\\
&\textit{curl curl } E_0 - k^2 E_0 = 0, && \textit{in } D, && (2.5b)\\
&\nu \times E = \nu \times E_0, && \textit{on } \partial D, && (2.5c)\\
&\nu \times \textit{curl } E = \nu \times \textit{curl } E_0, && \textit{on } \partial D. && (2.5d)
\end{aligned}
$$

The above problem can be rewritten into a fourth order formulation. Following [13], we let $u = E - E_0$ and $v = NE - E_0$. Then we have that

$$E = (N - I)^{-1}(v - u), \quad E_0 = (I - N)^{-1}(Nu - v).$$

Subtracting (2.5b) from (2.5a), we obtain

$$\text{curl curl } u = k^2 v,$$

and therefore

$$E = (N - I)^{-1} \left(\frac{1}{k^2} \text{curl curl } u - u \right). \tag{2.6}$$

Substituting for E in (2.5a) and taking the boundary conditions (2.5c) and (2.5d) into account, we end up with a fourth order differential equation for $u \in \mathcal{U}_0(D)$ satisfying

$$(\text{curl curl } - k^2 N)(N - I)^{-1}(\text{curl curl } u - k^2 u) = 0. \tag{2.7}$$

Therefore the variational formulation for the transmission eigenvalue problem can be stated as: finding $k^2 \neq 0$ and $u \in \mathcal{U}_0(D)$ such that

$$\big((N - I)^{-1} \left(\text{curl curl } - k^2 I\right) u, (\text{curl curl } - k^2 N)\phi\big) = 0, \quad \text{for all } \phi \in \mathcal{U}_0(D). \tag{2.8}$$

The following theorem shows that, under certain conditions on the index of refraction, there exists an infinite countable set of transmission eigenvalues.

THEOREM 2.3. *(Theorem 2.10 of [2]) Assume that $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ satisfies either one of the following assumptions:*

$$
\begin{aligned}
&1)\, 1 + \alpha \leq n_* \leq \bar{\xi} \cdot N(x)\xi \leq n^* < \infty,\\
&2)\, 0 < n_* \leq \bar{\xi} \cdot N(x)\xi \leq n^* < 1 - \beta,
\end{aligned}
$$

for every $\xi \in \mathbb{C}^3$ such that $\|\xi\| = 1$ and some constants $\alpha, \beta > 0$. Then there exists an infinite countable set of transmission eigenvalues with $+\infty$ as the only accumulation point.

Next we recall theorems which provide lower and upper bounds for transmission eigenvalues. We will test the efficiency of these inequalities later.

THEOREM 2.4. *(Theorem 4.33 in [3]) Let $k_{1,D,N(x)}$ be the first transmission eigenvalue and let α and β be positive constants. Denote by k_{1,D,n_*} and k_{1,D,n^*} the first transmission eigenvalue for $N = n_* I$ and $N = n^* I$, respectively.*

1. *If* $\|N(x)\|_2 \geq \alpha > 1$, *then* $0 < k_{1,D,n^*} \leq k_{1,D,N(x)} \leq k_{1,D,n_*}$ *for all* $x \in D$.
2. *If* $0 < \|N(x)\|_2 \leq 1-\beta$, *then* $0 < k_{1,D,n_*} \leq k_{1,D,N(x)} \leq k_{1,D,n^*}$ *for all* $x \in D$.

The bounds in terms of transmission eigenvalues on balls are obtained in [5]. Let B_{r_1} be the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} the smallest ball of radius r_2 such that $D \subset B_{r_2}$. We denote by k_{1,n_*} and k_{1,n^*} the first transmission eigenvalue for the unit ball with index of refraction n_* and n^*, respectively. For a given $0 < \epsilon \leq r_1$ let $m(\epsilon) \in \mathbb{N}$ be the number of balls B_ϵ of radius ϵ that are contained in D. Then we have the following estimate.

THEOREM 2.5. *(Corollary 2.11 in [5]) Assume that* $N \in L^\infty(D, \mathbb{R}^{d\times d})$, $d = 2, 3$, *and let* $k_{1,D,N(x)}$ *be the first transmission eigenvalue.*

1. *If* $1 + \alpha \leq n_* \leq \bar{\xi} \cdot N(x)\xi \leq n^* < \infty$ *for every* $\xi \in \mathbb{C}^d$ *such that* $\|\xi\| = 1$, *and some constant* $\alpha > 0$, *then*
$$0 < \frac{k_{1,n^*}}{r_2} \leq k_{1,D,N(x)} \leq \frac{k_{1,n_*}}{r_1}. \tag{2.9}$$
Furthermore, there exist at lease $m(\epsilon)$ *transmission eigenvalues in the interval* $[k_{1,n^*}/r_2, k_{1,n_*}/\epsilon]$.
2. *If* $0 < n_* \leq \bar{\xi} \cdot N(x)\xi \leq n^* < 1 - \beta$ *for every* $\xi \in \mathbb{C}^d$ *such that* $\|\xi\| = 1$ *and some constant* $\beta > 0$, *then*
$$0 < \frac{k_{1,n_*}}{r_2} \leq k_{1,D,N(x)} \leq \frac{k_{1,n^*}}{r_1}. \tag{2.10}$$
Furthermore, there exist at lease $m(\epsilon)$ *transmission eigenvalues in the interval* $[k_{1,n_*}/r_2, k_{1,n^*}/\epsilon]$.

Note that the above two theorems hold for arbitrarily small positive numbers α and β which is a rather mild requirement on the index of refraction. We refer the readers to [2, 7] for more results on the existence of the transmission eigenvalues. The task of the current paper is to develop efficient numerical methods for the computation of a few lowest non-zero Maxwell's transmission eigenvalues.

3. Transmission eigenvalues on balls. In this section we derive the transmission eigenvalues on balls with constant index of refraction. The eigenvalue problem on a ball has theoretical importance (see Theorem 2.5) and will also serve as a benchmark problem in Section 6. We assume that the index of refraction $N = N_0 I$ where N_0 is a scalar constant. Let $u = j_n(k\rho)Y_n^m(\hat{x})$ and $v = j_n(k\sqrt{N_0}\rho)Y_n^m(\hat{x})$ where j_n is the spherical Bessel's function of order n and Y_n^m is the spherical harmonic (see, e.g., [9]). Here $\hat{x} = x/|x|$ and $\rho = |x|$. Note that u and v are solutions of the Helmholtz equation (see p. 235 of [20]). Then the following are solutions to the Maxwell's equations (2.5a) and (2.5b), respectively,

$$\tilde{M}_u = \text{curl }\{xu\}, \quad \tilde{N}_u = \frac{1}{ik}\text{curl }\{\tilde{M}_u\}, \quad n > 1,$$
$$\tilde{M}_v = \text{curl }\{xv\}, \quad \tilde{N}_v = \frac{1}{ik}\text{curl }\{\tilde{M}_v\}, \quad n > 1.$$

Using the curl in spherical coordinates (ρ, θ, ϕ), we have solutions for Maxwell's equations of TE modes

$$\begin{aligned}\tilde{M}_u &= -\frac{\partial u}{\partial \theta}e_\phi + \frac{1}{\sin\theta}\frac{\partial u}{\partial \phi}e_\theta \\ &= -j_n(k\rho)\frac{\partial Y_n^m(\hat{x})}{\partial \theta}e_\phi + \frac{1}{\sin\theta}j_n(k\rho)\frac{\partial Y_n^m(\hat{\mathbf{x}})}{\partial \phi}e_\theta.\end{aligned}$$

Taking the curl of the above equation and dropping the constant $1/ik$, we have solutions of TM modes

$$
\begin{aligned}
(\text{curl } \tilde{M}_u)_\rho &= \frac{1}{\rho \sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\sin\theta\left[-j_n(k\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\theta}\right]\right) - \frac{1}{\sin\theta} j_n(k\rho)\frac{\partial^2 Y_n^m(\hat{x})}{\partial\phi^2}\right\} \\
&= \frac{j_n(k\rho)}{\rho\sin\theta}\left\{-\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_n^m(\hat{\mathbf{x}})}{\partial\theta}\right) - \frac{1}{\sin\theta}\frac{\partial^2 Y_n^m(\hat{x})}{\partial\phi^2}\right\}, \\
(\text{curl } \tilde{M}_u)_\theta &= \frac{1}{\rho}\left\{-\frac{\partial}{\partial\rho}\left(-\rho j_n(k\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\theta}\right)\right\} \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\rho)\right)\frac{\partial Y_n^m(\hat{x})}{\partial\theta}, \\
(\text{curl } \tilde{M}_u)_\phi &= \frac{1}{\rho\sin\theta}\frac{\partial}{\partial\rho}\left(\rho j_n(k\rho)\right)\frac{\partial Y_n^m(\hat{x})}{\partial\phi}.
\end{aligned}
$$

Note that similar results hold for $\tilde{M}_v$. For TE mode solutions, in order to satisfy the boundary conditions (2.5c) and (2.5d), the wave number k^2's need to satisfy

$$
\begin{vmatrix} j_n(k\rho) & j_n(k\sqrt{N_0}\rho) \\ \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\rho)\right) & \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\sqrt{N_0}\rho)\right) \end{vmatrix} = 0, \quad n \geq 1. \tag{3.1}
$$

The zeros of (3.1) give the first group, i.e., TE modes, of the Maxwell's transmission eigenvalues. We refer the readers to Example 3.2 of [7] for a detailed derivation of (3.1).

Next we consider the TM modes by letting $E_u = \nabla \times \tilde{M}_u$ and $E_v = \nabla \times \tilde{M}_v$. Simple calculation shows that

$$
\begin{aligned}
(\text{curl } E_u)_\rho &= 0, \\
(\text{curl } E_u)_\theta &= \frac{k^2}{\sin\theta} j_n(k\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\phi}, \\
(\text{curl } E_u)_\phi &= k^2 j_n(k\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\theta},
\end{aligned}
$$

and

$$
\begin{aligned}
(\text{curl } E_v)_\rho &= 0, \\
(\text{curl } E_v)_\theta &= \frac{k^2 N_0}{\sin\theta} j_n(k\sqrt{N_0}\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\phi}, \\
(\nabla \times E_v)_\phi &= k^2 N_0 j_n(k\sqrt{N_0}\rho)\frac{\partial Y_n^m(\hat{x})}{\partial\theta}.
\end{aligned}
$$

Similar to the TE modes, the transmission eigenvalues for TM modes are k^2's such that

$$
\begin{vmatrix} \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\rho)\right) & \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho j_n(k\sqrt{N_0}\rho)\right) \\ k^2 j_n(k\rho) & k^2 N_0 j_n(k\sqrt{N_0}\rho) \end{vmatrix} = 0, \quad n \geq 1. \tag{3.2}
$$

This set of transmission eigenvalues gives the second group of the Maxwell's transmission eigenvalues. Note that the multiplicities of the transmission eigenvalues for TE and TM modes are $3, 5, 7, \ldots$, which correspond to the number of spherical harmonics of order $n = 1, 2, 3, \ldots$

Table 3.1

Maxwell transmission eigenvalues (real) for the unit ball with $N = 16I$ determined by locating the zeros of the determinants in (3.1) and (3.2).

i	Transmission eigenvalue (k^2)	Type	Multiplicity
1	1.1654	TM	3
2	1.4608	TE	3
3	1.4751	TM	5
4	1.7640	TE	5
5	1.7775	TM	7
6	2.0611	TE	7

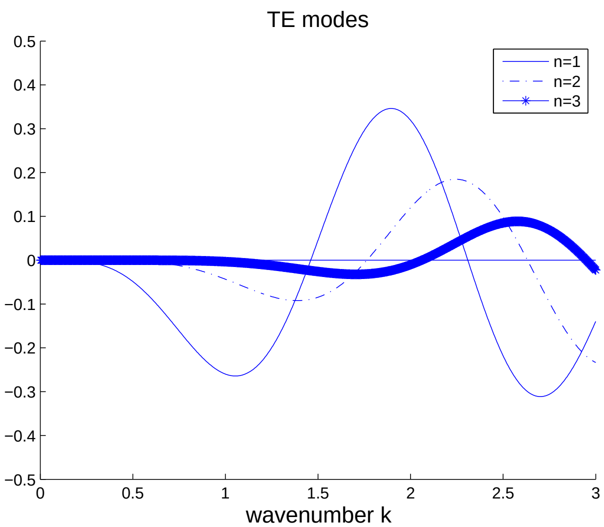

Fig. 3.1. *Graphs of the determinant in (3.1) as a function of wave number k for $n = 1, 2, 3$. Zeros of the determinants are transmission eigenvalues for the unit ball with $N_0 = 16$ (TE modes).*

Let D be the unit ball and set $N_0 = 16$. In Fig. 3.1 and Fig. 3.2 we show the plots of the determinants corresponding to TE and TM modes, respectively. By searching for the zeros of the determinants in (3.1) and (3.2), we obtain the Maxwell's transmission eigenvalues for the unit ball which are shown in Table 3.1.

Note that the smallest transmission eigenvalue belongs to the TM modes. This is similar to the standard Maxwell's eigenvalue problem. The smallest Maxwell's eigenvalue for the unit ball belongs to the TM mode [1].

One can also search in the complex plane of zeros of the determinants defined in (3.1) and (3.2). In Fig. 3.3, we plot the absolute values of the two determinants for the first TE and TM modes. The zeros on the real axis coincide with the values in Table.3.1. In addition, the plots also indicate the likely existence of complex Maxwell's transmission eigenvalues.

4. FEMs for Maxwell's transmission eigenvalues. Now we propose two finite element methods to compute transmission eigenvalues. The first one is a curl conforming finite element method based on the equations (2.5a)-(2.5d) directly (see also [18, 11]). The second one is a mixed finite element method for the fourth order

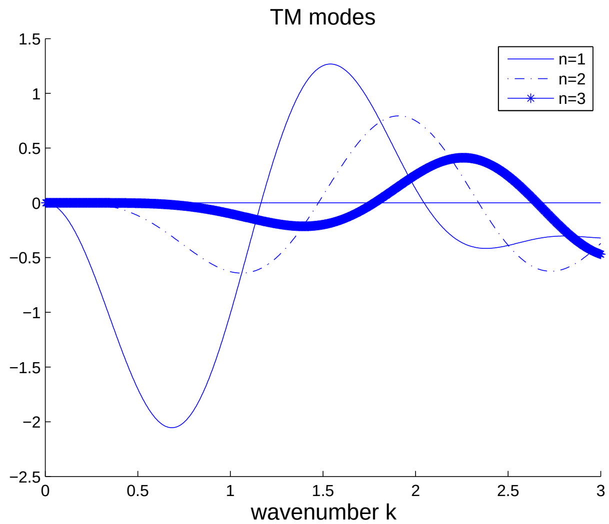

FIG. 3.2. *Graphs of the the determinant in (3.2) as a function of wave number k for $n = 1, 2, 3$. Zeros of the determinants are transmission eigenvalues for the unit ball with $N_0 = 16$ (TM modes).*

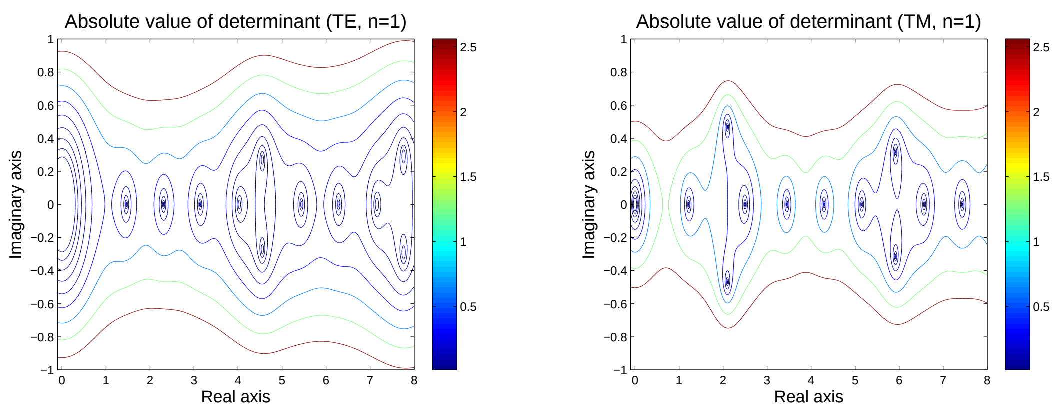

FIG. 3.3. *Contour plots of absolute values of the determinants for the first modes. The centers of the circles are the locations of transmission eigenvalues. We see that the plots also indicate the likely existence of complex Maxwell's transmission eigenvalues. Left: TE mode. Right: TM mode.*

formulation. We do not yet have theoretical convergence results.

4.1. A curl conforming finite element method. Multiplying by suitable test functions and integrating by parts, a variational formulation of (2.5a)-(2.5d) can be stated as follows (see Definition 4.1 in [18]). Find $k^2 \neq 0$, $E_0 \in H(\text{curl}, D)$ satisfying

$$(\text{curl } E_0, \text{curl } \phi) - k^2(E_0, \phi) = 0, \quad \text{for all } \phi \in H_0(\text{curl}, D), \tag{4.1}$$

and $E \in H(\text{curl}, D)$ satisfying

$$(\text{curl } E, \text{curl } \gamma) - k^2(NE, \gamma) = (\text{curl } E_0, \text{curl } \gamma) - k^2(E_0, \gamma), \tag{4.2}$$

for all $\gamma \in H(\text{curl}, D)$ together with the essential boundary condition $E = E_0$ on ∂D. In (4.2) we have enforced the boundary condition (2.5d) weakly.

The curl conforming finite element method is based on the above formulation. Let $\mathcal{T}$ be a regular tetrahedral mesh for D.

Then S_h denote the lowest-order curl conforming element space of Nédélec [21, 20] which is sometimes called the Whitney element [28]. To describe the S_h, we introduce the following polynomial space on an element K is given by

$$R_1 = \{u(x) = a + b \times x, \quad a, b \in \mathbb{C}^3\}.$$

The six constants in the definition of R_1 are determined from the moments $\int_e u \cdot \tau \mathrm{d}s$ on the six edges e of K, and these edge degrees of freedom ensure that the global space is curl conforming. Here τ is the unit tangential vector along the edge. Then

$$S_h = \{u \in H(\text{curl}; \Omega) \mid u|_K \in R_1 \text{ for every tetrahedron } K \in \mathcal{T}\}. \tag{4.3}$$

We define a subspace of S_h given by

$$S_h^0 = \{\xi_h \in S_h, \nu \times \xi_h = 0 \text{ on } \partial D\} \subset H_0(\text{curl}; \Omega). \tag{4.4}$$

The boundary condition $\nu \times \xi_h = 0$ can be simply satisfied by setting the degree of freedom associated to the boundary edges to zero.

Let $T = \dim S_h$, $K = \dim S_h^0$, and $P = T - K$. Let $\xi_1, \ldots, \xi_T$ be a basis for S^h and $\xi_1, \ldots, \xi_K$ be a basis for S_h^0. Thus we have $S_h = \text{span}\{\xi_j\}_{j=1}^T$ and $S_h^0 = \text{span}\{\xi_j\}_{j=1}^K$. In addition, we define $S_h^B = \text{span}\{\xi_j\}_{j=K+1}^T$.

Let w_h and v_h be the discrete approximations for E and E_0, respectively. We can write

$$\begin{aligned} w_h &= w_{0,h} + w_{B,h} \text{ where } w_{0,h} \in S_h^0 \text{ and } w_{B,h} \in S_h^B, \\ v_h &= v_{0,h} + w_{B,h} \text{ where } v_{0,h} \in S_h^0. \end{aligned}$$

First we choose a test function $\xi_h \in S_h^0$ and obtain

$$(\text{curl } (v_{0,h} + w_{B,h}), \text{curl } \xi_h) - k^2 (v_{0,h} + w_{B,h}, \xi_h) = 0, \tag{4.5}$$

for all $\xi_h \in S_h^0$. In the same way, we have

$$(\text{curl } (w_{0,h} + w_{B,h}), \text{curl } \xi_h) - k^2 (N(w_{0,h} + w_{B,h}), \xi_h) = 0, \tag{4.6}$$

for all $\xi_h \in S_h^0$. Rearranging terms in (4.2), we obtain

$$(\text{curl } (E - E_0), \text{curl } \gamma) - k^2 (NE - E_0, \gamma) = 0,$$

for all $\gamma \in H(\text{curl}, D)$. In the discrete case, for all $\gamma_h \in S_h^B$, we have

$$(\text{curl } (w_{0,h} - v_{0,h}), \text{curl } \gamma_h) - k^2 (N(w_{0,h} + w_{B,h}) - (v_{0,h} + w_{B,h}), \gamma_h) = 0. \tag{4.7}$$

Rearranging the terms in (4.5), (4.6) and (4.7), we obtain

$$\begin{aligned} (\text{curl } (v_{0,h} + w_{B,h}), \text{curl } \xi_h) &= k^2 (v_{0,h} + w_{B,h}, \xi_h), \\ (\text{curl } (w_{0,h} + w_{B,h}), \text{curl } \xi_h) &= k^2 (N(w_{0,h} + w_{B,h}), \xi_h), \\ (\text{curl } (w_{0,h} - v_{0,h}), \text{curl } \gamma_h) &= k^2 (N(w_{0,h} + w_{B,h}) - (v_{0,h} + w_{B,h}), \gamma_h), \end{aligned}$$

for all $\xi_h \in S_h^0$ and $\gamma_h \in S_h^B$.

We define the following matrices:

Matrix	Dimension	Definition
A	$K \times K$	interior space stiffness matrix, $A_{j,\ell} = (\nabla \times \xi_j, \nabla \times \xi_\ell)$
B_N, B_1	$K \times P$	boundary/interior mass matrices, $(B_N)_{j,\ell} = (N\xi_j, \xi_\ell)$, $(B_1)_{j,\ell} = (\xi_j, \gamma_\ell)$
C_N, C_1	$P \times P$	boundary space mass matrices, $(C_N)_{j,\ell} = (N\xi_j, \xi_\ell)$ $(C_1)_{j,\ell} = (\xi_j, \xi_\ell)$
D	$K \times P$	boundary/interior stiffness matrix, $D_{j,\ell} = (\nabla \times \xi_j, \nabla \times \xi_\ell)$
M_N, M_1	$K \times K$	interior space mass matrices, $(M_N)_{j,\ell} = (N\xi_j, \xi_\ell)$ $(M_1)_{j,\ell} = (\xi_j, \xi_\ell)$

Then the discrete problem we now have to solve is the generalized eigenvalue problem

$$\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x} \tag{4.8}$$

where $\vec{x}$ has dimension $2K + P$ corresponding to $w_{0,h}, v_{0,h}$ and $w_{B,h}$. The matrices $\mathcal{A}$ and $\mathcal{B}$ are given blockwise by

$$\mathcal{A} = \begin{pmatrix} A & 0 & D \\ 0 & A & D \\ D^T & -D^T & 0 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} M_N & 0 & B_N \\ 0 & M_1 & B_1 \\ B_N^T & -B_1^T & C_N - C_1 \end{pmatrix}.$$

While it is possible to change variables to make $\mathcal{A}$ and $\mathcal{B}$ symmetric, neither would be positive definite. So (4.8) is not a standard generalized eigenproblem.

4.2. A mixed finite element method. The second method is based on a mixed formulation for the fourth order problem (2.7) which is similar to the mixed finite element approach for the biharmonic equation [8, 19]. Recalling that $u = E - E_0$, we showed in (2.6) that $E = (N - I)^{-1}(\text{curl curl} - k^2 I)u$. Hence we have that

$$(\text{curl curl } - k^2 N)E = 0,$$
$$(\text{curl curl } - k^2 I)u = (N - I)E.$$

The mixed formulation can be stated as: find $(k^2, u, E) \in \mathbb{C} \times H_0(\text{curl}, D) \times H(\text{curl}, D)$ such that

$$(\text{curl } E, \text{curl } \phi) = k^2(NE, \phi), \quad \forall \phi \in H_0(\text{curl}, D),$$
$$(\text{curl } u, \text{curl } \varphi) - ((N - I)E, \varphi) = k^2(u, \varphi), \quad \forall \varphi \in H(\text{curl}, D).$$

Given finite dimensional spaces $S_h \subset H(\text{curl}, D)$ and $S_h^0 \subset H_0(\text{curl}, D)$ such that $S_h^0 \subset S_h$, the discrete problem is to find $(k_h^2, u_h, E_h) \in \mathbb{C} \times S_h^0 \times S_h$ such that

$$(\text{curl } E_h, \text{curl } \phi_h) = k_h^2(NE_h, \phi_h), \quad \forall \phi_h \in S_h^0,$$
$$(\text{curl } u_h, \text{curl } \varphi_h) - ((N - I)E_h, \varphi_h) = k_h^2(u_h, \varphi_h), \quad \forall \varphi \in S_h.$$

In the numerical tests, we again use the lowest order curl conforming edge elements of Nédélec, i.e., (4.3) and (4.4). Let $\psi_1, \ldots, \psi_K$ be a basis for S_h^0 and

$\psi_1, \ldots, \psi_K, \psi_{K+1}, \ldots, \psi_T$ be a basis for S_h. Let $u_h = \sum_{i=1}^K u_i\psi_i$ and $E_h = \sum_{i=1}^T E_i\psi_i$. Then the corresponding matrix problem is

$$S_{K\times T}E_h = k_h^2 M^N_{K\times T}E_h,$$
$$S_{T\times K}u_h - M^{N-I}_{T\times T}E_h = k_h^2 M_{T\times K}u_h,$$

where

Matrix	Dimension	Definition
$S_{K\times T}$	$K \times T$	stiffness matrix $S^{i,j}_{K\times T} = (\nabla\times\psi_i, \nabla\times\psi_j)$
$S_{T\times T}$	$T \times T$	stiffness matrix $S^{i,j}_{T\times T} = (\nabla\times\psi_i, \nabla\times\psi_j)$
$M_{K\times T}$	$K \times T$	mass matrix $M^{i,j}_{K\times T} = (\psi_i, \psi_j)$
$M^N_{T\times K}$	$T \times K$	mass matrix $(M^N_{T\times K})^{i,j} = (N\psi_i, \psi_j)$
$M^{N-I}_{T\times T}$	$T \times T$	mass matrix $(M^{N-I}_{T\times T})^{i,j} = ((N-I)\psi_i, \psi_j)$

We end up with the generalized eigenvalue problem

$$\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x} \tag{4.9}$$

where $\vec{x} = (E_h, u_h)^T$ and the matrices $\mathcal{A}$ and $\mathcal{B}$ are given blockwise by

$$\mathcal{A} = \begin{pmatrix} S_{K\times T} & 0_{K\times K} \\ -M^{N-I}_{T\times T} & S_{T\times K} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} M^N_{K\times T} & 0_{K\times K} \\ 0_{T\times T} & M_{T\times K} \end{pmatrix}.$$

At the continuous level the "biharmonic" problem (2.8) provides a weak form of the transmission eigenvalue problem that exactly respects the regularity requirements of the definition of the transmission eigenvalues. If we assume that E and E_0 are in $H(\text{curl}, D)$, then at the continuous level the curl conforming method and the mixed method we have outlined have, of course, the same spectrum in $(0, \infty)$. This equivalence carries over to the discrete problems (one discrete system can easily be derived from the other). As for the Maxwell's eigenvalue problem, $k_h = 0$ is an eigenvalue of large multiplicity for the discrete problem (see Section 4.7 of [20]). These eigenvalues are not physically relevant and should be excluded. Experimentally we find that the eigenspaces for $k_h = 0$ and $k_h = \infty$ differ between the two finite element methods.

The mixed method is easier to describe and implement since we have no need to impose the essential boundary condition on the difference of two fields. Both finite element methods have the advantage of using the standard linear edge elements which are easy to implement. We find that the curl conforming method performs slightly better in the Arnoldi process described in the next section, but this observation does not yet have any theoretical underpinning.

The generalized eigenvalue problem is non-Hermitian and the associated matrices are large and sparse. Direct methods are expensive even on a coarse mesh for two dimensional problems [11]. Therefore efficient computation of a few lowest transmission eigenvalues, which are important in algorithms to estimate material property in inverse scattering [4, 25], is a challenging problem.

5. An adaptive Arnoldi method. In this section, we will apply an adaptive technique based on the Arnoldi method [12, 24] for the generalized eigenvalue problem obtained in the last section which is large, sparse and non-Hermitian. In particular, we employ the Matlab Arnoldi solver 'sptarn' which can be integrated into our finite element code easily. 'sptarn' uses Arnoldi iteration with spectral transformation and requires an interval in which to search for the eigenvalues. On the one hand, this is a rather appealing feature since we only need a few lowest transmission eigenvalues. Moreover, it avoids computing the smallest eigenvalue of the generalized systems (4.8) and (4.9) corresponding to the non-physical case of $k = 0$. On the other hand, the interval needs to be kept rather small in order to guarantee efficiency. Otherwise, 'sptarn' will not return within reasonable amount of time. Fortunately we are able to overcome this difficulty by coupling an iterative scheme with an estimation of the transmission eigenvalues. We refer the readers to [16] for a similar treatment for Helmholtz transmission eigenvalue problem.

To this end we first recall the Faber-Krahn type inequality from [3].

THEOREM 5.1. *(Theorem 4.29 of [3])*

1. *Assume that the imaginary part $\mathcal{I}(N(x)) = 0$ and $\|N(x)\|_2 \geq \delta > 1$ for all $x \in D$ and some constant δ. Then,*

$$\sup_D \|N\|_2 \geq \frac{\lambda_1(D)}{k^2}, \tag{5.1}$$

where k is a transmission eigenvalue and $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

2. *Assume that the imaginary part $\mathcal{I}(N(x)) = 0$ and $0 \leq \beta \leq \|N(x)\|_2 \leq \delta < 1$ for all $x \in D$ and some constant β. Then, if k is a transmission eigenvalue,*

$$k^2 \geq \lambda_1(D), \tag{5.2}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D.

The above theorem provides a lower bound for transmission eigenvalues in terms of the first Dirichlet eigenvalue and $\sup_D \|N\|_2$. In the following we will only consider Case 1 of the above theorem, i.e., $\|N(x)\|_2 \geq \delta > 1$. The other case can be done in exactly the same way. From (5.1), we have

$$k_1^2 \geq \frac{\lambda_1(D)}{\sup_D \|N\|_2}. \tag{5.3}$$

In fact, $\sup_D \|N\|_2$ can be obtained from the given data easily and we can compute the first Dirichlet eigenvalue using standard linear finite elements. In particular, the discrete Dirichlet eigenvalue problem is simply the following generalized eigenvalue problem

$$S\vec{x} = \lambda M\vec{x} \tag{5.4}$$

where S and M are the stiffness matrix and the mass matrix, respectively.

To compute a few lowest non-zero transmission eigenvalues, we start searching the transmission eigenvalues with a small interval to the right of the lower bound given in (5.3) (or (5.2)). If we successfully find transmission eigenvalues, we stop the process. Otherwise, we shift to the right, double the size of the interval and start a new search using 'sptarn' and continue this process until we find the desired transmission eigenvalues. The size of the search interval, denoted by s, should be rather small at the beginning, say, $1.0e-3$ and we can slowly increase it. From our experience, to maintain efficiency the interval cannot be too large. This is due to the fact that if there are too many eigenvalues in the interval, the efficiency of 'sptarn' will be significantly downgraded. Note that 'sptarn' also computes complex eigenvalues. Since only real transmission eigenvalues are of interest, we simply discard the complex ones. This process implicitly assumes that the Faber-Krahn lower bound is also a lower bound for the first non-zero discrete transmission eigenvalue, a fact we have not yet verified although our numerical experiments suggest it is true.

Assuming a tetrahedral mesh $\mathcal{T}$ is already generated for D, the following adaptive algorithm computes N_e lowest transmission eigenvalues efficiently.

Algorithm for computing TEs

```
input a tetrahedral mesh for D and the initial size of the search interval s
input the index of refraction N(x) and sup_D ||N(x)||_2
input the number of transmission eigenvalues N_e to be computed
construct matrices A and B
compute λ_1(D)
set TE = ∅, lb = λ_1(D) / sup_D ||N(x)||_2, rb = lb + s
while length(TE) < N_e
    [V, D] = sptarn(A, B, lb, rb)
     delete complex values in D
    TE = TE ∪ D
    lb = rb, s = min(2s, 1), rb = lb + s
end
```

It is also possible to use the bounds in Theorem 2.5 to estimate a search interval. However, one needs to find balls inside and outside D and devise an effective way to compute transmission eigenvalues for balls with constant index of refraction. Since we use finite element to compute transmission eigenvalues, it is easier for us to compute the Dirichlet eigenvalues using the same mesh.

6. Numerical Examples. In this section we provide some numerical examples to show the viability of the proposed methods and test the efficiency of the inequalities in Section 2. We choose two domains: D_1 the unit ball centered at the origin and D_2 the unit cube given by $[0,1]\times[0,1]\times[0,1]$ (see Fig. 6.1). We only consider when $\|N(x)\|_2 \geq \alpha > 1$ since the case of $0 < \|N(x)\|_2 \leq 1-\beta$ is similar. We test three different cases for the index of refraction $N(x)$ corresponding to isotropic medium with constant index of refraction, anisotropic medium with constant index of refraction, and anisotropic medium with variable index of refraction:

$$1) \begin{pmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{pmatrix}, \; 2) \begin{pmatrix} 16 & 1 & 0 \\ 1 & 16 & 0 \\ 0 & 0 & 14 \end{pmatrix}, \; \text{and } 3) \begin{pmatrix} 16 & x & y \\ x & 16 & z \\ y & z & 14 \end{pmatrix}. \tag{6.1}$$

Note that due to the 3D nature of the problem and the desktop computer available for the numerical tests, we have to restrict the mesh size h to be larger than roughly 0.2.

6.1. Comparison of the two FEM methods. We compare the two finite element methods using the same meshes. The first example is the unit ball with index of refraction $N = 16I$. We use a mesh with mesh size $h \approx 0.4$. To make comparison, we compute the full spectrum of the generalized eigenvalue problems using Matlab's 'eig'. Both methods end up with the same degree of freedom (DoF) 2566. The curl conforming method computes 708 zero eigenvalues and the mixed method computes 228 zero eigenvalues. That the curl conforming method has a large eigenspace corresponding to $k_h = 0$ is unsurprising since $k = 0$ is a non-trivial

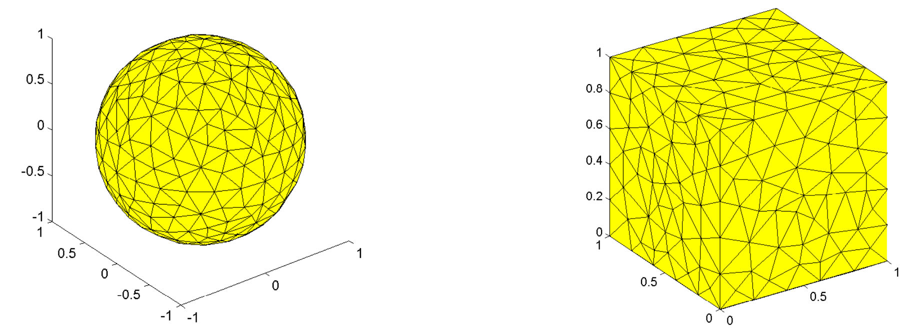

FIG. 6.1. *Two domains used for numerical examples and sample meshes. Left: the unit ball centered at the origin. Right: the unit cube given by* $[0,1]\times[0,1]\times[0,1]$.

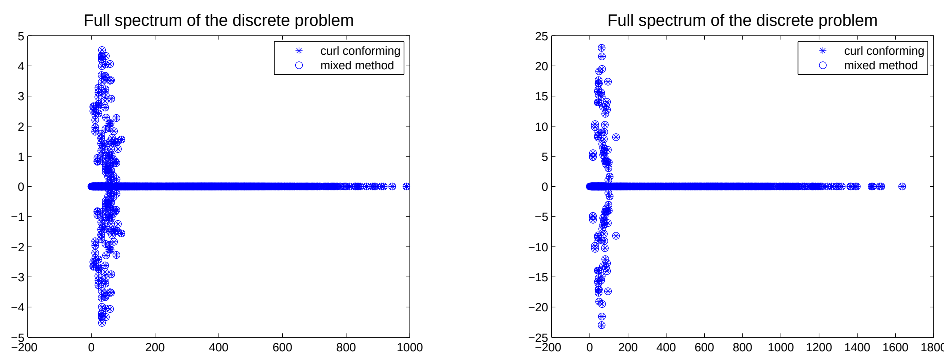

FIG. 6.2. *The full spectrum of the generalized eigenvalue problems when* $N = 16I$. *Left: the unit ball centered at the origin. Right: the unit cube given by* $[0,1]\times[0,1]\times[0,1]$.

transmission eigenvalue for (2.5) with infinite dimensional eigenspace. Unlike for the Helmholtz equation, the fourth order problem (2.7) also has $k = 0$ as an eigenvalue. The mixed method computes $k_h = 0$ as an eigenvalue, but also computes many eigenvalues $k_h = `inf'$ since $\mathcal{B}$ in (4.9) is singular. The rest of the spectrum in $(0,\infty)$ coincides (see Fig. 6.2), even for complex eigenvalues, as we claimed earlier.

If we use the Arnoldi method in the interval $[1,3]$ which contains 11 eigenvalues, the curl conforming method uses 4.38s (CPU time) which is slightly shorter than the mixed method with 4.98s (CPU time), providing a slight reason for preferring the curl conforming method.

We have repeated this experiment for the unit cube with index of refraction $N = 16I$. We use a mesh with mesh size $h \approx 0.3$. Both methods end up with the same number of DoF 1376. The curl conforming method computes 444 zero eigenvalues and mixed method computes 102 zero eigenvalues (as well as some infinite eigenvalues). The rest of the spectrum in $(0,\infty)$ also coincides (see Fig. 6.2). If we use the Arnoldi method in the interval $[3,5]$ which contains 3 eigenvalues, the curl conforming method uses 0.74s (CPU time) which is slightly shorter than the mixed method with 0.91s (CPU time). We summarize the result in Table 6.1. Note that for both examples,

Table 6.1

Comparison of the curl conforming method and the mixed method ($N = 16I$).

domain	unit ball			unit cube		
	DoF	# of zero	CPU time	DoF	# of zero	CPU time
curl conforming	2566	708	4.38s	1376	444	0.74s
mixed method	2566	228	4.98s	1376	102	0.91s

Table 6.2

Computed Maxwell's transmission eigenvalues for the unit ball with $N = 16I$. The mesh size $h \approx 0.2$. Comparing to the values in Table 3.1, the computed eigenvalues are accurate with correct multiplicities.

1.1741	1.1717	1.1721	1.4665	1.4667	1.4671
1.4824	1.4828	1.4828	1.4830	1.4836	1.7690
1.7690	1.7698	1.7700	1.7705	1.7857	1.7859
1.7862	1.7865	1.7867	1.7868	1.7872	

the number of zero eigenvalues of the mixed method is twice the number of boundary nodes. In addition, the difference between the number of zero eigenvalues computed by the two methods coincides with the number of edges on the domain boundary. Since the two methods compute the same non-zero spectrum and the curl conforming method is slightly more efficient, we will use the curl conforming method for the remainder of the paper.

6.2. The unit ball. We first show a few transmission eigenvalues for the unit ball with constant index of refraction $N = 16I$, i.e., case 1) of (6.1) (see Table 6.2) computed using $h \approx 0.2$. These values coincides rather well with the exact transmission eigenvalues shown in Table 3.1 and have correct multiplicities.

Since we have exact values for this case, we can look at the convergence rate of the lowest transmission eigenvalue. This is done by carrying out the computation on a series of meshes with decreasing mesh size h. We plot the errors against the mesh size h in log scale in Fig. 6.3 where second order convergence can be seen clearly (see Table 6.3 for the actual h and the errors).

Next we check Theorem 2.4 for case 3) of (6.1). Straightforward calculation shows that

$$n_* \approx 13.5697 \leq \hat{\xi} \cdot N(x)\xi \leq 17.0000 \approx n^*, \quad \text{for all } x \in D_1. \tag{6.2}$$

Using a mesh with mesh size $h \approx 0.2$, we find that

$$k_{1,D_1,n^*} \approx 1.1381 < k_{1,D_1,N(x)} \approx 1.1857 < k_{1,D_1,n_*} \approx 1.2877,$$

i.e., Theorem 2.4 gives a reasonable estimate for $k_{1,D_1,N(x)}$ for this example.

6.3. The unit cube. Now we consider the unit cube, i.e., $D_2 = [0,1] \times [0,1] \times [0,1]$. First, we check Theorem 2.4 for case 3) of (6.1). Again, straightforward calculation shows that

$$n_* \approx 13.2679 \leq \hat{\xi} \cdot N(x)\xi \leq 17.5616 \approx n^*, \quad \text{for all } x \in D_2. \tag{6.3}$$

Using a mesh with mesh size $h \approx 0.2$, we compute $k_{1,D_2,N(x)} \approx 2.0527$ and have that

$$k_{1,D_2,n^*} \approx 1.9920 < k_{1,D_2,N(x)} \approx 2.0527 < k_{1,D_2,n_*} \approx 2.2187.$$

Table 6.3

The errors of the lowest Maxwell's transmission eigenvalues for the unit ball with $N = 16I$. The exact value is from Table. 3.1.

mesh size	computed eigenvalue	exact eigenvalue	error
$h \approx 0.66$	1.2145	1.1654	0.0491
$h \approx 0.50$	1.1970	1.1654	0.0316
$h \approx 0.40$	1.1837	1.1654	0.0183
$h \approx 0.29$	1.1761	1.1654	0.0107
$h \approx 0.22$	1.1720	1.1654	0.0066

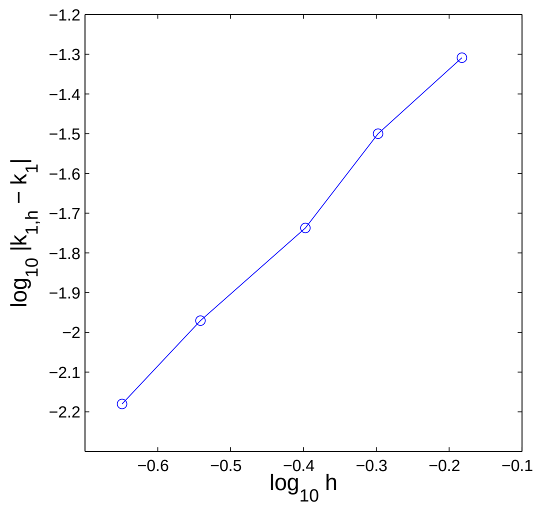

Fig. 6.3. *Convergence of the lowest transmission eigenvalues for the unit ball with $N = 16I$. Here h denotes the mesh size. Second order convergence is observed.*

Next, we check Theorem 2.5. It is obvious that the ball B_1 with radius $r_1 = 1/2$ is the largest ball such that $B_{r_1} \subset D_2$ and the ball B_2 with radius $r_2 = \sqrt{2}$ is the smallest ball such that $D_2 \subset B_{r_2}$. When the index of refraction is given by case 2) of (6.1), we have that

$$n_* = 14, \quad n^* = 17.$$

Using the the result of Section 3, we have that

$$k_{1,n^*} = 1.1277, \quad k_{1,n_*} = 1.2539.$$

The finite element method gives that $k_{1,D_2,N(x)} \approx 2.0411$ and we have that

$$\frac{k_{1,n^*}}{\sqrt{2}} \approx 0.7974 \leq k_{1,D,N(x)} \approx 2.0411 \leq \frac{k_{1,n_*}}{r_1} \approx 2.5078.$$

Now let $\epsilon = 1/4$. Then we can put $m(1/4) = 4$ balls $B_{1/4}$ with radius $1/4$ in D_2. According to Theorem 2.5, there are *at least* $m(1/4) = 4$ transmission eigenvalues in the interval

$$\left[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}\right] \approx [0.7974, 5.0156].$$

The numerical method computes 16 transmission eigenvalues in $[0.7974, 3.1623]$.

When the index of refraction is given by case 3) of (6.1), we have that

$$n_* = 13.2679, \quad n^* = 17.5616.$$

Once again using the result of Section 3, we have that

$$k_{1,n^*} = 1.1081, \quad k_{1,n_*} = 1.2918.$$

The finite element method gives $k_{1,D_2,N(x)} \approx 2.0527$ and we have that

$$\frac{k_{1,n^*}}{\sqrt{2}} \approx 0.7835 \leq k_{1,D,N(x)} \approx 2.0527 \leq \frac{k_{1,n_*}}{r_1} \approx 2.5836.$$

Similarly, according to Theorem 2.5, there are *at least* $m(1/4) = 4$ transmission eigenvalues in the interval

$$\left[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}\right] \approx [0.7835, 5.1672].$$

The numerical method computes 19 transmission eigenvalues in $[0.7835, 3.1623]$.

7. Conclusions and future work. We consider the problem of computing a few lowest real Maxwell's transmission eigenvalues which play an important role in inverse scattering theory. The transmission eigenvalues of balls with constant index of refraction are derived exactly by using special functions. Two finite element methods using the lowest order edge element are proposed. Since the resulting generalized matrix eigenvalue problem is large, sparse, and non-Hermitian, an adaptive Arnoldi method is devised which makes the use of a Faber-Krahn type inequality for transmission eigenvalues. The numerical results show that the proposed method is both efficient and effective.

The numerical examples in this paper are computed using Matlab and their sizes are limited by the amount of memory available. To handle larger problems would require an implementation of the Arnoldi method for parallel computers. Although by no means straightforward due to the need for inner products, there has been a substantial effort in this direction [14]. Thus we expect that the proposed methods can be extended using a parallel implementation such that larger problems can be treated rapidly.

Due to the fact that the transmission eigenvalue problem is neither elliptic nor self-adjoint, we are not able to prove convergence so far. Another method based on the fourth order formulation whose convergence can be shown is under consideration.

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