C^0 IPG Method for Biharmonic Eigenvalue Problems

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Abstract

We investigate the C^0 interior penalty Galerkin (C^0 IPG) method for biharmonic eigenvalue problems with the boundary conditions of the clamped plate, the simply supported plate and the Cahn-Hilliard type. We prove the convergence of the method and present numerical results to illustrate its performance. We also compare the C^0 IPG method with the Argyris C^1 finite element method, the Ciarlet-Raviart mixed finite element method, and the Morley nonconforming finite element method.

Keywords biharmonic eigenvalue problems, C^0 interior penalty Galerkin method, clamped, simply supported, Cahn-Hilliard

1 Introduction

In this paper we consider the numerical solution of several eigenvalue problems for the biharmonic operator. Such problems appear for example in mechanics (vibration and buckling of plates [22, 14, 23, 30]) and inverse scattering theory (the transmission eigenvalue problem [33]). Finite element methods for eigenvalue problems usually are based on numerical methods for the corresponding source problems, and there are three classical approaches to discretizing the biharmonic equation in the literature. The first approach uses conforming finite elements, for example, the Argyris finite element method [2] or the partition of unity finite element method [34, 29, 16]. These methods require globally continuously differentiable finite element spaces, which are difficult to construct and implement (in particular for three dimensional problems). The second approach uses classical nonconforming finite elements such as the Adini element [1] or the Morley element [27, 31, 32]. A disadvantage is that such elements do not come in a natural hierarchy and existing nonconforming elements only involve low order polynomials that are not efficient for capturing smooth solutions. The third classical approach to discretizing the biharmonic operator uses mixed finite element methods [15, 4, 26] that only require continuous Lagrange finite element spaces. However, for the boundary conditions of simply supported plates, some mixed finite element methods can result in spurious solutions on non-convex domains (Sapondjan paradox [28]). This is also the case for the boundary conditions of the Cahn-Hilliard type that appear in mathematical models for phase separation phenomena. The solution of the saddle point problems resulting from the use of a mixed finite element method is also more involved than that for a direct discretization of the fourth order operator.

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An alternative to the three classical approaches is provided by the C^0 interior penalty Galerkin (C^0 IPG) method developed in the last decade [17, 12, 8]. This is a discontinuous Galerkin method based on standard continuous finite element spaces usually used for second order elliptic problems. The lowest order methods in this approach are almost as simple as classical nonconforming finite element methods and are much simpler than finite element methods using continuously differentiable basis functions. Unlike classical nonconforming finite element methods, higher order finite elements can be used in this approach to capture smooth solutions efficiently. Furthermore, the C^0 IPG method converges for the biharmonic source problem with boundary conditions of the clamped plate, the simply supported plate and the Cahn-Hilliard type. It also preserves the symmetric positive-definiteness of the continuous problems. This last property is very attractive for eigenvalue problems since it means that the convergence for the eigenvalue problem can be derived from the convergence for the source problem by using the classical spectral approximation theory. In contrast, the convergence of mixed finite element methods for the source problem does not necessarily lead to convergence for the eigenvalue problem unless the mixed method is chosen carefully [7].

In this paper we extend the C^0 IPG method to biharmonic eigenvalue problems. We show that the method converges for all three types of boundary conditions, and we present numerical results that validate the theory. We also compare the performance of the C^0 IPG method, the Argyris C^1 finite element method, the Ciarlet-Raviart mixed finite element method and the Morley nonconforming finite element method.

We note that numerical results for a related C^0 discontinuous Galerkin method were presented in [35] for the plate vibration and buckling problems on a square with the boundary conditions of simply supported plates. However the convergence of the method for the eigenvalue problems was not addressed.

The rest of the paper is organized as follows. In Section 2 we introduce the biharmonic eigenvalue problems for the boundary conditions of the clamped plate, the simply support plate, and the Cahn-Hilliard type. In Section 3 we define the C^0 IPG method for the biharmonic eigenvalue problems and establish its convergence. Numerical examples of the C^0 IPG method are presented in Section 4. In Section 5 we compare the C^0 IPG method with the Argyris C^1 finite element method, the Ciarlet-Raviart mixed finite element method and the Morley finite element method. We end the paper with some concluding remarks in Section 6.

2 Biharmonic eigenvalue problems

Let Ω denote a bounded Lipschitz polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, and let n denote the unit outward normal. We consider biharmonic eigenvalue problems with three types of boundary conditions.

Clamped Plate (CP)

$$u = \frac{\partial u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{2.1}$$

Simply Supported Plate (SSP)

$$u = \triangle u = 0 \qquad \text{on } \partial\Omega \tag{2.2}$$

Cahn-Hilliard Type (CH)

$$\frac{\partial u}{\partial n} = \frac{\partial \triangle u}{\partial n} = 0 \qquad \text{on } \partial\Omega \tag{2.3}$$

The biharmonic eignevalue problem for plate vibration is to find $u \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$\Delta^2 u = \lambda u \qquad \text{in } \Omega$$

together with the boundary conditions (2.1), (2.2) or (2.3). We shall refer to them as the **V-CP** problem, the **V-SSP** problem and **V-CH** problem respectively.

The biharmonic eigenvalue problem for plate buckling is to find $u \neq 0$ and $\lambda \in \mathbb{R}$ such that

$$\Delta^2 u = -\lambda \Delta u \qquad \text{in } \Omega$$

together with the boundary conditions (2.1), (2.2) or (2.3). We shall refer to them as the **B-CP** problem, the **B-SSP** problem and **B-CH** problem respectively.

Let the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ be defined by

$$a(u,v)=\int_\Omega D^2u : D^2v\,dx, \tag{2.4}$$

where $D^2u : D^2v = \sum_{i,j=1}^2 u_{x_ix_j}v_{x_ix_j}$ is the Frobenius inner product of the Hessian matrices of u and v, and

$$b(u,v)=\begin{cases}(u,v)=\displaystyle\int_\Omega uv\,dx & \text{for plate vibration,}\\ (\nabla u,\nabla v)=\displaystyle\int_\Omega \nabla u\cdot\nabla v\,dx & \text{for plate buckling.}\end{cases} \tag{2.5}$$

The weak formulation of the biharmonic eigenvalue problem is to seek $(u,\lambda)\in V\times\mathbb{R}$ such that $u\neq 0$ and

$$a(u,v)=\lambda b(u,v)\qquad \forall\, v\in V, \tag{2.6}$$

where

$$V=H_0^2(\Omega) \qquad \text{for } \textbf{V-CP} \text{ and } \textbf{B-CP}, \tag{2.7}$$
$$V=H^2(\Omega)\cap H_0^1(\Omega) \qquad \text{for } \textbf{V-SSP} \text{ and } \textbf{B-SSP}, \tag{2.8}$$
$$V=\{v\in H^2(\Omega): \partial v/\partial n=0 \text{ on } \partial\Omega \text{ and } (v,1)=0\} \qquad \text{for } \textbf{V-}\text{ CH and } \textbf{B-CH}. \tag{2.9}$$

Remark 2.1. *Since the bilinear form $a(\cdot,\cdot)$ is symmetric positive-definite on V for all three types of boundary conditions, the biharmonic eigenvalues being considered are positive. Note that we have excluded the trivial eigenvalue 0 from the* **CH** *problem by imposing the zero mean constraint.*

In the rest of this section, we list some facts which are helpful for the validation of numerical methods. For the **V-CP** problem on the unit square, an accurate lower bound for the first eigenvalue is 1294.933940 given by Wieners [36]. An accurate upper bound is 1294.9339796 given by Bjørstad and Tjøstheim [5].

For the **V-SSP** problem on convex domains, the biharmonic eigenvalues are just the squares of the eigenvalues for the Laplace operator with the homogeneous Dirichlet boundary condition. The **V-SSP** eigenvalues for the unit square are therefore given by

$$4\pi^4, 25\pi^4, 25\pi^4, 64\pi^4, 100\pi^4, 100\pi^4, \ldots \tag{2.10}$$

with the corresponding eigenfunctions

$$\sin(\pi x_1)\sin(\pi x_2), \sin(2\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\sin(2\pi x_2), \sin(2\pi x_1)\sin(2\pi x_2), \sin(3\pi x_1)\sin(\pi x_2),$$
$$\sin(\pi x_1)\sin(3\pi x_2), \ldots$$

Similarly, for the **V-CH** problem on convex domains, the positive biharmonic eigenvalues are given by the square of the positive eigenvalues for the Laplace operator with the homogeneous Neumann boundary condition. Therefore the **V-CH** eigenvalues on the unit square are given by

$$\pi^4, \pi^4, 4\pi^4, 16\pi^4, 16\pi^4, 25\pi^4, 25\pi^4, \ldots \tag{2.11}$$

with the corresponding eigenfunctions

$$\cos(\pi x_1), \cos(\pi x_2), \cos(\pi x_1)\cos(\pi x_2), \cos(2\pi x_1), \cos(2\pi x_2), \cos(2\pi x_1)\cos(\pi x_2), \cos(\pi x_1)\cos(2\pi x_2), \ldots$$

We will also consider an L-shaped domain whose vertices are

$$(0,0), (1/2,0), (1/2,1/2), (1,1/2), (1,1), (0,1).$$

For the **V-SSP** problem on the L-shape domain, some of the eigenvalues are from (2.10) because the restrictions of the corresponding eigenfunctions on the L-shaped domain also satisfy the boundary conditions in (2.2). For example, the eigenfunction for the unit square

$$\sin(2\pi x_1)\sin(2\pi x_2)$$

is also an eigenfunction for the L-shaped domain with the same eigenvalue. Similarly, for the **V-CH** problem, the eigenfunctions

$$\cos(2\pi x_1) \text{ and } \cos(2\pi x_2),$$

for the unit square are also eigenfunctions for the L-shaped domain.

For the **B-CP** problem on the unit square, an accurate approximation 52.34469116 for the first eigenvalue is given in [5]. For the **B-SSP** problem on the unit square the first eigenvalue is the simple eigenvalue $2\pi^2 \approx 19.73920880$ with eigenfunction $\sin(\pi x_1)\sin(\pi x_2)$. For the **B-CH** problem on the unit square, the first eigenvalue is the double eigenvalue $\pi^2 \approx 9.869604401$ whose eigenspace is spanned by the functions $\cos(\pi x_1)$ and $\cos(\pi x_2)$.

3 The C^0 IPG method for biharmonic eigenvalue problems

Let $\mathcal{T}_h$ be a regular triangulation of Ω with mesh size h and $\tilde{V}_h \subset H^1(\Omega)$ be the $\mathbb{P}_k$ Lagrange finite element space ($k \geq 2$) associated with $\mathcal{T}_h$. Let $\mathcal{E}_h$ be the set of the edges in $\mathcal{T}_h$. For edges $e \in \mathcal{E}_h$ that are the common edge of two adjacent triangles $T_\pm \in \mathcal{T}_h$ and for $v \in \tilde{V}_h$, we define the jump of the flux to be

$$[\![\partial v/\partial n_e]\!] = \frac{\partial v_{T_+}}{\partial n_e}\Big|_e - \frac{\partial v_{T_-}}{\partial n_e}\Big|_e,$$

where n_e is the unit normal pointing from T_- to T_+. We let

$$\frac{\partial^2 v}{\partial n_e^2} = n_e \cdot (D^2 v) n_e$$

and define the average normal-normal derivative to be

$$\left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{1}{2}\left(\frac{\partial^2 v_{T_+}}{\partial n_e^2} + \frac{\partial^2 v_{T_-}}{\partial n_e^2} \right).$$

For $e \in \partial\Omega$, we take n_e to be the unit outward normal and define

$$[\![\partial v/\partial n_e]\!] = -\frac{\partial v}{\partial n_e} \quad \text{and} \quad \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} = \frac{\partial^2 v}{\partial n_e^2}.$$

Let $\mathbb{R}_+$ be the set of positive real numbers. The C^0 IPG method for the biharmonic eigenvalue problem is to find $(u_h, \lambda_h) \in V_h \times \mathbb{R}_+$ such that $u_h \neq 0$ and

$$a_h(u_h, v) = \lambda_h b(u_h, v) \qquad \forall\, v \in V_h, \tag{3.1}$$

where the choices of V_h and $a_h(\cdot,\cdot)$ depend on the boundary conditions.

CP For this boundary condition the choices for V_h and $a_h(\cdot,\cdot)$ are given by

$$V_h = \tilde{V}_h \cap H_0^1(\Omega), \tag{3.2}$$

$$\begin{aligned} a_h(w,v) = \sum_{T\in\mathcal{T}_h} \int_T D^2 w : D^2 v\,dx + \sum_{e\in\mathcal{E}_h} \int_e \left\{\!\!\left\{ \frac{\partial^2 w}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] + \left\{\!\!\left\{ \frac{\partial^2 v}{\partial n_e^2} \right\}\!\!\right\} \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] ds \\ + \sigma \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e \left[\!\!\left[\frac{\partial w}{\partial n_e} \right]\!\!\right] \left[\!\!\left[\frac{\partial v}{\partial n_e} \right]\!\!\right] ds, \end{aligned} \tag{3.3}$$

where $\sigma > 0$ is a (sufficiently large) penalty parameter.

SSP For this boundary condition we use the same V_h in (3.2) and the bilinear form

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T D^2 w : D^2 v\,dx + \sum_{e\in\mathcal{E}_h^i}\int_e \left\{\!\!\left\{\frac{\partial^2 w}{\partial n_e^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right] + \left\{\!\!\left\{\frac{\partial^2 v}{\partial n_e^2}\right\}\!\!\right\}\left[\!\!\left[\frac{\partial w}{\partial n_e}\right]\!\!\right]\,ds$$
$$+\,\sigma\sum_{e\in\mathcal{E}_h^i}\frac{1}{|e|}\int_e\left[\!\!\left[\frac{\partial w}{\partial n_e}\right]\!\!\right]\left[\!\!\left[\frac{\partial v}{\partial n_e}\right]\!\!\right]\,ds, \tag{3.4}$$

where $\mathcal{E}_h^i$ is the set of the edges interior to Ω.

CH For this boundary condition we use the same bilinear form $a_h(\cdot,\cdot)$ defined in (3.3) and take

$$V_h = \left\{v\in\tilde{V}_h : (v,1)=0\right\}. \tag{3.5}$$

The convergence of the C^0 IPG method for these eigenvalue problems is based on the convergence of the C^0 IPG method for the corresponding source problems.

Let W be the space $L^2(\Omega)$ for the plate vibration problems, the space $H_0^1(\Omega)$ for the **B-CP** and **B-SSP** problems and the space $\{v\in H^1(\Omega) : (v,1)=0\}$ for the **B-CH** problem. We will denote by $\|f\|_b$ the norm induced by the bilinear form $b(\cdot,\cdot)$ defined in (2.5), i.e.,

$$\|f\|_b^2 = b(f,f).$$

Given $f\in W$, the weak formulation for the source problem is to find $u\in V$ such that

$$a(u,v) = b(f,v) \qquad \forall\, v\in V, \tag{3.6}$$

where the bilinear form $a(\cdot,\cdot)$ is defined in (2.4). For the **V-CH** source problem, we also assume that f satisfies the constraint $(f,1)=0$.

The corresponding C^0 IPG method for (3.6) is to find $u_h\in V_h$ such that

$$a_h(u_h,v) = b(f,v) \qquad \forall\, v\in V_h, \tag{3.7}$$

where V_h and $a_h(\cdot,\cdot)$ are defined by

1. Equations (3.2) and (3.3) respectively for the **CP** boundary conditions,
2. Equations (3.2) and (3.4) respectively for the **SSP** boundary conditions, and
3. Equations (3.5) and (3.3) respectively for the **CH** boundary conditions.

The following lemma summarizes the results for the source problems obtained in [12, 10, 9].

Lemma 3.1. *The biharmonic source problem* (3.6) *and the discrete source problem* (3.7) *are uniquely solvable for the boundary conditions of* **CP**, **SSP** *and* **CH**. *In addition there exists $\beta > 0$ such that*

$$\|u-u_h\|_h \le Ch^{\beta}\|f\|_b, \tag{3.8}$$
$$\|u-u_h\|_b \le Ch^{2\beta}\|f\|_b, \tag{3.9}$$

where $u\in V$ (resp. $u_h\in V_h$) is the solution of (3.6) (*resp.* (3.7)), *and the mesh-dependent energy norm $\|\cdot\|_h$ is defined by*

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h}|e|^{-1}\|[\![\partial v/\partial n_e]\!]\|_{L^2(e)}^2 \tag{3.10}$$

for the boundary conditions of **CP** *and* **CH**, *and*

$$\|v\|_h^2 = \sum_{T\in\mathcal{T}_h}|v|_{H^2(T)}^2 + \sum_{e\in\mathcal{E}_h^i}|e|^{-1}\|[\![\partial v/\partial n_e]\!]\|_{L^2(e)}^2 \tag{3.11}$$

for the boundary conditions of **SSP**.

Remark 3.2. *Let V be the Sobolev space for the biharmonic problem defined in* (2.7)*,* (2.8) *or* (2.9) *and V_h be the corresponding finite element space. Then $\|\cdot\|_h$ defined by* (3.10) *is a norm on the space $V+V_h$ for the boundary conditions of* **CP** *and* **CH***, and $\|\cdot\|_h$ defined by* (3.11) *is a norm on the space $V+V_h$ for the boundary conditions of* **SSP***. Moreover in all three cases we have a Poincaré-Friedrichs inequality* [13]

$$\|v\|_b \leq C\|v\|_h \qquad \forall\, v \in V+V_h. \tag{3.12}$$

Remark 3.3. *The exponent β in* (3.8) *and* (3.9) *is given by $\beta = \min(\alpha, k-1)$, where α is index of elliptic regularity that appears in the elliptic regularity estimate* [6, 19, 20]

$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C_{\Omega,\alpha}\|f\|_b$$

for the solution u of the source problem (3.6)*. It is determined by the angles at the corners of Ω and the boundary conditions. For the* **CP** *boundary conditions* (2.1)*, α belongs to $(\frac{1}{2}, 1]$ and $\alpha = 1$ if Ω is convex. For the* **SSP** *boundary conditions* (2.2) *and the* **CH** *boundary conditions* (2.3)*, α belongs to $(0, 1]$ in general, $\alpha = 2$ for a rectangular domain, and α is any number strictly less than $1/3$ for an L-shaped domain.*

For the convergence analysis of the C^0 IPG method for the biharmonic eigenvalue problems, we need two (bounded) solution operators $T : W \longrightarrow V\,(\subset W)$ and $T_h : W \longrightarrow V_h\,(\subset W)$ on the Hilbert space $\big(W, b(\cdot,\cdot)\big)$, which are defined by

$$\begin{aligned} a(Tf, v) &= b(f, v) \qquad \forall\, v \in V, \\ a_h(T_h f, v) &= b(f, v) \qquad \forall\, v \in V_h. \end{aligned}$$

Note that (2.6) is equivalent to $Tu = (1/\lambda)u$, (3.1) is equivalent to $T_h u_h = (1/\lambda_h)u_h$, and the estimates (3.8)–(3.9) can be rewritten as

$$\|(T - T_h)f\|_h \leq Ch^{\beta}\|f\|_b \qquad \forall\, f \in W, \tag{3.13}$$

$$\|(T - T_h)f\|_b \leq Ch^{2\beta}\|f\|_b \qquad \forall\, f \in W. \tag{3.14}$$

The operator T is symmetric, positive-definite and compact due to the compact embedding of V into W. Therefore the spectrum of T consists of a sequence of positive eigenvalues $\mu_1 \geq \mu_2 \geq \ldots$ decreasing to 0, and the numbers $\lambda_j = 1/\mu_j$ are the biharmonic eigenvalues that increase to ∞.

The convergence of the discrete eigenfunctions to the continuous eigenfunctions will be measured by the gaps between the corresponding eigenspaces. Given two subspaces X and Y of a normed space $(Z, \|\cdot\|)$, the gap $\delta(X,Y)$ between them is defined by (cf. [24])

$$\delta(X, Y) = \max\{\hat{\delta}(X, Y), \hat{\delta}(Y, X)\}, \quad \text{where} \quad \hat{\delta}(X, Y) = \sup_{x \in X, \|x\|=1} \inf_{y \in Y} \|x - y\|. \tag{3.15}$$

The convergence of the C^0 IPG method for the biharmonic eigenvalue problems follows from (3.13), (3.14) and the classical spectral approximation theory (cf. [24, 3] and the references therein). The theorem below follows immediately from the results in [24, Section 5.4.3] and [3, Section 2.7].

Theorem 3.4. *Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the biharmonic eigenvalues, $\lambda = \lambda_j = \ldots = \lambda_{j+m-1}$ be a biharmonic eigenvalue with multiplicity m, and V_λ be the corresponding m-dimensional eigenspace. Let $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \ldots$ be the discrete eigenvalues obtained by the C^0 IPG method. Then we have, as $h \to 0$,*

$$|\lambda_{h,l} - \lambda| \leq Ch^{2\beta}, \quad l = j, j+1, \ldots, j+m-1.$$

In addition, if $V_\lambda \subset V$ is the space spanned by the eigenfunctions corresponding to the biharmonic eigenvalues $\lambda_j, \ldots, \lambda_{j+m-1}$ and $V_{h,\lambda} \subset V_h$ is the space spanned by the eigenfunctions corresponding to the discrete eigenvalues $\lambda_{h,j}, \ldots, \lambda_{h,j+m-1}$, then we have, as $h \to 0$, $\delta(V_\lambda, V_{h,\lambda}) = O(h^\beta)$ in $(W, \|\cdot\|_h)$ and $\delta(V_\lambda, V_{h,\lambda}) = O(h^{2\beta})$ in $(W, \|\cdot\|_b)$.

Remark 3.5. *We can apply the classical theory because we use the Hilbert space $\big(W, b(\cdot,\cdot)\big)$ and V_h is a subspace of W. This would not be possible if we use the space V in* (2.7)–(2.9)*.*

Remark 3.6. *The convergence of the method in* [35] *for eigenvalue problems can also be established analogously by the classical spectral approximation theory.*

4 Numerical examples of the C^0 IPG method

In this section we present numerical results for two domains to illustrate the performance of the C^0 IPG method for the biharmonic eigenvalue problems. For simplicity we implement the C^0 IPG for $k = 2$. The penalty parameter σ is taken to be 50 in all the computations.

We first consider the unit square. In Table 1 we display the first biharmonic eigenvalues for the **V-CP** problem, the **V-SSP** problem and the **V-CH** problem, computed by the C^0 IPG method on a series of unstructured meshes generated by uniform refinement. We note that the first **V-CP** eigenvalue obtained in [36] is $1,294.93398$. The first **V-SSP** eigenvalue is $4\pi^4 \approx 389.6363$ and the first **V-CH** eigenvalue is $\pi^4 \approx 97.4091$. Therefore the C^0 IPG method provides good approximations in all three cases.

Table 1: The first **V-CP**, **V-SSP** and **V-CH** eigenvalues of the unit square.

h	1/10	1/20	1/40	1/80
CP(1)	1,377.1366	1,318.5091	1,301.3047	1296.5904
SSP(1)	395.1181	391.1631	390.0452	389.7422
CH(1)	98.2067	97.6410	97.4711	97.4251

The second domain is the L-shaped domain. In Table 2 we present the first biharmonic plate vibration eigenvalues computed by the C^0 IPG method on a series of uniformly refined unstructured meshes. We also include the results for the third eigenvalues of **V-SSP** and **V-CH**, whose exact values are $64\pi^4 \approx 6234.1818$ and $16\pi^4 \approx 1558.5455$ respectively. They are approximated correctly with less than 1% relative error at the finest meshes. Comparing Table 1 and Table 2, we see that the convergence for the L-shaped domain is slower.

Table 2: The first **V-CP**, **V-SSP** and **V-CH** eigenvalues of the L-shaped domain.

h	1/10	1/20	1/40	1/80
CP(1)	7,834.5030	7,104.1915	6,854.7447	6,763.0157
SSP(1)	2,870.9514	2,748.1841	2,693.7255	2,663.3927
SSP(3)	6,327.5449	6,573.0063	6,259.2682	6,240.6958
CH(1)	177.4750	174.1519	172.3741	171.1519
CH(3)	1,603.9472	1,571.3380	1,562.0031	1,559.4471

We define the relative error of the approximate eigenvalue by

$$R_i = \frac{|\lambda_{h_i} - \lambda_{h_{i+1}}|}{\lambda_{h_{i+1}}},$$

where λ_{h_i} is a fixed eigenvalue computed by the C^0 IPG method on the mesh with mesh size h_i. In Fig. 1 we plot the convergence history of the C^0 IPG method. For the unit square, the convergence rates are $O(h^2)$ as predicted by the theory in the previous section. For the L-shaped domain, there is a decrease in the convergence rate due to the reentrant corner, which is also consistent with the theoretical result.

In Fig. 2 we present the 2D surface plots of the eigenfunctions corresponding to the first biharmonic eigenvalues of the unit square and the L-shaped domain for the **V-CP** problem and the **V-SSP** problem. The eigenfunctions for **V-CP** exhibit the correct rotational symmetry, which is consistent with the fact that the first **V-CP** eigenvalue is a simple eigenvalue for both domains (cf. Table **??** and Table **??**). This is also true for the **V-SSP** problem (cf. Table **??** and Table **??**). Moreover, the computed **V-SSP** eigenfunction for the first biharmonic eigenvalue on the unit square should approximate a multiple of $\sin(\pi x_1)\sin(\pi x_2)$ and this is observed.

In Fig. 3 we present the 2D surface plots of eigenfunctions for the **V-CH** problem. As was mentioned at the end of Section 2, the first eigenvalue of the **V-CH** problem on the square is π^2 (with multiplicity 2) and the eigenspace is

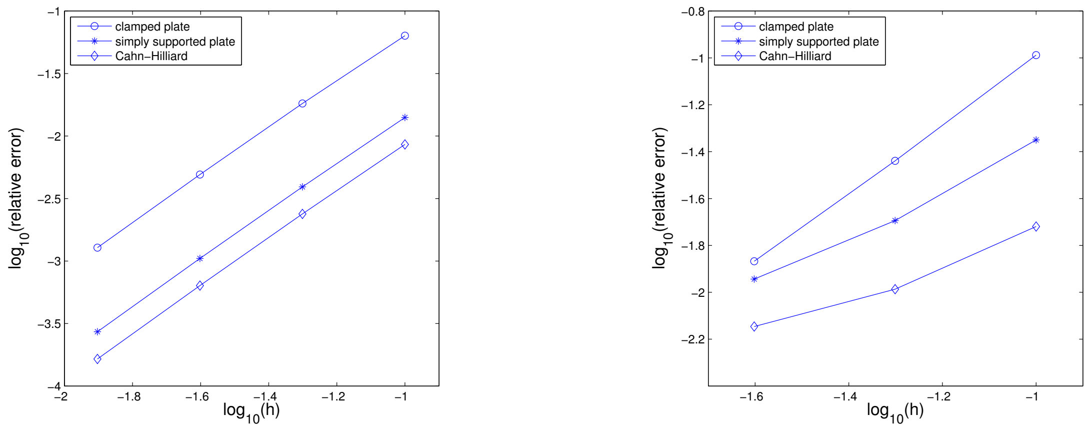

Figure 1: Relative errors of the first biharmonic plate vibration eigenvalues. Left: the unit square. Right: the L-shaped domain.

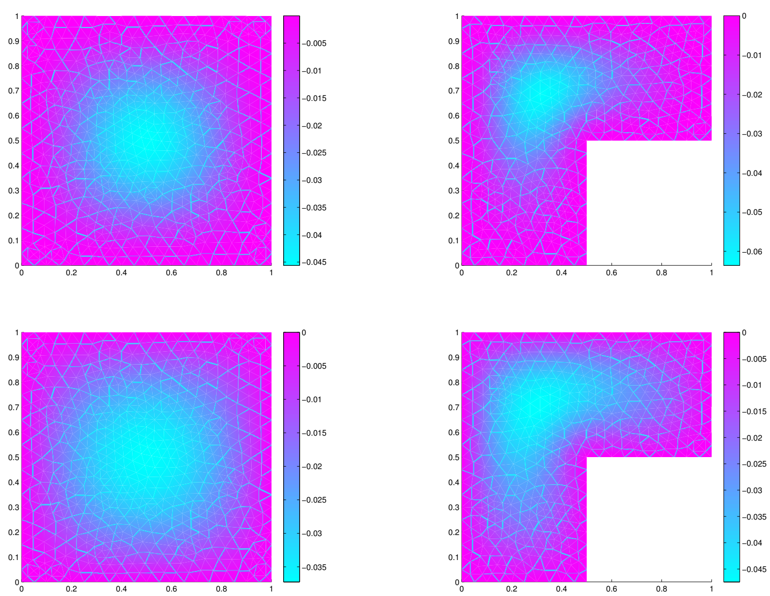

Figure 2: Eigenfunctions corresponding to the first biharmonic plate vibration eigenvalues. First row: **V-CP** eigenfunctions. Second row: **V-SSP** eigenfunctions.

spanned by the two functions $\cos(\pi x_1)$ and $\cos(\pi x_2)$. In the top row of Fig. 3, the plane wave features of $\cos(\pi x_1)$ and $\cos(\pi x_2)$ are clearly observed.

The 2D surface plot of the computed eigenfunction for the first **V-CH** eigenvalue for the L-shaped domain is displayed on the second row of Fig. 3. It is observed that the computed eigenfunction is anti-symmetric with respect to the line connecting the re-entrant corner and the upper left corner, which is consistent with the zero mean constraint and with the fact that the first **V-CH** eigenvalue is a simple eigenvalue (cf. Table **??**).

The 2D surface plots for some other **V-SSP** and **V-CH** eigenfunctions on the L-shaped domain are presented in

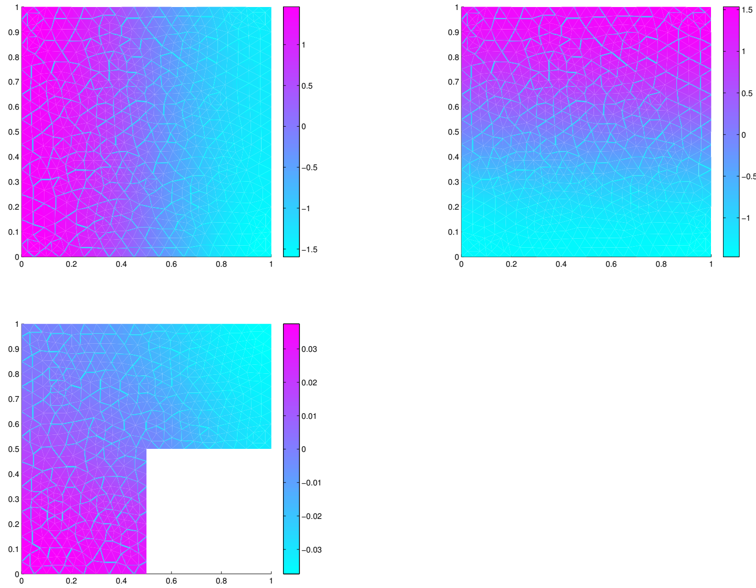

Figure 3: Eigenfunctions corresponding to the first two **V-CH** eigenvalues for the unit square (first row) and the first **V-CH** eigenvalue for the L-shaped domain (second row).

Fig. 4. As was mentioned at the end of Section 2, $\sin(2\pi x_1)\sin(2\pi x_2)$ is also a **V-SSP** eigenfunction for the L-shaped domain with the simple eigenvalue $64\pi^4 \approx 6234.1818$, which turns out to be the 3rd eigenvalue. The 2D surface plot of the computed eigenfunction for this eigenvalue is displayed in the first row of Fig. 4, where the same symmetry as the function $\sin(2\pi x_1)\sin(2\pi x_2)$ is observed.

The functions $\cos(2\pi x_1)$ and $\cos(2\pi x_2)$ span the eigenspace of the double (3rd and 4th) **V-CH** eigenvalue $16\pi^4 \approx 1558.5455$. In Fig. 4, the plane wave feature of $\cos(2\pi x_1)$ and $\cos(2\pi x_2)$ is clearly visible.

Next we present some numerical results for the **B-CP** problem, the **B-SSP** problem and the **B-CH** problem. Tables 3 and 4 display the first eigenvalues on a series of uniformly refined meshes for the unit square and the L-shaped domain. The approximate eigenvalue for the **B-CP** on the unit square agree with the approximation obtained in [5], and the approximate eigenvalues for **B-SSP** (resp. **B-CH**) problem on the unit square also agrees with $2\pi^2 \approx 19.73920880$ (resp. $\pi^2 \approx 9.869604401$).

Table 3: The first **B-CP**, **B-SSP** and **B-CH** eigenvalues for the unit square.

h	1/10	1/20	1/40	1/80
BCP(1)	55.4016	53.2067	52.5757	52.4045
BSSP(1)	20.0244	19.8193	19.7607	19.7448
BCH(1)	9.9541	9.893	9.8758	9.8712

The convergence histories of the first eigenvalue for the plate buckling problem on the unit square and the L-shaped domain are presented in Fig. 5, which exhibit similar behavior as the plate vibration problem.

The behavior of the eigenfunctions for the buckling problems are similar to the eigenfunctions for vibration problems and are therefore not presented here.

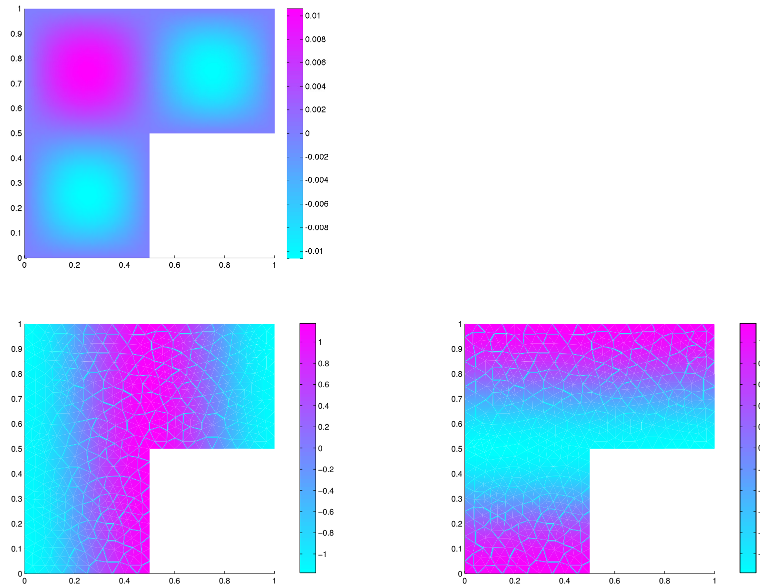

Figure 4: Eigenfunctions for the L-shaped domain. Top: the 3rd **V-SSP** eigenfunction. Bottom: the normalized 3rd and 4th **V-CH** eigenfunctions.

Table 4: The first **B-CP**, **B-SSP** and **B-CH** eigenvalues of the L-shaped domain.

h	1/10	1/20	1/40	1/80
BCP(1)	148.0750	135.0775	130.8045	129.3580
BSSP(1)	65.8585	63.3735	62.2093	61.6123
BCH(1)	15.3809	14.8899	14.6087	14.4305

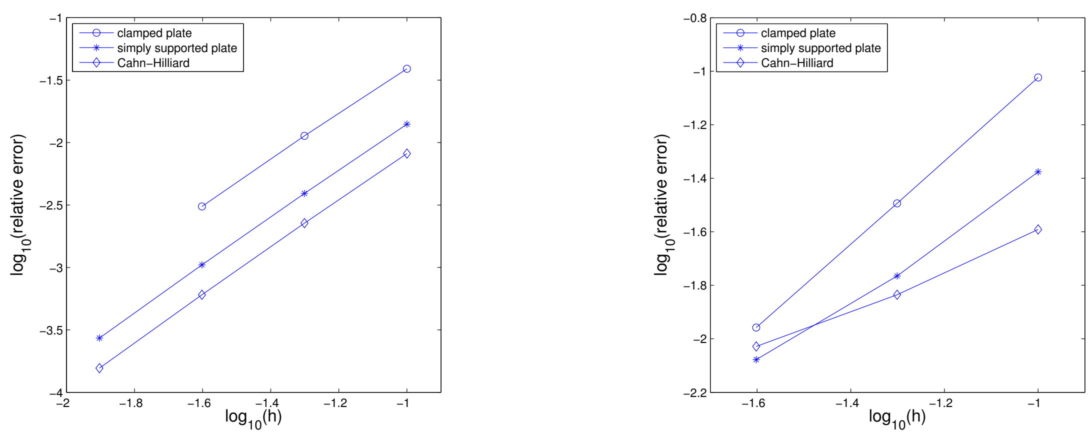

Figure 5: Relative errors of the first **B-CP**, **B-SSP** and **B-CH** eigenvalues. Left: the unit square. Right: the L-shaped domain.

5 Comparison with other methods

In this section we compare the quadratic C^0 IPG method with the Argyris C^1 finite element method [2], the Ciarlet-Raviart mixed finite element method [15], and the Morley nonconforming finite element method [27].

The quintic Argyris element is an H^2 conforming finite element. The 21 degrees of freedom for the Argyris element consist of the function values and the first and second order partial derivatives at the three vertices, and the normal derivatives at the midpoints of the three edges. Let V_h be the Argyris finite element space such that $V_h \subset V$, where V is defined in (2.7)–(2.9) for the three types of boundary conditions. The discrete biharmonic eigenvalue problem for the Argyris finite element method is to find $(u_h, \lambda_h) \in V_h \times \mathbb{R}_+$ such that $u_h \neq 0$ and

$$(\triangle u_h, \triangle v_h) = \lambda_h b(u_h, v_h) \qquad \forall\, v \in V_h.$$

The Ciarlet-Raviart mixed finite element method for the biharmonic eigenvalue problems is based on the following weak formulation: Find $\lambda \in \mathbb{R}_+$ and nontrivial $(p, u) \in Q \times V$ such that

$$\int_\Omega pq dx - \int_\Omega \nabla q \cdot \nabla u \, dx = 0 \qquad \forall\, q \in Q, \tag{5.1a}$$

$$-\int_\Omega \nabla p \cdot \nabla v \, dx = -\lambda b(u, v) \qquad \forall\, v \in V, \tag{5.1b}$$

where

$$\begin{aligned}
Q &= H^1(\Omega) \quad \text{and} \quad V = H^1_0(\Omega) \quad \text{for the } \textbf{CP} \text{ boundary conditions},\\
Q &= H^1_0(\Omega) \quad \text{and} \quad V = H^1_0(\Omega) \quad \text{for the } \textbf{SSP} \text{ boundary conditions},\\
Q &= H^1(\Omega) \quad \text{and} \quad V = H^1(\Omega) \quad \text{for the } \textbf{CH} \text{ boundary conditions}.
\end{aligned}$$

The discrete eigenvalue problem is to find $\lambda_h \in \mathbb{R}_+$ and nontrivial $(p_h, u_h) \in Q_h \times V_h$ such that

$$\int_\Omega p_h q dx - \int_\Omega \nabla q \cdot \nabla u_h dx = 0 \qquad \forall\, q \in Q_h, \tag{5.2a}$$

$$-\int_\Omega \nabla p_h \cdot \nabla v dx = -\lambda_h b(u_h, v) \qquad \forall\, v \in V_h, \tag{5.2b}$$

where $Q_h \subset Q$ and $V_h \subset V$ are standard $\mathbb{P}_1$ Lagrange finite element spaces.

Let u be a biharmonic eigenfunction. If $p = -\Delta u$ belongs to $H^1(\Omega)$, then (p, u) will satisfy the weak formulation (5.1) and hence the discrete eigenvalue problem (5.2) defines a Galerkin method for such an eigenfunction. This is the case for the **CP**, **SSP** and **CH** biharmonic eigenvalue problems if Ω is convex [6, 19, 20]. However, as far as we know, only the convergence of the Ciarlet-Raviart finite element method for the **CP** eigenvalue problem on convex domains has been established in [25].

The Morley nonconforming finite element method is based on a triangular quadratic element, whose 6 degrees of freedom consist of the function values at the 3 vertices and the normal derivatives at the midpoints of the 3 edges. The Morley finite element space V_h is determined by the boundary conditions. For the **CP** boundary conditions, we set all the degrees of freedom on $\partial\Omega$ to be zero. For the **SSP** boundary conditions, we set only the degrees of freedom at the vertices in $\partial\Omega$ to be zero. For the **CH** boundary conditions we set only the degrees of freedom at the midpoints of the edges along $\partial\Omega$ to be zero, and we also impose the zero mean condition on V_h.

The Morley finite element method is to find $(u_h, \lambda_h) \in V_h \times \mathbb{R}_+$ such that $u_h \neq 0$ and

$$\sum_{T \in \mathcal{T}_h} \int_T D^2 u_h : D^2 v \, dx = \lambda_h b_h(u_h, v) \qquad \forall\, v \in V_h,$$

where

$$b_h(w, v) = \begin{cases} \displaystyle\int_\Omega wv \, dx & \text{for plate vibration problems,} \\ \displaystyle\sum_{T \in \mathcal{T}_h} \int_T \nabla w \cdot \nabla v \, dx & \text{for plate buckling problems.} \end{cases}$$

The convergence analysis of the Morley finite element method for the plate vibration and plate buckling problems was carried out in [30].

The numerical results of the four methods for the plate vibration problem on the unit square, the L-shaped domain and with the three types of boundary conditions are presented in Tables 5 and 6. The mesh size used in the computations is $h \approx 0.0125$.

Table 5: The first eigenvalues for the unit square.

	V-CP	**V-SSP**	**V-CH**	**B-CP**	**B-SSP**	**B-CH**
C^0IPG	1,296.5904	389.7422	97.4251	52.4045	19.7448	9.8712
Argyris	1,295.0271	389.6365	97.4096	52.3469	19.7392	9.8695
Mixed	1,295.4749	389.7557	97.4248	52.3671	19.7422	9.8704
Morley	1,294.4736	389.5991	97.4049	52.3301	19.7383	9.8694

Table 6: The first eigenvalues for the L-shaped domain.

	V-CP	**V-SSP**	**V-CH**	**B-CP**	**B-SSP**	**B-CH**
C^0IPG	6,763.0157	2,663.3927	171.1519	129.3580	61.6123	14.4305
Argyris	6,744.0019	2,691.9825	175.5528	129.0132	61.9109	14.6288
Mixed	6,694.5532	1,491.0917	34.9277	128.4905	38.6147	5.9099
Morley	6,673.1009	2,481.0174	156.2246	127.7805	59.1396	13.9426

We observe that the numerical results for the quadratic C^0 IPG method and the Argyris method are comparable in all six cases. In view of the smooth nature of the eigenfunctions on the unit square and the high order of the finite element, the Argyris method provides very accurate approximation of the biharmonic eigenvalues on the unit square. Therefore the quadratic C^0 IPG method is quite efficient for the unit square. This can also be seen by comparing the eigenvalues in Table 5 with the ones in [36]. The Ciarlet-Raviart mixed finite element method converges on the unit square for all three types of boundary conditions.

For the L-shaped domain, we observe the Ciarlet-Raviart mixed finite element method also converges for the **V-CP** problem. For the boundary conditions of **SSP** and **CH**, the results show spurious eigenvalues generated by the Ciarlet-Raviart mixed finite element method.

Comparing with the C^0 IPG method, the performance of the Morley finite element method is slightly better when the eigenfunction is very smooth and slightly worse when the eigenfunction is less smooth. The approximate eigenvalues generated by the Morley finite element method is consistently less than the approximations generated by the Argyris finite element method, which agrees with the discussion in [21].

Numerical results for the first eigenvalues of the plate buckling problems are also shown in Tables 5 and 6. For the unit square, the results from all four methods with respect to all three boundary conditions are consistent. For the L-shaped domain, the results from the C^0IPG method, the Argyris finite element method and the Morley finite element method are consistent for all three boundary conditions, whereas the Ciarlet-Raviart mixed finite element method is consistent with the other methods only for the **CP** boundary conditions and generates spurious eigenvalues for the other two boundary conditions.

The behavior of the Ciarlet-Raviart mixed finite element method on nonconvex domains with respect to the boundary conditions of **CP**, **SSP** and **CH** can be given a heuristic explanation as follows. Since an eigenfunction u for the **CP** eigenvalue problem always belongs to $H^{2+\alpha}(\Omega)$ for some $\alpha \in (\frac{1}{2}, 1]$, we can replace (5.1) by another weak formulation: Find $\lambda \in \mathbb{R}_+$ and nontrivial $(p, u) \in H^\alpha(\Omega) \times H_0^1(\Omega)$ such that

$$\int_\Omega pq\,dx - (\nabla q, \nabla u) = 0 \qquad \forall\, q \in H^1(\Omega), \tag{5.3a}$$

$$-\langle \nabla p, \nabla v \rangle = -\lambda b(u, v) \qquad \forall\, v \in H_0^{2-\alpha}(\Omega), \tag{5.3b}$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $H^{\alpha-1}(\Omega)$ and $H^{1-\alpha}(\Omega)$. Since the P_1 finite element spaces satisfy $Q_h \subset H^1(\Omega) \subset H^\alpha(\Omega)$ and $V_h \subset H_0^{2-\alpha}(\Omega) \subset H_0^1(\Omega)$, we can treat (5.2) as a Petrov-Galerkin method for the **V-CP** and **B-CP** eigenvalue problems based on (5.3). This explains why the Ciarlet-Raviart method converges for the **V-CP** and **B-CP** eigenvalue problems on the L-shaped domain. On the other hand, since $p = -\Delta u$ may only belong to $H^\alpha(\Omega)$ for some $\alpha \in (0, \frac{1}{2})$ if u is an eigenfunction for the biharmonic eigenvalue problems with the **SSP** or the **CH** boundary conditions, a similar Petrov-Galerkin interpretation for (5.2) is not valid because V_h is not a subspace of $H^{2-\alpha}(\Omega)$ when $\alpha < \frac{1}{2}$.

6 Conclusion

We have demonstrated that the C^0 IPG method is a provably accurate scheme for approximating biharmonic eigenvalue problems. It is robust with respect to different boundary conditions, which is a significant advantage over the Ciarlet-Raviart mixed finite element method, because the latter produces spurious eigenvalues on nonconvex domains for the boundary conditions of the simply supported plate and the Cahn-Hilliard type. Its performance is also comparable to the more complicated Argyris C^1 finite element method.

The results in this paper can be extended to three dimensions where the advantage over C^1 finite element methods would be even more obvious, and they can also be extended to domains with curved boundaries where the isoparametric version of the C^0 IPG method [11] can be applied, while the constructions of C^1 finite element space for such domains are much more complicated.

The application of the C^0 IPG method to transmission eigenvalue problems and the development of an adaptive C^0 IPG method for biharmonic eigenvalue problems are ongoing projects.

From the numerical results in Section 5, we see that the Ciarlet-Raviart mixed finite element method converges on nonconvex domains for the boundary conditions of the clamped plate. As far as we know this method has only been analyzed for convex domains [15, 4, 18] or smooth domains [26] even for the source problem. It would be interesting to see if one can turn the heuristic argument at the end of Section 5 into a rigorous analysis of the Ciarlet-Raviart mixed finite element method on nonconvex domains.

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