A Multi-Level Method for Transmission Eigenvalues of Anisotropic Media

Xia Ji[a], Jiguang Sun[b,*]

[a] LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing, 100190, China

[b] Department of Mathematical Sciences, Michigan Technological University, Houghton, MI 49931, USA,

Abstract

In this paper, we propose a multi-level finite element method for the transmission eigenvalue problem of anisotropic media. The problem is non-standard and non-self-adjoint with important applications in inverse scattering theory. We employ a suitable finite element method to discretize the problem. The resulting generalized matrix eigenvalue problem is large, sparse and non-Hermitian. To compute the smallest real transmission eigenvalue, which is usually an interior eigenvalue, we devise a multi-level method using Arnoldi iteration. At the coarsest mesh, the eigenvalue is obtained using Arnoldi iteration with an adaptive searching technique. This value is used as the initial guess for Arnoldi iteration at the next mesh level. This procedure is then repeated until the finest mesh level. Numerical examples are presented to show the viability of the method.

Keywords: Transmission Eigenvalues, Anisotropic Media, Arnoldi Iteration, Finite Element Method

1. Introduction

The transmission eigenvalue problem is a challenging research topic arising from the inverse scattering theory for inhomogeneous media. It was first introduced by Colton and Monk [1] and Kirsch [2]. Physically, if the wave number is a transmission eigenvalue, an incident field with unit norm can be constructed such that the corresponding scattered field is arbitrarily small, i.e., the target (inhomogeneous medium) becomes "transparent". Due to its importance in the theory of qualitative methods, the transmission eigenvalue problem received significant attention recently, for example, see [3, 4, 5, 6]. Recent development on the existence of transmission eigenvalues started with Päivärinta and Sylvester

*Corresponding author at: LSEC, Institute of Computational Mathematics, Chinese Academy of Sciences, Beijing, 100190, China. Tel: +86-10-62627309.

Email addresses: jixia@lsec.cc.ac.cn (Xia Ji), jiguangs@mtu.edu (Jiguang Sun)

[7]. The authors proved the existence provided that the contrast in the medium is large enough. Cakoni et al. [8] showed the existence under the assumption that the contrast in the medium does not change sign and is bounded away from zero. We refer the readers to [9] and references therein for the recent theoretical development.

From a practical point of view, Cakoni et al. [4, 10] showed that transmission eigenvalues can be determined from scattering data and used to give a lower bound for the index of refraction. A similar result of the Maxwell's equation was given in [11] for anisotropic media where the transmission eigenvalues are used to obtain upper and lower bounds on the norm of the index of refraction. Other techniques using transmission eigenvalues to estimate index of refraction are contained in [12, 13]. In particular, a novel inverse scattering method using the detection of transmission eigenvalues to reconstruct the shape of the target is proposed in [14].

Since the problem is neither elliptic nor self-adjoint, numerical computation of the interior transmission problem and the associated eigenvalue problem is challenging. Efficient computational tools are of great importance because theoretical results are still partial and numerical evidence might lead theorists to the right direction. Moreover, numerical methods for computing transmission eigenvalues are needed in optimization type algorithms to estimate the index of refraction [11, 13]. There are only a few papers dealing with the numerical computation of transmission eigenvalues. Finite element methods for transmission eigenvalues were first proposed in [15]. Two iterative methods based on the fourth order reformulation of the transmission eigenvalues were proposed in [16]. To the authors' knowledge, this is the only paper containing some convergence analysis. However, the H^2-conforming Argyris elements were used for the fourth order problem which complicates the implementation. Ji et al. [17] proposed a mixed finite element method and an adaptive algorithm to solve the resulted generalized matrix eigenvalue problem. The main ingredient is the Arnoldi iteration for non-Hermitian matrice [18, 19, 20]. This idea was further extended to the case of Maxwell's equations by Monk and Sun [21].

In this paper, we consider the computation of transmission eigenvalues of anisotropic media. In particular, we would like to compute the smallest (real) transmission eigenvalue which is essential in the reconstruction of the index of refraction [13]. Note that, for anisotropic media, due to the fact that the transmission eigenvalue problem can not be written as a fourth order problem, the methods proposed in [16] and the mixed methods in [17, 21] do not work. We shall devise finite element methods based on the original second order systems for the transmission eigenvalue problem. The generalized matrix eigenvalue problem resulting from the finite element discretization is large, sparse, and most importantly, non-Hermitian. It is prohibitive to use direct methods even for a rather coarse mesh in 2D cases. Moreover, as we are trying to obtain more accurate results, the adaptive Anorldi method proposed in [17, 21] is not enough. This motivates us to devise a multi-level method using several meshes from coarser to finer. The meshes do not need to have any hierarchy relation. We use the Arnoldi iteration with suitable searching technique to obtain the eigenvalue at the coarsest mesh. At the next mesh level, this value is used as the initial guess (spectral transformation) for Arnoldi iteration. This procedure is then repeated until the finest mesh.

This paper is organized as follows. Section 2 is for the transmission eigenvalue problem for Helmholtz equation. We empoly a continuous finite element method. The Maxwell's transmission eigenvalue problem is treated in Section 3 using a curl-conforming edge element. To handle the resulting generalized matrix eigenvalue problems, a multi-level approach is proposed in Section 4. The numerical results are presented in Section 5. Finally, we draw some conclusions and discuss some future works in Section 6.

2. The Helmholtz transmission eigenvalues

The transmission eigenvalue problem arises in the study of inverse scattering problems of inhomogeneous media. In the following, we will illustrate briefly how the transmission eigenvalue problem is derived.

Starting with the direct scattering problem for anisotropic media of bounded support, let $D \subset \mathbb{R}^2$ be an open bounded domain with a Lipschitz boundary $\Gamma = \partial D$. Let A be a 2×2 matrix valued function with $L^\infty(D)$ entries and $n \in L^\infty(D)$. In addition, A is symmetric in $\bar{D}$ such that $\xi \cdot \mathrm{Im}(A)\xi \leq 0$ and $\xi \cdot \mathrm{Re}(A)\xi \geq \gamma|\xi|^2$ for all $\xi \in \mathbb{R}^2$ with $\gamma > 0$.

The direct scattering problem of an anisotropic medium can be stated as follows. Let k be the wave number and the incident wave be given by $u^i = e^{ikx\cdot d}$. Find the total field $u(x)$ such that

$$\nabla \cdot A\nabla u + k^2 n(x) u = 0, \qquad \text{in } D, \tag{1a}$$

$$\Delta u + k^2 u = 0, \qquad \text{in } \mathbb{R}^2 \setminus D, \tag{1b}$$

$$u(x) = e^{ikx\cdot d} + u^s(x), \qquad \text{on } \mathbb{R}^2, \tag{1c}$$

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{1d}$$

where u^s is the scattered field, $x \in \mathbb{R}^2, r = |x|$, $d \in \Omega := \{\hat{x} \in \mathbb{R}^2; |\hat{x}| = 1\}$. The Sommerfeld radiation condition (1d) is assumed to hold uniformly with respect to $\hat{x} = x/|x|$.

It is well-known that this problem has a unique solution $u \in H^1_{loc}(\mathbb{R}^2)$. The scattered field has the asymptotic behavior

$$u^s(x) = \frac{e^{ikr}}{\sqrt{r}} u_\infty(\hat{x}, d) + O(r^{-3/2}), \tag{2}$$

as $r \to \infty$ uniformly in $\hat{x}$, where u_∞ is the far field pattern.

We can define the far field operator $F : L^2(\Omega) \to L^2(\Omega)$ based on the far field pattern u_∞

$$(Fg)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d) g(d) \mathrm{d}s(d), \quad \hat{x} \in \Omega. \tag{3}$$

The far field operator F has fundamental importance in the study of qualitative methods, for example, the linear sampling method (see Section 3.3 of [22]). The sampling methods for reconstructing the support of an inhomogeneous medium fail, if the interrogating frequency corresponds to a transmission eigenvalue.

For the case of anisotropic media, F is injective with dense range provided that k is not a transmission eigenvalue which we will define next. We refer the readers to [3, 22, 23] for the mathematical derivation and interpretation of the problem.

The transmission eigenvalue problem for anisotropic media is to find $k \neq 0$, such that there exist non-trivial solutions w and v satisfying

$$\nabla \cdot A \nabla w + k^2 n(x) w = 0, \qquad \text{in } D, \tag{4a}$$

$$\Delta v + k^2 v = 0, \qquad \text{in } D, \tag{4b}$$

$$w - v = 0, \qquad \text{on } \Gamma, \tag{4c}$$

$$\frac{\partial w}{\partial \nu_A} - \frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \Gamma, \tag{4d}$$

where $\frac{\partial w}{\partial \nu_A}$ is the conormal derivative defined by

$$\frac{\partial u}{\partial \nu_A}(x) := \nu(x) \cdot A(x) \nabla u(x), \quad x \in \Gamma\ ,$$

with ν the unit outward normal to Γ. The wave numbers k's for which the transmission eigenvalue problem has non-trivial solutions are called transmission eigenvalues.

2.1. Transmission eigenvalues on disks

Now we consider a simple case where D is a disk with radius R, $A = aI$ for some constant a and n is also a constant. The main concern here is to derive the exact transmission eigenvalues which will serve as a benchmark.

The solutions of (4a) and (4b) can be written as

$$w = J_m(kr\sqrt{n/a}) \cos m\theta, \qquad v = J_m(kr) \cos m\theta, \quad m = 0, 1, 2, \ldots$$

To satisfy the boundary condition (4c), one needs to set

$$w = \frac{J_m(kR)}{J_m(kR\sqrt{n/a})} J_m(kr\sqrt{n/a}) \cos m\theta.$$

Ignoring the trigonometric functions and using the identity,

$$J_m' = -J_{m+1} + \frac{m}{x} J_m,$$

we have that

$$\frac{\partial v}{\partial r} = k\left(-J_{m+1}(kr) + \frac{m}{kr}J_m(kr)\right),$$

$$a\frac{\partial w}{\partial r} = a\frac{J_m(kR)}{J_m(kR\sqrt{\frac{n}{a}})}\left[k\sqrt{\frac{n}{a}}\left(-J_{m+1}(kr\sqrt{n/a}) + \frac{m}{kr\sqrt{n/a}}J_m(kr\sqrt{n/a})\right)\right].$$

To satisfy the boundary condition (4d), we need to enforce

$$\begin{vmatrix} J_m(kR) & J_m(kR\sqrt{n/a}) \\ \frac{1}{r}\frac{\partial}{\partial r}\left(J_m(kr)\right)_{r=R} & a\frac{1}{r}\frac{\partial}{\partial r}\left(J_m(kr\sqrt{n/a})\right)_{r=R} \end{vmatrix} = 0, \quad m \geq 0. \tag{5}$$

Then the transmission eigenvalues are the roots of (5).

2.2. Finite element method

If $A = I$, we can rewrite (4) as a fourth order problem. Let $z = v - w \in H_0^2(D)$. Then we have

$$(\triangle + k^2 n(x))z = -k^2(n(x)-1)v,$$

i.e., $(n(x)-1)^{-1}(\triangle + k^2 n(x))z = -k^2 v$. Applying $(\triangle + k^2)$ to both sides of the above equation, we obtain

$$(\triangle + k^2)\frac{1}{n(x)-1}(\triangle + k^2 n(x))z = 0.$$

The transmission eigenvalue problem can be stated as: find $(k^2, z) \in \mathbb{C} \times H_0^2(D)$ such that

$$\left(\frac{1}{n(x)-1}(\triangle + k^2 n(x))z, (\triangle + k^2)\phi\right) = 0, \quad \text{for all } \phi \in H_0^2(D). \tag{6}$$

Then one can apply mixed methods or H^2 conforming finite elements [16, 17]. However, this is impossible for general A.

In the following, we shall describe a continuous finite element method for transmission eigenvalues of anisotropic media. Multiplying (4a) by a test function ϕ and integrating by part, we obtain

$$(A\nabla w, \nabla \phi) - k^2(nw, \phi) - \left\langle \frac{\partial w}{\partial \nu_A}, \phi \right\rangle = 0, \tag{7}$$

where $\langle \cdot, \cdot \rangle$ is the boundary integral on $\Gamma = \partial D$. Similarly, multiplying (4b) by a test function ϕ and integrating by part, we obtain

$$(\nabla v, \nabla \phi) - k^2(v, \phi) - \left\langle \frac{\partial v}{\partial \nu}, \phi \right\rangle = 0. \tag{8}$$

Subtracting (8) from (7) and employing the boundary condition (4d), we have

$$(A\nabla w - \nabla v, \nabla \phi) - k^2((nw - v), \phi) = 0. \tag{9}$$

For discretization, we use standard Lagrange finite elements. We define

$$
\begin{aligned}
S_h &= \text{the space of continuous piecewise } p\text{-degree finite elements on } D,\\
S_h^0 &= S_h \cap H_0^1(D)\\
&= \text{the subspace of functions in } S_h \text{ with vanishing DoF on } \partial D,\\
S_h^B &= \text{the subspace of functions in } S_h \text{ with vanishing DoF in } D,
\end{aligned}
$$

where DoF stands for degrees of freedom. We explicitly enforce the Dirichlet boundary condition (4c) by letting

$$
\begin{aligned}
w_h &= w_{0,h} + w_{B,h} \text{ where } w_{0,h} \in S_h^0 \text{ and } w_{B,h} \in S_h^B,\\
v_h &= v_{0,h} + w_{B,h} \text{ where } v_{0,h} \in S_h^0.
\end{aligned}
$$

In (7), letting the test function $\xi_h \in S_h^0$, we obtain the standard weak formulation for w_h as

$$
(A\nabla(w_{0,h} + w_{B,h}), \nabla\xi_h) - k^2(n(w_{0,h} + w_{B,h}), \xi_h) = 0, \tag{10}
$$

for all $\xi_h \in S_h^0$.

Analogously, letting the test function $\eta_h \in S_h^0$, we obtain the weak formulation for v_h as

$$
(\nabla(v_{0,h} + w_{B,h}), \nabla\eta_h) - k^2((v_{0,h} + w_{B,h}), \eta_h) = 0, \tag{11}
$$

for all $\eta_h \in S_h^0$.

For (9), letting $\phi_h \in S_h^B$, we have

$$
\begin{aligned}
&(A\nabla(w_{0,h} + w_{B,h}), \nabla\phi_h) - (\nabla(v_{0,h} + w_{B,h}), \nabla\phi_h)\\
&\qquad -k^2\left(n(w_{0,h} + w_{B,h}) - (v_{0,h} + w_{B,h}), \phi_h\right) = 0.
\end{aligned} \tag{12}
$$

Let N_h, N_h^0, and N_h^B be the dimensions of S_h, S_h^0 and S_h^B, respectively. In addition, we choose $\{\xi_1, \ldots, \xi_{N_h}\}$ be the finite element basis for S_h such that $\{\xi_1, \ldots, \xi_{N_h^0}\}$ is a basis for S_h^0. We define the following matrices

S_A	stiffness matrix, $(S_A)_{j,\ell} = (A\nabla\xi_j, \nabla\xi_\ell)$
S	stiffness matrix, $(S)_{j,\ell} = (\nabla\xi_j, \nabla\xi_\ell)$
M_n	mass matrices, $(M_n)_{j,\ell} = (n\xi_j, \xi_\ell)$
M	mass matrices, $(M)_{j,\ell} = (\xi_j, \xi_\ell)$

Combining (10), (11), and (12), the discrete problem is to solve the following generalized eigenvalue problem

$$
\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x}, \tag{13}
$$

where the matrices $\mathcal{A}$ and $\mathcal{B}$ are given block-wisely by

$$
\mathcal{A} = \begin{pmatrix}
S_A^{N_h^0\times N_h^0} & 0 & S_A^{N_h^0\times N_h^B}\\
0 & S^{N_h^0\times N_h^0} & S^{N_h^0\times N_h^B}\\
(S_A^{N_h^0\times N_h^B})^T & (-S^{N_h^0\times N_h^B})^T & S_A^{N_h^B\times N_h^B} - S^{N_h^B\times N_h^B}
\end{pmatrix},
$$

and

$$\mathcal{B} = \begin{pmatrix} M_n^{N_h^0\times N_h^0} & 0 & M_n^{N_h^0\times N_h^B} \\ 0 & M^{N_h^0\times N_h^0} & M^{N_h^0\times N_h^B} \\ (M_n^{N_h^0\times N_h^B})^T & -(M^{N_h^0\times N_h^B})^T & M_n^{N_h^B\times N_h^B} - M^{N_h^B\times N_h^B} \end{pmatrix}.$$

It is obvious that $\mathcal{A}$ and $\mathcal{B}$ are non-symmtric. If one makes the substitution $B = -A$ and $m = -n$ [24], the transmission eigenvalue problem is equivalent to the following problem: find $k \neq 0$ and $(w, v) \neq 0$ such that

$$\nabla \cdot B\nabla w + k^2 m w = 0, \tag{14a}$$

$$\nabla \cdot \nabla v + k^2 v = 0, \tag{14b}$$

$$w - v = 0, \tag{14c}$$

$$\frac{\partial w}{\partial \nu_B} + \frac{\partial v}{\partial \nu} = 0. \tag{14d}$$

Using the same technique as above, the matrices $\mathcal{A}$ and $\mathcal{B}$ are given block-wisely by

$$\mathcal{A} = \begin{pmatrix} S_B^{N_h^0\times N_h^0} & 0 & S_B^{N_h^0\times N_h^B} \\ 0 & S^{N_h^0\times N_h^0} & S^{N_h^0\times N_h^B} \\ (S_B^{N_h^0\times N_h^B})^T & (S^{N_h^0\times N_h^B})^T & S_B^{N_h^B\times N_h^B} + S^{N_h^B\times N_h^B} \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} M_m^{N_h^0\times N_h^0} & 0 & M_m^{N_h^0\times N_h^B} \\ 0 & M^{N_h^0\times N_h^0} & M^{N_h^0\times N_h^B} \\ (M_m^{N_h^0\times N_h^B})^T & (M^{N_h^0\times N_h^B})^T & M_m^{N_h^B\times N_h^B} + M^{N_h^B\times N_h^B} \end{pmatrix}.$$

with

S_B	stiffness matrix, $(S_B)_{j,\ell} = (B\nabla\xi_j, \nabla\xi_\ell)$
S	stiffness matrix, $(S)_{j,\ell} = (\nabla\xi_j, \nabla\xi_\ell)$
M_m	mass matrices, $(M_m)_{j,\ell} = (m\xi_j, \xi_\ell)$
M	mass matrices, $(M)_{j,\ell} = (\xi_j, \xi_\ell)$

Thus (13) becomes a generalized eigenvalue problem for symmetric matrices.

3. The Maxwell's transmission eigenvalues

We consider the Maxwell's transmission eigenvalue problem in this section. Let $D \subset \mathbb{R}^3$ be a bounded connected region with Lipschitz boundary ∂D. Let ν be the unit outward normal to ∂D. The Hilbert space $H(\text{curl}, D)$ is defined as

$$H(\text{curl}, D) = \{u \in (L^2(D))^3 : \text{curl } u \in (L^2(D))^3\},$$

equipped with the scalar product

$$(u, v)_{\text{curl}} = (u, v) + (\text{curl } u, \text{curl } v),$$

where $(\cdot,\cdot)$ is the L^2 inner product on D. We also define

$$\begin{aligned}
H_0(\mathrm{curl}, D) &= \{u \in H(\mathrm{curl}, D) : \nu \times u = 0 \text{ on } \partial D\},\\
\mathcal{U}(D) &= \{u \in H(\mathrm{curl}, D) : \mathrm{curl}\, u \in H(\mathrm{curl}, D)\},\\
\mathcal{U}_0(D) &= \{u \in H_0(\mathrm{curl}, D) : \mathrm{curl}\, u \in H_0(\mathrm{curl}, D)\}.
\end{aligned}$$

Let N be a 3×3 matrix valued function defined on D such that $N \in L^\infty(D, \mathbb{R}^{3\times 3})$.

Definition 3.1. *A real matrix field N is said to be bounded positive definite on D if $N \in L^\infty(D, \mathbb{R}^{3\times 3})$ and there exists a constant $\gamma > 0$ such that*

$$\xi \cdot N\xi \geq \gamma|\xi|^2, \quad \forall \xi \in \mathbb{R}^3, \qquad a.e. \text{ in } D.$$

We assume that N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive definite real matrix fields on D as in [22].

Let the electromagnetic incident plane wave be given by

$$E^i(x, d, p) = \frac{i}{k}\mathrm{curl}\,\mathrm{curl}\, p\, e^{ikx\cdot d}, \quad H^i(x, d, p) = \mathrm{curl}\, p\, e^{ikx\cdot d},$$

where $d \in \mathbb{R}^3$ is a unit vector giving the direction of propagation, and the vector p is the polarization.

Letting A be a bounded positive definite 3×3 matrix field on D, the scattering by an anisotropic medium leads to the following problem: Find the interior electric and magnetic fields E, H and the scattered electric and magnetic field E^s, H^s satisfying

$$\mathrm{curl}\, E^s - ikH^s = 0, \qquad \text{in } \mathbb{R}^3 \setminus D, \tag{15a}$$

$$\mathrm{curl}\, H^s + ikE^s = 0, \qquad \text{in } \mathbb{R}^3 \setminus D, \tag{15b}$$

$$A\,\mathrm{curl}\, E - ikH = 0, \qquad \text{in } D, \tag{15c}$$

$$\mathrm{curl}\, H + ikN(x)H = 0, \qquad \text{in } D, \tag{15d}$$

$$\nu \times (E^s + E^i) - \nu \times E = 0, \qquad \text{on } \partial D, \tag{15e}$$

$$\nu \times (H^s + H^i) - \nu \times AH = 0, \qquad \text{on } \partial D, \tag{15f}$$

with the Silver-Müller radiation condition

$$\lim_{r\to\infty}(H^s \times x - rE^s) = 0, \tag{16}$$

where $r = |x|$. Under suitable conditions on A, N, and D, the well-posedness of the above problem is known (Theorem 4.2 of [22]).

The scattered fields have the following asymptotic behavior

$$E^s(x, d, p) = \frac{e^{ikr}}{r} E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r^2}\right), \qquad r \to \infty, \tag{17a}$$

$$H^s(x, d, p) = \frac{e^{ikr}}{r} \hat{x} \times E_\infty(\hat{x}, d, p) + O\left(\frac{1}{r^2}\right), \qquad r \to \infty, \tag{17b}$$

where $\hat{x} = x/r$ and E_∞ is the electric far field pattern [23]. Given E_∞, one can define the far field operator $F : L_t^2(\Omega) \to L_t^2(\Omega)$ by

$$(Fg)(\hat{x}) := \int_\Omega E_\infty(\hat{x}, d, g(d))\, \mathrm{d}s, \tag{18}$$

where $\Omega = \{\hat{x} \in \mathbb{R}^3; |\hat{x}| = 1\}$ and $L_t^2(\Omega) := \{u \in (L^2(\Omega))^3 : \nu \cdot u = 0 \text{ on } \Omega\}$.

Similar to the Helmholtz case, the far field operator F has fundamental importance in the study of qualitative methods. It is well-known that F has dense range provided k is not a transmission eigenvalue. We refer the readers to [3, 22, 23] for more disscussion of the corresponding inverse scattering problems.

In terms of electric fields, the Maxwell's transmission eigenvalue problem for an anisotropic medium can be formulated as follows [25].

Definition 3.2. *A value of $k^2 \neq 0$ is called a transmission eigenvalue if there exist non-trivial fields $E, E_0 \in (L^2(D))^3$ with $E - E_0 \in \mathcal{U}_0(D)$ such that*

$$\mathit{curl}\ A\ \mathit{curl}\ E - k^2 N E = 0, \qquad \text{in } D, \tag{19a}$$

$$\mathit{curl}\ \mathit{curl}\ E_0 - k^2 E_0 = 0, \qquad \text{in } D, \tag{19b}$$

$$\nu \times E = \nu \times E_0, \qquad \text{on } \partial D, \tag{19c}$$

$$\nu \times A\mathit{curl}\ E = \nu \times \mathit{curl}\ E_0, \qquad \text{on } \partial D. \tag{19d}$$

3.1. Transmission eigenvalues of balls

We consider the transmission eigenvalues for the special case when D is a ball. Suppose that $A = aI$ and $N = N_0 I$ for some constants a and N_0. The transmission eigenvalues can be derived exactly and will serve as a benchmark. The solutions of (19a) and (19b) are given by

$$\tilde{M}_u = \text{curl } \{xu\}, \quad \tilde{N}_u = \frac{1}{ik}\text{curl } \{\tilde{M}_u\},$$
$$\tilde{M}_v = \text{curl } \{xv\}, \quad \tilde{N}_v = \frac{1}{ik}\text{curl } \{\tilde{M}_v\},$$

where $u = j_n(kr)Y_n^m(\hat{x})$ and $v = j_n(kr\sqrt{N_0/a})Y_n^m(\hat{x})$, j_n is the spherical Bessel's function of order $n, n \geq 1$, Y_n^m is the spherical harmonic [23], and $r = |x|$.

Similar to the derivation in [21], for TE mode, to satisfy the boundary conditions (19c) and (19d), the wave number k^2's need to satisfy

$$\begin{vmatrix} j_n(kr) & j_n(kr\sqrt{N_0/a}) \\ \frac{1}{r}\frac{\partial}{\partial r}\left(rj_n(kr)\right) & a\frac{1}{r}\frac{\partial}{\partial r}\left(rj_n(kr\sqrt{N_0/a})\right) \end{vmatrix} = 0, \quad n \geq 1. \tag{20}$$

For TM mode, the wave number k^2's need to satisfy

$$\begin{vmatrix} \frac{1}{r}\frac{\partial}{\partial r}\left(rj_n(kr)\right) & \frac{1}{r}\frac{\partial}{\partial r}\left(rj_n(kr\sqrt{N_0/a})\right) \\ k^2 j_n(kr) & k^2 N_0 j_n(kr\sqrt{N_0/a}) \end{vmatrix} = 0, \quad n \geq 1. \tag{21}$$

The transmission eigenvalues k's are the roots of (20) and (21).

3.2. Finite element method

Equation (19) can be rewritten into a fourth order problem when $A = I$. Following [26], we let $u = E - E_0$ and $v = NE - E_0$. Then we have that

$$E = (N - I)^{-1}(v - u), \quad E_0 = (I - N)^{-1}(Nu - v).$$

Subtracting (19b) from (19a), we obtain curl curl $u = k^2 v$ and therefore

$$E = (N - I)^{-1}\left(\frac{1}{k^2}\text{curl curl } u - u\right). \tag{22}$$

Substituting for E in (19a) and taking the boundary conditions (19c) and (19d) into account, we end up with a fourth order differential equation. Find $(k, u) \in \mathbb{R} \times \mathcal{U}_0(D)$ satisfying

$$(\text{curl curl } - k^2 N)(N - I)^{-1}(\text{curl curl } u - k^2 u) = 0. \tag{23}$$

The variational formulation for the transmission eigenvalue problem can be stated as finding $k^2 \neq 0$ and $u \in \mathcal{U}_0(D)$ such that

$$\left((N - I)^{-1}\left(\text{curl curl } - k^2 I\right) u, (\text{curl curl } - k^2 N)\phi\right) = 0, \tag{24}$$

for all $\phi \in \mathcal{U}_0(D)$. A mixed method for the above problem is proposed in [21]. Again, it is not possible to write the problem as a fourth order problem for general A.

In the following, we shall describe a finite element method based on the curl-conforming edge element for general A. Multiplying by suitable test functions and integrating by parts, a variational formulation of (19a)-(19d) can be stated as follows (see Definition 4.1 in [25]). Find $k^2 \neq 0$, $E_0 \in H(\text{curl}, D)$ satisfying

$$(\text{curl } E_0, \text{curl } \phi) - k^2(E_0, \phi) = 0 \tag{25}$$

for all $\phi \in H_0(\text{curl}, D)$, and $E \in H(\text{curl}, D)$ satisfying

$$(A\,\text{curl } E, \text{curl } \gamma) - k^2(NE, \gamma) = (\text{curl } E_0, \text{curl } \gamma) - k^2(E_0, \gamma), \tag{26}$$

for all $\gamma \in H(\text{curl}, D)$ together with the essential boundary condition $E = E_0$ on ∂D. In (26) we have enforced the boundary condition (19d) weakly.

The curl-conforming edge element method is based on the above formulation. Let $\mathcal{T}$ be a regular tetrahedral mesh for D, S_h denote the first family curl-conforming edge element space of Nédélec [27, 28]. Let

$$P_l = \{\text{polynomials of maximum total degree } l\},$$
$$\tilde{P}_l = \{\text{homogeneous polynomials of total degree exactly } l\}.$$

We denote by S_l a subspace of homogeneous vector polynomials of degree l by

$$\mathcal{S}_l = \left\{p \in (\tilde{P}_l)^3 | x \cdot p = 0\right\}.$$

and R_l by

$$R_l = (P_{l-1})^3 \oplus \mathcal{S}_l.$$

The curl-conforming edge element space is defined as

$$S_h = \{u \in H(\mathrm{curl}, D) | u|_K \in R_l \text{ for all } K \in \mathcal{T}\}.$$

The degrees of freedom for finite elements in S_h are associated with the edges e, faces f and the volume of an element $K \in \mathcal{T}$. Letting τ denote a unit vector parallel to e and ν denote the unit outward normal to f, the degrees of freedom are defined as

$$M_e(u) = \left\{ \int_e u \cdot \tau q \,\mathrm{d}s, \quad \text{for all } q \in P_{l-1}(e) \text{ for each edge } e \text{ of } K \right\},$$
$$M_f(u) = \left\{ \int_f u \times \nu \cdot g \,\mathrm{d}A, \quad \text{for all } g \in (P_{l-2}(f))^2 \text{ for each face } f \text{ of } K \right\},$$
$$M_K(u) = \left\{ \int_K u \cdot g \,\mathrm{d}x, \quad \text{for all } g \in (P_{l-3}(K))^3 \right\}.$$

In the linear case of $l = 1$, one has that

$$R_1 = \{u(x) = a + b \times x, \quad a, b \in \mathbb{C}^3\}.$$

The six constants in the definition of R_1 are determined from the moments $\int_e u \cdot \tau \mathrm{d}s$ on the six edges e of K, and these edge degrees of freedom ensure that the global space is curl-conforming. Then the linear edge element space S_h is given by

$$\{u \in H(\mathrm{curl}; \Omega) \mid u|_K \in R_1 \text{ for all } K \in \mathcal{T}\}. \tag{27}$$

We define a subspace of S_h given by

$$S_h^0 = \{\xi_h \in S_h, \nu \times \xi_h = 0 \text{ on } \partial D\} \subset H_0(\mathrm{curl}; \Omega), \tag{28}$$

and $S_h^B = S_h \setminus S_h^0$. The boundary condition $\nu \times \xi_h = 0$ can be simply satisfied by setting the degrees of freedom associated to the boundary edges to zero.

Let w_h and v_h be the discrete approximations for E and E_0, respectively. We write

$$\begin{aligned} w_h &= w_{0,h} + w_{B,h} \text{ where } w_{0,h} \in S_h^0 \text{ and } w_{B,h} \in S_h^B, \\ v_h &= v_{0,h} + w_{B,h} \text{ where } v_{0,h} \in S_h^0. \end{aligned}$$

First we choose a test function $\xi_h \in S_h^0$ and obtain

$$(A\mathrm{curl}\ (w_{0,h} + w_{B,h}), \mathrm{curl}\ \xi_h) - k^2(N(w_{0,h} + w_{B,h}), \xi_h) = 0, \tag{29}$$

for all $\xi_h \in S_h^0$. Similarly, we have

$$(\mathrm{curl}\ (v_{0,h} + w_{B,h}), \mathrm{curl}\ \xi_h) - k^2(v_{0,h} + w_{B,h}, \xi_h) = 0, \tag{30}$$

for all $\xi_h \in S_h^0$. Rearranging terms in (26), we end up with

$$(A\text{curl } E - \text{curl } E_0, \text{curl } \phi) - k^2(NE - E_0, \phi) = 0,$$

for all $\phi \in H(\text{curl}, D)$. In the discrete case, for all $\phi_h \in S_h^B$, we have

$$\begin{aligned}(A\,\text{curl } (w_{0,h} + w_{B,h}) - \text{curl } (v_{0,h} + w_{B,h}), \text{curl } \phi_h)\\ -k^2(N(w_{0,h} + w_{B,h}) - (v_{0,h} + w_{B,h}), \phi_h) = 0.\end{aligned} \tag{31}$$

Let $N_h = \dim S_h$, $N_h^0 = \dim S_h^0$, and $N_h^B = N_h - N_h^0$. Let $\{\xi_1, \ldots, \xi_{N_h}\}$ be a basis for S_h and $\{\xi_1, \ldots, \xi_{N_h^0}\}$ be a basis for S_h^0. We define the following matrices

C_A	curl matrix, $(C_A)_{j,\ell} = (A\,\nabla \times \xi_j, \nabla \times \xi_\ell)$
C	curl matrix, $(C)_{j,\ell} = (\nabla \times \xi_j, \nabla \times \xi_\ell)$
M_N	mass matrix, $(M_N)_{j,\ell} = (N\xi_j, \xi_\ell)$
M	mass matrix, $(M)_{j,\ell} = (\xi_j, \xi_\ell)$

The discrete problem is the following generalized eigenvalue problem

$$\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x}, \tag{32}$$

where $\mathcal{A}$ and $\mathcal{B}$ are given block-wisely by

$$\mathcal{A} = \begin{pmatrix} C_A^{N_h^0\times N_h^0} & 0 & C_A^{N_h^0\times N_h^B} \\ 0 & C^{N_h^0\times N_h^0} & C^{N_h^0\times N_h^B} \\ (C_A^{N_h^0\times N_h^B})^T & -(C^{N_h^0\times N_h^B})^T & C_A^{N_h^B\times N_h^B} - C^{N_h^B\times N_h^B} \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} M_N^{N_h^0\times N_h^0} & 0 & M_N^{N_h^0\times N_h^B} \\ 0 & M^{N_h^0\times N_h^0} & M^{N_h^0\times N_h^B} \\ (M_N^{N_h^0\times N_h^B})^T & -(M^{N_h^0\times N_h^B})^T & M_N^{N_h^B\times N_h^B} - M^{N_h^B\times N_h^B} \end{pmatrix},$$

Note that neither $\mathcal{A}$ nor $\mathcal{B}$ is symmetric. Similar to the Helmholtz case, a change of variable $B = -A$ and $M = -N$ would make the generalized eigenvalue problem symmetric, i.e.

$$\mathcal{A} = \begin{pmatrix} C_B^{N_h^0\times N_h^0} & 0 & C_B^{N_h^0\times N_h^B} \\ 0 & C^{N_h^0\times N_h^0} & C^{N_h^0\times N_h^B} \\ (C_B^{N_h^0\times N_h^B})^T & (C^{N_h^0\times N_h^B})^T & C_B^{N_h^B\times N_h^B} + C^{N_h^B\times N_h^B} \end{pmatrix},$$

and

$$\mathcal{B} = \begin{pmatrix} M_M^{N_h^0\times N_h^0} & 0 & M_M^{N_h^0\times N_h^B} \\ 0 & M^{N_h^0\times N_h^0} & M^{N_h^0\times N_h^B} \\ (M_M^{N_h^0\times N_h^B})^T & (M^{N_h^0\times N_h^B})^T & M_M^{N_h^B\times N_h^B} + M^{N_h^B\times N_h^B} \end{pmatrix},$$

with

C_B	stiffness matrix, $(C_B)_{j,\ell} = (B\,\nabla \times \xi_j, \nabla \times \xi_\ell)$
C	stiffness matrix, $(C)_{j,\ell} = (\nabla \times \xi_j, \nabla \times \xi_\ell)$
M_m	mass matrix, $(M_M)_{j,\ell} = (M\xi_j, \xi_\ell)$
M	mass matrix, $(M)_{j,\ell} = (\xi_j, \xi_\ell)$

However, this nicer structure does not seem to be helpful for Arnoldi iteration as we shall see later.

4. A multi-level approach of the generalized eigenvalue problem

In this section, we discuss how to solve the generalized eigenvalue problem

$$\mathcal{A}\vec{x} = k^2\mathcal{B}\vec{x}, \tag{33}$$

which is large, sparse, and most annoyingly, non-Hermitian. It is prohibitive to use direct methods even for a rather coarse mesh in 2D cases [15]. In [17, 21], an adaptive Arnoldi method is proposed. The method works fine for mid-size problem ($\approx 10,000$ DoF). For larger problems, in particular, 3D problems, the method takes too long searching for the smallest transmission eigenvalue which, in general, is an interior eigenvalue. This is due to the fact that the Arnoldi iteration works efficiently only if the spectral transformation (initial guess) is very close to the true eigenvalue. This motivates us to devise a more efficient solver based on multiple meshes from coarse to fine.

We generate a series of meshes for D, not necessarily to be hierarchy meshes. The only requirement is that the mesh sizes decrease gradually. The first step is to obtain the smallest real transmission eigenvalue at the coarsest mesh level. Since the generalized eigenvalue problem is rather small, we can empoly an improved version of the algorithm in [21]. The method combines the Arnoldi iteration and an estimation of the smallest real transmission eigenvalue using the Faber-Krahn type inequalities, i.e., a lower bound for the smallest real transmission eigenvalue.

In [9], Cakoni et al. have proved the following Faber-Krahn type inequality for the Helmholtz equation. Define

$$\begin{aligned}
A_* &:= \inf_{x\in D} \inf_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) > 0,\\
A^* &:= \sup_{x\in D} \sup_{\xi\in\mathbb{R}^2, |\xi|=1} (\xi \cdot A(x)\xi) < \infty,\\
n_* &:= \inf_{x\in D} n(x) > 0,\\
n^* &:= \sup_{x\in D} n(x) < \infty.
\end{aligned}$$

Suppose $\int_D (n-1)dx \neq 0$ and $A^* < 1$ or $A_* > 1$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$. Moreover, the nonzero eigenvalue of smallest magnitude k_1 satisfies the Faber-Krahn type estimate

$$|k_1|^2 \geq \frac{A_*(1-\sqrt{A^*})}{C_p \max(n^*,1)(1+\sqrt{n^*})}, \quad \text{if} \quad A^* < 1, \tag{34}$$

$$|k_1|^2 \geq \frac{(1-1/\sqrt{A_*})}{C_p \max(n^*,1)(1+\sqrt{n_*})}, \quad \text{if} \quad A_* > 1, \tag{35}$$

where C_p is the reciprocal of the smallest Dirichlet eigenvalue for D.

The above inequality provides a lower bound for transmission eigenvalues as long as we have computed the smallest Dirichlet eigenvalue. In fact, this can be done easily since we have the necessary stiffness and mass matrices. The discrete Dirichlet eigenvalue problem is simply the following generalized eigenvalue problem

$$S^{N_h^0 \times N_h^0} \vec{x} = \lambda M^{N_h^0 \times N_h^0} \vec{x}, \tag{36}$$

where $S^{N_h^0 \times N_h^0}$ and $M^{N_h^0 \times N_h^0}$ are the stiffness matrix and the mass matrix, respectively.

For the Maxwell's case, the above inequalities holds with

$$n_* := \inf_{x \in D} \inf_{\xi \in \mathbb{R}^3, |\xi|=1} (\xi^T N(x) \xi) > 0,$$

$$n^* := \sup_{x \in D} \sup_{\xi \in \mathbb{R}^3, |\xi|=1} (\xi^T N(x) \xi) < \infty,$$

and C_p is the reciprocal of the smallest Maxwell's eigenvalue. Similar to the Helmholtz case, the discrete Maxwell's eigenvalue problem is the following generalized eigenvalue problem

$$C^{N_h^0 \times N_h^0} \vec{x} = \lambda M^{N_h^0 \times N_h^0} \vec{x}, \tag{37}$$

where $C^{N_h^0 \times N_h^0}$ and $M^{N_h^0 \times N_h^0}$ are the curl matrix and the mass matrix, respetively. Note that the smallest Maxwell's eigenvalue is the smallest non-zero eigenvalue of (37). We refer the readers to [28, 29] and the references therein for the computation of the Maxwell's eigenvalue which itself is a non-trivial problem.

On the coarest mesh level, we start with the lower bounds in the above Faber-Krahn type inequalities, and use Arnoldi iteration, in particular, the Matlab *eigs* to compute the eigenvalues. Since Arnoldi method computes complex eigenvalues as well, we need to exclude them. If we find the smallest eigenvalue, the method stops. Otherwise, we use a larger initial guess and repeat the process until the smallest eigenvalue is found. Note that we are guaranteed to get it due to the result on the existence of real transmission eigenvalues [9]. We present the program flow in **Algorithm 1**.

After the smallest eigenvalue is obtained on the coarsest mesh, as the second step, the value is used as the initial guess for the next mesh. This is repeated until we are done with the finest mesh. **Algorithm 2** gives the multi-level procedure. Suppose we have meshes $L_1, \ldots, L_J$ from coarse to fine. The detailed algorithms are given below.

Algorithm 1: On the coarsest mesh L_1

Input: coarsest mesh L_1 for D
Input: $n(x), A(x)$ and n^*, n_*, A^*, A_*
Input: the increment of the spectral transformation Δk
Output: the smallest transmission eigenvalue
construct mass and stiff matrices
construct matrices $\mathcal{A}, \mathcal{B}$ from mass and stiff matrices
compute C_p and the initial guess k_0 using (34) or (35)
While
  $k = eigs(\mathcal{A}, \mathcal{B}, 1, k_0);$
  If no real eigenvalue found
    $k_0 = k_0 + \Delta k;$
  Else
    Return the smallest real eigenvalue k
  End
End

Algorithm 2: Multi-level algorithm

Input the index of refraction $n(x), A(x)$ and n^*, n_*, A^*, A_*
Do k = Algorithm1 on mesh L_1
Set $i = 1$
While $i < J$
  $i = i + 1$
  Input the regular triangular mesh L_i for D
  construct matrices $\mathcal{A}, \mathcal{B}$
  $k = eigs(\mathcal{A}, \mathcal{B}, 1, k);$
End

5. Numerical Examples

In this section, we present some preliminary examples to compute the smallest transmission eigenvalue. For simplicity, we use linear elements for all the examples.

Table 1: The computed smallest transmission eigenvalue of the disk.

meshes	DoF	Case I	Case II	Case III
L1	1018	5.8700	4.9259	3.9904
L2	4066	5.8214	4.8802	3.9646
L3	16258	5.8093	4.8686	3.9581
L4	65026	5.8062	4.8657	3.9565
L5	260098	5.8055	4.8649	3.9561

Table 2: The computed smallest transmission eigenvalues of the unit square.

meshes	DoF	Case I	Case II	Case III
L1	1298	5.3505	4.4275	3.6063
L2	5186	5.3116	4.3970	3.5878
L3	20738	5.3018	4.3892	3.5831
L4	82946	5.2994	4.3873	3.5819
L5	331778	5.2988	4.3868	3.5816

5.1. 2D examples

We first consider a couple of 2D examples. Let $n = 1$ and we choose three cases for A

$$\text{(I)}\ \frac{1}{4}I, \quad \text{(II)} \begin{pmatrix} 1/2 & 0 \\ 0 & 1/8 \end{pmatrix}, \quad \text{(III)} \begin{pmatrix} 1/6 & 0 \\ 0 & 1/8 \end{pmatrix}.$$

The first example is a disk with radius 1/2. For Case I, we first use the result (the smallest root of (5)) in Section 2 to obtain the exact smallest transmission eigenvalue which is 5.8052. We generate five levels of meshes, denoted by L1, L2, L3, L4, and L5 from the coarsest level to the finest level, respectively. The result is shown in Table 1. The second column denotes the degrees of freedom of the generalized matrix eigenvalue problem. It is clear that the computed eigenvalue converges numerically. In particular, for Case I, the value is consistent with the exact solution.

The second example is the unit square. We also generate five level meshes. The result is shown in Table 2.

The third example is an L-shape domain given by

$$(-0.5, 0.5) \times (-0.5, 0.5) \setminus [0, 0.5] \times [-0.5, 0].$$

Again, we consider five level meshes and show the numerical result in Table 3. We can see that the computed smallest eigenvalue converges numerically.

Table 3: The computed smallest transmission eigenvalues of the L-shape domain.

meshes	DoF	Case I	Case II	Case III
L1	978	6.8438	6.0899	4.3490
L2	3906	6.7586	5.9757	4.3151
L3	15618	6.7364	5.9456	4.3060
L4	62466	6.7304	5.9377	4.3035
L5	249858	6.7288	5.9355	4.3028

Table 4: The computed smallest transmission eigenvalues of the unit ball.

meshes	DoF	Case I	Case II	Case III
L1	6580	1.1837	1.2236	0.9691
L2	21282	1.1761	1.2172	0.9609
L3	49792	1.1720	1.2139	0.9576
L4	169136	1.1687	1.2112	0.9552

5.2. 3D Examples

We consider three groups of A and N, i.e.,
I) $A = I, N = 16I$;
II) $A = 1/2I, N = 8I$;
III)

$$A = \begin{pmatrix} 5/9 & 0 & 9 \\ 0 & 1/2 & 0 \\ 0 & 0 & 5/11 \end{pmatrix}, \quad N = \begin{pmatrix} 8 & 1 & 2 \\ 1 & 10 & 3 \\ 2 & 3 & 12 \end{pmatrix}.$$

The first example is the unit ball with four levels of tetrahedral meshes for it. The smallest transmission eigenvalues can be obtained exactly using the result in Section 3, i.e., roots of (20) and (21) for Case I and Case II, respectively. They are 1.1654 and 1.2093. We show the result in Table 4. The values are consistent with the exact values and those obtained in [21].

Next example is a "hockey puck" given by $x^2 + y^2 \leq 9$, $-1/2 \leq z \leq 1/2$. We generate three levels of meshes. The result is shown in Table 5.

The third example is the unit ball with a hole inside. The hole is a small ball centered at origin with radius $\delta = 0.1$. Three levels of meshes are used. The result is shown in Table 6.

Table 5: The computed smallest transmission eigenvalues of the "hockey puck".

meshes	DoF	Case I	Case II	Case III
L1	17874	1.0675	1.1501	1.0431
L2	39870	1.0666	1.1500	1.0430
L3	141816	1.0590	1.1440	1.0385

Table 6: The computed smallest transmission eigenvalues of the unit ball with a hole inside.

meshes	DoF	Case I	Case II	Case III
L1	24656	1.1866	1.2335	0.9699
L2	92837	1.1827	1.2308	0.9672
L3	292088	1.1802	1.2286	0.9653

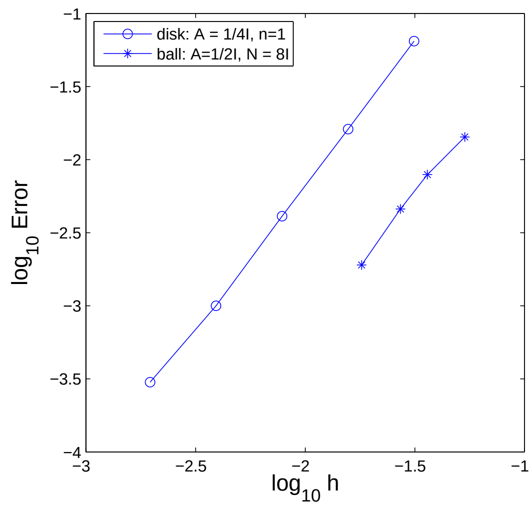

Figure 1: Convergence rate of the finite element method is roughly $O(h^2)$.

5.3. Convergence and comparison

Since we have exact transmission eigenvalues for the cases of a disk with $A = 1/4I$ and $n = 1$ and the unit ball with $A = 1/2I$ and $N = 8I$, we plot the convergence rate in Fig. 1. Due to the fact that we use linear elements for both cases, we have second order convergence as expected.

For both the Helmholz case and the Maxwell's case, using suitable substitutions, we are able to end up with generalized eigenvalue problems with symmetric $\mathcal{A}$ and $\mathcal{B}$ as described in Sections 2 and 3. In Table. 7, we show the time (in second) used by Arnoldi method for symmetric pairs and non-symmetric pairs with $A = I$ and $N = 16I$ for 3D cases. It can be seen that there is no significant difference between the third and fourth columns. The result verifies that Arnoldi iteration does not take the advantage of the nicer symmetric structure of $\mathcal{A}$ and $\mathcal{B}$.

Table 7: The time (in seconds) used for symmetric pairs and non-symmetric $\mathcal{A}$ and $\mathcal{B}$.

Problem	DoF	Symmetric	Non-symmetric
Unit Ball	49792	36.01	36.64
Hockey Puck	39870	11.82	12.01
Unit Ball with Cavity	24656	9.44	9.41

6. Conclusions and future works

In this paper, we study finite element methods for transmission eigenvalues for anisotropic media. The generalized matrix eigenvalue problems from the finite element discretization are sparse, large, and non-Hermitian. We propose a multi-level Arnoldi method for the matrix eigenvalue problems. At the coarsest mesh, the smallest real eigenvalue is obtained using Arnoldi iteration with some suitable searching technique. Then this eigenvalue serves as the spectral transformation for the problem on the finer mesh. The accurate initial guess makes the algorithm efficient and robust.

The convergence of the proposed finite element methods is still open. Faster techniques for even larger non-Hermitian generalized matrix eigenvalue problems are desirable. These are interesting and important projects for future works.

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