Analytical and Computational Methods for Transmission Eigenvalues

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Abstract

The interior transmission problem is a boundary value problem that arises in the scattering of time-harmonic waves by an inhomogeneous medium of compact support. The associated transmission eigenvalue problem has important applications in qualitative methods in inverse scattering theory. In this paper, we first establish optimal conditions for the existence of transmission eigenvalues for a spherically stratified medium and give numerical examples of the existence of both real and complex transmission eigenvalues in this case. We then propose three finite element methods for the computation of the transmission eigenvalues for the cases of a general non-stratified medium and use these methods to investigate the accuracy of recently established inequalities for transmission eigenvalues.

1 Introduction

In recent years the interior transmission eigenvalue problem has come to play an important role in inverse scattering theory [13, 12]. This is due to the fact that transmission eigenvalues can be determined from the far field data of the scattered waved and used to obtain estimates for the material properties of the scattering object [3, 4, 7]. For the case of scattering of acoustic waves by a bounded simply connected inhomogeneous medium $D \subset \mathbb{R}^3$, the interior transmission eigenvalue problem is to find $k \in \mathbb{C}$, $w, v \in L^2(D)$, $w - v \in H^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0, \quad \text{in } D, \tag{1.1a}$$

$$\Delta v + k^2 v = 0, \quad \text{in } D, \tag{1.1b}$$

$$w - v = 0, \quad \text{on } \partial D, \tag{1.1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial D, \tag{1.1d}$$

where ν is the unit outward normal to the smooth boundary ∂D and the index of refraction $n(x)$ is positive. Values of $k \neq 0$ such that there exists a nontrivial

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solution to (1.1) are called transmission eigenvalues. Research on transmission eigenvalues has focused on two main themes: 1) conditions on $n \in L^\infty(D)$ such that transmission eigenvalues exist and form a discrete set [10, 18, 16, 8, 15], and 2) the determination of upper and lower bounds for $n(x)$ from a knowledge of the transmission eigenvalues corresponding to (1.1) [13, 7, 8, 5, 6]. The purpose of this paper is to contribute to both of these two themes as well as to provide the first numerical study of the computation of transmission eigenvalues.

In order for transmission eigenvalues to form a discrete set it is clearly necessary that $n(x)$ is not identically equal to one. For general $n \in L^\infty(D)$ the sharpest conditions to date on $n(x)$ for transmission eigenvalues to exist and form a discrete set are that $n(x)$ is either greater than or less than one in $\bar{D}$ [7, 6]. This is clearly not optimal since for the case when $n(x) = n(r)$ depends only on $r = |x|$ it can be shown [13] that transmission eigenvalues exist and form a discrete set provided

$$\int_0^a \sqrt{n(r)}\, dr \neq a \tag{1.2}$$

where D is the ball $\{x : |x| < a\}$. In Section 2 of this paper we will show that condition (1.2) can be replaced by the sharper (and optimal!) condition that $n(r)$ is not identically equal to one for $0 \leq r \leq a$. Our proof is also valid for radially symmetric disks in $\mathbb{R}^2$. In Section 3 we will numerically compute transmission eigenvalues for the case when $n(x)$ is constant and D is a disk in $\mathbb{R}^2$.

In two recent papers [7, 6], Cakoni, Gintides and Haddar obtained upper and lower bounds on $n(x)$ in terms of transmission eigenvalues for balls with constant index of refraction. In particular, they proved the following theorem:

Theorem 1.1. *Let $n(x) \in L^\infty(D)$ and let B_1 be the largest ball such that $B_1 \subset D$ and B_2 the smallest ball such that $D \subset B_2$. Then*

1) If $1+\gamma \leq n_* \leq n(x) \leq n^* < \infty$ then

$$0 < k_{1,B_2,n^*} \leq k_{1,D,n(x)} \leq k_{1,B_1,n_*}.$$

2) If $0 < n_* \leq n(x) \leq n^* < 1-\beta$ then

$$0 < k_{1,B_2,n_*} \leq k_{1,D,n(x)} \leq k_{1,B_1,n^*}.$$

Here k_{1,B_i,n_*} and k_{1,B_i,n^*}, $i = 1,2$ are the first (real) transmission eigenvalues corresponding to the ball B_i with constant index of refraction n_* and n^* respectively, $k_{1,D,n(x)}$ is the first transmission eigenvalue of D with index of refraction $n(x)$.

As opposed to previous estimates obtained via Faber-Krahn type inequalities [13, 5], these bounds are clearly sharp for $n(x)$ equal to a constant and D a ball. In particular, Theorem 1.1 shows that for constant index of refraction the first transmission eigenvalue depends monotonically on the index of refraction. Thus from a knowledge of the first transmission eigenvalue for D and $n(x)$ and the balls B_1 and B_2 we can obtain (in case 1 of Theorem 1.1) a lower bound for $\sup n$ and an upper bound for $\inf n$ with similar estimates holding in case 2 of Theorem 1.1. In order to compare these new estimates with Faber-Krahn type inequalities, in Section 4 of this paper we will devise finite element methods to compute transmission eigenvalues and in Section 5 will use these results to test the accuracy of the estimates in the above theorem.

2 Transmission Eigenvalues for Spherically Stratified Media

2.1 Existence of Transmission Eigenvalues

We consider the interior transmission eigenvalue problem (1.1) when $n(x) = n(r)$ is spherically stratified, D is the ball $\{x : |x| < a\}$ and $n \in C^2[0, a]$. In this case we can expand v and w in a series of spherical harmonics

$$v(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m j_l(kr) Y_l^m(\hat{x}), \tag{2.3a}$$

$$w(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_l^m y_l(r) Y_l^m(\hat{x}), \tag{2.3b}$$

where $r = |x|$, $\hat{x} = x/|x|$, j_l is a spherical Bessel function of order l and y_l is a real valued solution of

$$y'' + \frac{2}{r} y' + \left(k^2 n(r) - \frac{l(l+1)}{r^2} \right) y = 0 \tag{2.4}$$

normalized such that $y_l(r)$ behaves like $j_l(kr)$ as $r \to 0$. From [11] we can represent this solution in the form

$$y_l(r) = j_l(kr) + k^2 \int_0^r G(r, s, k) j_l(ks) ds \tag{2.5}$$

where G is real valued and twice continuously differentiable for $0 \le s \le r$ and is an even entire function of k of finite exponential type. Setting $f_l(r) = r y_l(r)$ we see from (2.4) that f_l satisfies

$$f'' + \left(k^2 n(r) - \frac{l(l+1)}{r^2} \right) f = 0 \tag{2.6}$$

and from Section 9.4 of [12] (There is a misprint in Theorem 9.9 of [12]: λ should be $\lambda = l + 1/2$) we can deduce that for fixed $r > 0$ f_l is a bounded function of k as $k \to \infty$. Hence for fixed $r > 0$, y_l is an entire function of k of finite exponential type that is bounded for k on the positive real axis.

Theorem 2.1. *Assume that $n(x) = n(r)$ is spherically stratified, D is the ball $\{x : |x| < a\}$ and $n \in C^2[0, a]$. Then if $n(r)$ is not identically equal to one there exist a countably infinite number of transmission eigenvalues for* (1.1).

Proof. Assume that $n = n(r)$ satisfies the hypothesis of the theorem. A necessary and sufficient condition for k to be a transmission eigenvalue is that k is a zero of the determinant

$$d_l(k) := \det \begin{vmatrix} y_l(a) & -j_l(ka) \\ y_l'(a) & -k j_l'(ka) \end{vmatrix}$$

for some non-negative integer l (c.f. Section 9.4 of [12]). Since the spherical Bessel functions are entire functions of k of finite exponential type and bounded for k on the positive real axis, by the above discussion we see that $d_l(k)$ also has this property. Furthermore, by the series expansion of j_l [12], we see that $d_l(k)$ is an even function of k and $d_l(0) = 0$. Hence if $d_l(k)$ does not have a countably infinite number of zeros, by the Hadamard factorization theorem [20] $d_l(k)$ must be identically zero. We will now show that $d_l(k)$ is not identically zero for every l unless $n(r)$ is identically equal to one.

Assume that $d_l(k)$ is identically zero for every non-negative integer l. Noticing that $j_l(kr)Y_l^m(\hat{x})$ is a Herglotz wave function, it follows from the proof of Theorem 8.16 in [12] that

$$\int_0^a j_l(k\rho)y_l(\rho)\rho^2 m(\rho)\,d\rho = 0 \tag{2.7}$$

for all k where $m(r) := 1 - n(r)$. Hence, using the Taylor series expansion of $j_l(k\rho)$ and (2.5) we see that

$$\int_0^a \rho^{2l+2} m(\rho)\,d\rho = 0$$

for all non-negative integers l. By Muntz's theorem [20] we now have that $m(r) = 0$, i.e., $n(r) = 1$ and the proof is complete. □

2.2 Non-existence of Purely Imaginary Transmission Eigenvalues

Having shown that transmission eigenvalues exist in the case of a spherically stratified medium, it would be of interest to determine where they are located in the complex k plane. We will show later that numerical evidence suggests that complex eigenvalues exist in the case of a spherically stratified medium. However it can be shown that in general if $n(x)$ is never equal to one then purely imaginary transmission eigenvalues do not exist.

Theorem 2.2. *Assume $n(x) > 1$ for $x \in \bar{D}$ or $n(x) < 1$ for $x \in \bar{D}$. Then there are no purely imaginary transmission eigenvalues.*

Proof. From [18] we see that a weak formulation of (1.1) is to find a nontrivial solution $u \in H_0^2(D)$ and $k \in \mathbb{C}$ such that

$$\int_D \frac{1}{n-1}(\triangle u + k^2 u)(\triangle \bar{v} + k^2 n \bar{v})dx = 0 \tag{2.8}$$

for all $v \in H_0^2(D)$. Let $n(x) > 1$ for $x \in \bar{D}$ and assume, contrary to the statement of the theorem, that there exist purely imaginary transmission eigenvalues. Then $\frac{1}{n-1} \geq \sigma > 0$ and following [8] we define

$$A_\tau(u,v) = \left(\frac{1}{n-1}(\triangle u + \tau u), (\triangle v + \tau v)\right) + \tau^2(u,v), \tag{2.9}$$

$$B(u,v) = (\nabla u, \nabla v), \tag{2.10}$$

where $\tau = k^2$. Then (2.8) can be written as

$$A_\tau(u,v) - \tau B(u,v) = 0 \text{ for all } v \in H_0^2(D).$$

If k is pure imaginary, $\tau = -\sigma < 0$ with $\sigma > 0$. Setting $v = u$, we have

$$\begin{aligned} 0 &= A_\tau(u,u) + \sigma B(u,u) \\ &\geq \sigma^2(u,u) + \sigma(\nabla u, \nabla u) \end{aligned}$$

and this implies $u = 0$, a contradiction.

Similarly, if $n < 1$, then $\frac{n}{1-n} \geq \sigma > 0$. Let

$$\begin{aligned} \tilde{A}_\tau(u,v) &= \left(\frac{1}{n-1}(\triangle u + \tau n u), (\triangle v + \tau n v)\right) + \tau^2(nu, v) \\ &= \left(\frac{n}{1-n}(\triangle u + \tau u, (\triangle v + \tau v)\right) + (\triangle u, \triangle v). \end{aligned} \tag{2.11}$$

Then

$$\begin{aligned} 0 &= \tilde{A}_\tau(u,u) + \sigma B(u,u) \\ &\geq (\triangle u, \triangle u) + \sigma(\nabla u, \nabla u). \end{aligned}$$

By Poincare's inequality this again implies $u = 0$ and the proof is complete. $\square$

3 Computation of Transmission Eigenvalues in the Case of Spherically Stratified Media

3.1 Real Transmission Eigenvalues

Let D be a disk of radius R and let the index of refraction n be a positive real constant. Solutions of the Helmholtz equation $\triangle v + k^2 v = 0$ in D are

$$J_m(kr)\cos m\theta, \quad J_m(kr)\sin m\theta, \quad m \geq 0, \tag{3.12}$$

where J_m is first kind Bessel function of order m. Solutions of the Helmholtz equation $\triangle w + k^2 n w = 0$ in D are

$$J_m(k\sqrt{n}r)\cos m\theta, \quad J_m(k\sqrt{n}r)\sin m\theta, \quad m \geq 0. \tag{3.13}$$

For a fixed m, in order to make $v - w$ vanish on ∂D, one can choose

$$v = J_m(kr)\cos m\theta, \quad w = \frac{J_m(kR)}{J_m(k\sqrt{n}R)} J_m(k\sqrt{n}r)\cos m\theta, \quad m \geq 0.$$

Then the transmission eigenvalues are those k's such that

$$\frac{\partial v}{\partial r} = \frac{\partial w}{\partial r} \quad \text{on} \quad \partial D.$$

Using the recursive formula for the derivatives of Bessel's functions, we have

$$\begin{aligned} \frac{\partial J_m(kr)}{\partial r} &= k\left(J_{m-1}(kr) - \frac{m}{kr}J_m(kr)\right), \\ \frac{\partial J_m(k\sqrt{n}r)}{\partial r} &= k\sqrt{n}\left(J_{m-1}(k\sqrt{n}r) - \frac{m}{k\sqrt{n}r}J_m(k\sqrt{n}r)\right). \end{aligned}$$

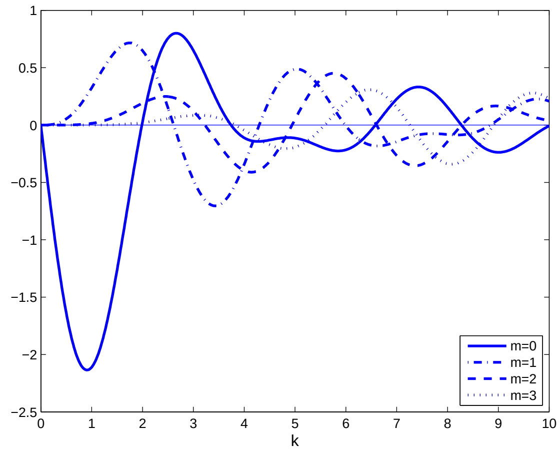

Figure 1: The plot of the determinants d_m against k for $m = 0, 1, 2, 3$. The transmission eigenvalues are the intersections of the curves and the x-axis.

Then the eigenvalues are k's such that

$$J_1(kR)J_0(k\sqrt{n}R) = \sqrt{n}J_0(kR)J_1(k\sqrt{n}R), \quad m = 0,$$

and

$$J_{m-1}(kR)J_m(k\sqrt{n}R) = \sqrt{n}J_m(kR)J_{m-1}(k\sqrt{n}R), \quad m \geq 1.$$

The case of $m = 0$ corresponds to the spherically stratified media for constant index of refraction [13].

Considering a simple case when $R = 1/2$ and $n = 16$, we have

$$J_1(k/2)J_0(2k) = 4J_0(k/2)J_1(2k), \quad m = 0,$$

and

$$J_{m-1}(k/2)J_m(2k) = 4J_m(k/2)J_{m-1}(2k), \quad m \geq 1.$$

In Fig. 1, we plot the value d_m against the wavenumber k where

$$d_m = J_1(k/2)J_0(2k) - 4J_0(k/2)J_1(2k), \quad m = 0, \tag{3.14}$$

$$d_m = J_{m-1}(k/2)J_m(2k) - 4J_m(k/2)J_{m-1}(2k), \quad m = 1, 2, 3. \tag{3.15}$$

The transmission eigenvalues are those k's where $d_m = 0$. From Fig. 1, we see that the distribution of the (real) transmission eigenvalues are quite complicated. In Table 1, we show the computed transmission eigenvalues using a root finding software. Note that the eigenvalues for $m > 0$ have multiplicity 2 since the above derivation works for both $\cos m\theta$ and $\sin m\theta$, $m > 0$ in (3.12) and (3.13).

3.2 Complex Transmission Eigenvalues

It is possible to search for transmission eigenvalues in the whole complex plane $\mathbb{C}$. Consider the case for $m = 0$ in the above example. Define

$$Z_0(k) = J_1(kR)J_0(k\sqrt{n}R) - \sqrt{n}J_0(kR)J_1(k\sqrt{n}R).$$

Table 1: Transmission eigenvalues corresponding to different m's on a disk with $R = 1/2$ and $n = 16$. These values are computed from (3.14) and (3.15) using a numerical root finding software.

$m = 0$	1.9880	3.7594	6.5810
$m = 1$	2.6129	4.2954	5.9875
$m = 2$	3.2240	4.9462	6.6083
$m = 3$	3.8248	5.5870	7.2591
$m = 4$	4.4556	6.2278	7.9099

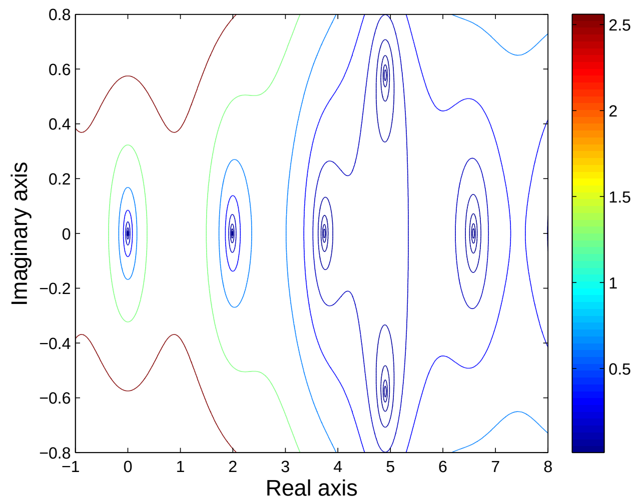

Figure 2: The contour plot of abs(Z_0) suggests the existence of complex transmission eigenvalues around $k = 4.901 \pm 0.5781i$.

Then the zeros of $Z_0(k)$, if they exist, are transmission eigenvalues, including the real k's given above. Again let $R = 1/2$ and $n = 16$. Using root finding software, we find that there exist a pair of complex transmission eigenvalues around $k = 4.901 \pm 0.5781i$ along with other real and complex eigenvalues. In Fig. 2 we plot abs(Z_0) in a neighborhood of the origin. It can be seen that in addition to real transmission eigenvalues, there are also complex transmission eigenvalues. Note that since we require that $n(x)$ is real, the complex transmission eigenvalues must appear in complex conjugate pairs.

4 Finite Element Methods

We now present three finite element methods to compute transmission eigenvalues for general $n(x)$. The first method is based on a fourth order formulation for the interior transmission problem and will be discretized by Argyris finite elements. Hence we call it the Argyris method. The second method is the continuous finite element method which uses standard finite elements and equations (1.1a)-(1.1d) directly. The third method is the mixed finite element method which is based on writing one of equations (1.1a) or (1.1b) as a first order system.

4.1 The Argyris Method

In this section we present a finite element method based on a fourth order formulation (2.8). In particular, let $X_h \subset H_0^2(D)$ be the finite element space associated with a regular mesh of Argyris triangles [14]. Then the discrete formulation for the transmission eigenproblem is: find $u_h \in X_h, u_h \neq 0$ and $k \in \mathbb{C}$ such that

$$\int_D \frac{1}{n-1}(\triangle u_h + k^2 u_h)(\triangle v_h + k^2 n v_h) dx = 0 \text{ for all } v_h \in X_h. \tag{4.16}$$

Let $\{\xi_j\}_{j=1}^{N_h}$ be a basis for X_h and $u_h = \sum_{i=1}^{N_h} u_j \xi_j$. Note that the $\{u_j\}$ are not nodal values of u_h, but instead correspond to the standard degrees of freedom for Argyris elements. We need the following matrices in the discrete case:

Matrix	Dimension	Definition
A	$N_h \times N_h$	stiffness matrix: $A_{j,\ell} = \int_D \frac{1}{n-1} \triangle \xi_j \cdot \triangle \xi_\ell \, dx$
B	$N_h \times N_h$	matrix: $B_{j,\ell} = \int_D \triangle \frac{1}{n-1} \xi_j \, \xi_\ell + \frac{n}{n-1} \triangle \xi_j \xi_\ell \, dx$
C	$N_h \times N_h$	mass matrix, $C_{j,\ell} = \int_D \frac{n}{n-1} \xi_j \, \xi_\ell \, dx$

This leads to a quadratic eigenvalue problem

$$(A + k^2 B + k^4 C)\vec{u} = 0 \tag{4.17}$$

where $\vec{u} = (u_1, u_2, \ldots, u_{N_h})^T$. The following theorem shows the existence of eigenvalues of the above quadratic eigenvalue problem.

Theorem 4.1. *Assume $n(x) > 1$ for $x \in \bar{D}$ or $n(x) < 1$ for $x \in \bar{D}$. Then there exist $2N_h$ eigenvalues of (4.17).*

Proof. Rather than solve the quadratic eigenvalue problem directly, we will use the linearization technique [19] and write the problem as a generalized eigenvalue problem. Let $\vec{x} = k^2 \vec{u}$ and $\vec{w} = (\vec{u}^T, \vec{x}^T)^T$. Then (4.17) can be rewritten as a generalized eigenvalue problem

$$\mathcal{A}\vec{w} = k^2 \mathcal{B}\vec{w} \tag{4.18}$$

where $\mathcal{A}$ and $\mathcal{B}$ are $2N_h \times 2N_h$ matrices given by

$$\mathcal{A} = \begin{pmatrix} 0 & I_{N_h} \\ -A & -B \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} I_{N_h} & 0 \\ 0 & C \end{pmatrix} \tag{4.19}$$

where I_{N_h} is the $N_h \times N_h$ identity matrix. Due to the properties of Argyris finite element space X_h [9], the matrix $\mathcal{B}$ is positive definite if $n(x) > 1$ for $x \in \bar{D}$. Thus there exist $2N_h$ eigenvalues of (4.18). The case of $n(x) < 1$ for $x \in \bar{D}$ can be treated similarly.

Define two auxiliary matrices

$$\mathcal{E}(k^2) = \begin{pmatrix} -(B + k^2 C) & -I_{N_h} \\ I_{N_h} & 0 \end{pmatrix}, \quad \mathcal{F}(k^2) = \begin{pmatrix} I_{N_h} & 0 \\ k^2 I_{N_h} & I_{N_h} \end{pmatrix}. \tag{4.20}$$

Thus we have

$$
\left(\begin{array}{cc} k^4C+k^2B+A & 0 \\ 0 & I_{N_h} \end{array}\right) = \mathcal{E}(k^2)(\mathcal{A}-k^2\mathcal{B})\mathcal{F}(k^2). \tag{4.21}
$$

Since both det$\mathcal{E}$ and det$\mathcal{F}$ are nonzero constants, the eigenvalues of k^4C+k^2B+A and $\mathcal{A}-k^2\mathcal{B}$ coincide. This implies the existence of $2N_h$ eigenvalues of the quadratic eigenvalue problem (4.17). $\square$

4.2 Continuous Finite Element Approximation

We now describe a method, which is based on an idea similar to [15], for using standard piecewise linear finite elements to discretize the interior transmission problem. Let

$$
\begin{array}{rcl}
S_h &=& \text{the space of continuous piecewise } p \text{ degree finite elements on } D,\\
S_h^0 &=& S_h \cap H_0^1(D)\\
&=& \text{the subspace of functions in } S_h \text{ that have vanishing DoF on } \partial D,\\
S_h^B &=& \text{the subspace of functions in } S_h \text{ that have vanishing DoF in } D,
\end{array}
$$

where DoF stands for degrees of freedom. We want to find a finite element approximation to the system (1.1). Let

$$
\begin{array}{rcl}
w_h &=& w_{0,h}+w_{B,h} \text{ where } w_{0,h}\in S_h^0 \text{ and } w_{B,h}\in S_h^B,\\
v_h &=& v_{0,h}+w_{B,h} \text{ where } v_{0,h}\in S_h^0.
\end{array}
$$

At this stage w_h is a general function in S_h but v_h is chosen so that $v_h = w_h$ on ∂D and hence (1.1c) is always satisfied. To satisfy (1.1a) we choose a test function $\xi_h \in S_h^0$ and derive the standard weak formulation for w_h as:

$$
\int_D \nabla(w_{0,h}+w_{B,h})\cdot\nabla\xi_h\,dx - k^2\int_D n(w_{0,h}+w_{B,h})\xi_h\,dx = 0 \quad \forall \xi_h\in S_h^0. \tag{4.22}
$$

In the same way equation (1.1b) is discretized by seeking v_h such that

$$
\int_D \nabla(v_{0,h}+w_{B,h})\cdot\nabla\eta_h\,dx - k^2\int_D (v_{0,h}+w_{B,h})\eta_h\,dx = 0 \quad \forall \eta_h\in S_h^0. \tag{4.23}
$$

We now need to implement (1.1d). Choosing $\gamma\in H^1(D)$ we multiply both equations (1.1a) and (1.1b) by this function and integrate by parts to obtain

$$
\begin{array}{rcl}
\displaystyle\int_D \nabla w\cdot\nabla\gamma\,dx - k^2\int_D nw\gamma\,dx - \int_{\partial D}\frac{\partial w}{\partial\nu}\gamma\,ds &=& 0,\\
\displaystyle\int_D \nabla v\cdot\nabla\gamma\,dx - k^2\int_D v\gamma\,dx - \int_{\partial D}\frac{\partial v}{\partial\nu}\gamma\,ds &=& 0.
\end{array}
$$

Subtracting these equations we get

$$
\int_D \nabla(w-v)\cdot\nabla\gamma\,dx - k^2\int_D (nw-v)\gamma\,dx = 0.
$$

This gives us (substituting the expansion for w_h in place of w and v_h for v and using a discrete test function):

$$
\int_D \nabla(w_{0,h}-v_{0,h})\cdot\nabla\gamma_h\,dx - k^2\int_D (n(w_{0,h}+w_{B,h})-(v_{0,h}+w_{B,h}))\gamma_h\,dx = 0 \quad \forall\gamma_h\in S_h^B
$$

where the boundary terms cancel in the first term on the left hand side above, but not in the second term.

We now write these equations in matrix form. Let $\{\xi_j\}_{j=1}^{N_h^0}$ denote a basis for the finite element space S_h^0. For piecewise linear elements N_h^0 is the number of vertices of the mesh in the interior of D. Let $\{\gamma_j\}_{j=1}^{N_h^B}$ denote a basis for S_h^B. In this case, for piecewise linear elements we can take this to be the standard nodal basis associated with vertices on ∂D. So a basis for S_h is $\{\xi_j\}_{j=1}^{N_h^0} \cup \{\gamma_j\}_{j=1}^{N_h^B}$. We can write $w_{0,h} = \sum_{j=1}^{N_h^0} w_j \xi_j$ and associate a vector of degrees of freedom $\vec{w} = (w_1, w_2, \cdots)^T$. Similarly $v_{0,h} = \sum_{j=1}^{N_h^0} v_j \xi_j$ and associate a vector of degrees of freedom $\vec{v} = (v_1, v_2, \cdots)^T$. Finally write $w_{B,h} = \sum_{j=1}^{N_h^B} u_j \gamma_j$ and associate a vector of degrees of freedom for w or v on ∂D by $\vec{u} = (u_1, u_2, \cdots)^T$.

We need the following matrices:

Matrix	Dimension	Definition
A	$N_h^0 \times N_h^0$	interior space stiffness matrix, $A_{j,\ell} = \int_D \nabla \xi_j \cdot \nabla \xi_\ell \, dx$
B_n, B_1	$N_h^0 \times N_h^B$	boundary/interior mass matrices, $(B_n)_{j,\ell} = \int_D n \xi_j \, \gamma_\ell \, dx$
		$(B_1)_{j,\ell} = \int_D \xi_j \, \gamma_\ell \, dx$
C_n, C_1	$N_h^B \times N_h^B$	boundary space mass matrices, $(C_n)_{j,\ell} = \int_D n \gamma_j \, \gamma_\ell \, dx$
		$(C_1)_{j,\ell} = \int_D \gamma_j \, \gamma_\ell \, dx$
D	$N_h^0 \times N_h^B$	boundary/interior stiffness matrix, $D_{j,\ell} = \int_D \nabla \xi_j \cdot \nabla \gamma_\ell \, dx$
M_n, M_1	$N_h^0 \times N_h^0$	interior space mass matrices, $(M_n)_{j,\ell} = \int_D n \xi_j \, \xi_\ell \, dx$
		$(M_1)_{j,\ell} = \int_D \xi_j \, \xi_\ell \, dx$

The eigenvalue problem we now have to solve is the generalized problem

$$\mathcal{A}\vec{x} = k^2 \mathcal{B}\vec{x} \tag{4.24}$$

where $\vec{x}$ has dimension $2N_h^0 + N_h^B$ and

$$\vec{x} = \begin{pmatrix} \vec{w} \\ \vec{v} \\ \vec{u} \end{pmatrix}.$$

The matrices $\mathcal{A}$ and $\mathcal{B}$ are given blockwise by

$$\mathcal{A} = \begin{pmatrix} A & 0 & D \\ 0 & A & D \\ D^T & -D^T & 0 \end{pmatrix}$$

and

$$\mathcal{B} = \begin{pmatrix} M_n & 0 & B_n \\ 0 & M_1 & B_1 \\ B_n^T & -B_1^T & C_n - C_1 \end{pmatrix}.$$

4.3 Mixed Finite Element Method

The transmission eigenvalue problem can be treated by a mixed finite element method [2, 1]. Introducing $\mathbf{u} = \nabla v$ and rewriting (1.1), we obtain

$$\Delta w + k^2 n w = 0 \qquad \text{in } D, \tag{4.25a}$$

$$\mathbf{u} - \nabla v = 0 \qquad \text{in } D, \tag{4.25b}$$

$$\nabla \cdot \mathbf{u} + k^2 v = 0 \qquad \text{in } D, \tag{4.25c}$$

$$w - v = 0, \qquad \text{on } \partial D, \tag{4.25d}$$

$$\frac{\partial w}{\partial \nu} - \mathbf{u} \cdot \nu = 0 \qquad \text{on } \partial D. \tag{4.25e}$$

Let

$$H(\text{div}; D) = \{\mathbf{u} \in (L^2(D))^2 | \nabla \cdot \mathbf{u} \in L^2(D)\}.$$

The weak formulation for this problem is to find $k \in \mathbb{C}$, $w \in H^1(D)$, $\mathbf{u} \in H(\text{div}, D)$, and $v \in L^2(D)$ such that

$$\int_D -\nabla w \cdot \nabla \phi + k^2 n w \phi \, dx + \int_{\partial D} \mathbf{u} \cdot \nu \phi \, ds = 0, \qquad \forall \phi \in H^1(D), \tag{4.26a}$$

$$\int_D \mathbf{u} \cdot \tau \, dx + \int_D v \nabla \cdot \tau \, dx - \int_{\partial D} \tau \cdot \nu w \, ds = 0, \quad \forall \tau \in H(\text{div}; D), \tag{4.26b}$$

$$\int_D \nabla \cdot \mathbf{u} q \, dx + k^2 \int_D v q \, dx = 0, \qquad \forall q \in L^2(D). \tag{4.26c}$$

Now suppose the discrete finite element spaces are given by $W_h \subset H^1(D)$, $U_h \subset H(\text{div}, D)$ and $V_h \subset L^2(D)$. In particular, we choose the standard piecewise linear finite element for W_h and Raviart-Thomas [17] element of the lowest order for U_h and V_h. Assume that $w_h = \sum_{i=1}^N w_i \xi_i$, $v_h = \sum_{i=1}^J v_i \eta_i$, $\mathbf{u}_h = \sum_{i=1}^K u_i \tau_i$, where $\xi_i, i = 1, \ldots, N$, $\eta_i, i = 1, \ldots, J$, $\tau_i, i = 1, \ldots, K$ are basis for W_h, U_h and V_h respectively. Let $\vec{v} = (v_1, v_2, \ldots, v_J)^T$, $\vec{w} = (w_1, w_2, \ldots, w_N)^T$ and $\vec{u} = (u_1, u_2, \ldots, u_K)^T$. We need the following matrices

Matrix	Dimension	Definition
S	$N \times N$	stiffness matrix, $S^{ij} = \int_D \nabla \xi_j \nabla \xi_i \, dx$
M_n	$N \times N$	mass matrix, $M_n^{i,j} = \int_D n \xi_j \xi_i \, dx$
M_u	$K \times K$	mass matrix, $M_u^{i,j} = \int_D \tau_j \tau_i \, dx$
M_v	$J \times J$	mass matrix, $M_v^{i,j} = \int_D n \tau_j \eta_i \, dx$
A	$N \times J$	$A^{i,j} = \int_D \nabla \cdot \tau_j w_i \, dx$
B_u	$K \times N$	$B_u^{i,j} = \int_{\partial D} u_j \cdot \nu \xi_i \, ds$

Now we have

$$\begin{pmatrix} -S + k^2 M_n & B_u & 0 \\ -B_u^T & M_u & A \\ 0 & A^T & k^2 M_v \end{pmatrix} \begin{pmatrix} \vec{w} \\ \vec{u} \\ \vec{v} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.27}$$

Solving $\vec{u}$ in terms of $\vec{w}$ and $\vec{v}$ we obtain

$$\vec{u} = M_u^{-1}(B_u^T \vec{w} - A\vec{v}).$$

Table 2: Transmission eigenvalues of a disk D with radius 1/2. The index of refraction n is 16. The mesh size $h \approx 0.1$. The total number of equations in the assembled matrices is denoted by N.

Argyris Method ($N = 2074$)	2.0076	2.6382	2.6396	3.2580	3.2598
Continuous Method ($N = 256$)	2.0301	2.6937	2.6974	3.3744	3.3777
Mixed Method ($N = 398$)	1.9912	2.6218	2.6234	3.2308	3.2397

Then we substitute $\vec{u}$ back into (4.27) and move the terms involving k^2 to the right-hand-side. The generalized matrix eigenproblem is now given by

$$\begin{pmatrix} -S + B_u M_u^{-1} B_u^T & -B_u M_u^{-1} A \\ A^T M_u^{-1} B_u^T & -A^T M_u^{-1} A \end{pmatrix} \begin{pmatrix} \vec{w} \\ \vec{v} \end{pmatrix} = -k^2 \begin{pmatrix} M_n & 0 \\ 0 & M_v \end{pmatrix} \begin{pmatrix} \vec{w} \\ \vec{v} \end{pmatrix}.$$

5 Numerical Results

We now use the above methods to compute transmission eigenvalues and verify some estimates for them. The numerical results show that both the continuous finite element method and the mixed method compute the degenerate eigenvalue $k = 0$. The continuous method provides better approximations in the sense that the norm of the computed degenerate eigenvalues are small (less than 10^{-6} on a coarse mesh). In contrast, the mixed method gives purely imaginary eigenvalues whose norm are relatively large. By Theorem 2.2, these eigenvalues are spurious. Note that the Argyris method does not compute zero transmission eigenvalues since $\|\triangle u\|_{L^2}$ is a norm on $H_0^2(D)$ [9].

The continuous method ends up with much sparser matrices than the other two methods. The implementation is also easy (only linear finite element is used). The Argyris method takes more effort since we need to compute the almost affine transformation for each triangle of the mesh. Because we need to solve $\vec{u}$ in terms of $\vec{w}$ and $\vec{v}$, the final matrix eigenvalue problem of the mixed method is no longer sparse.

In all cases we have to compute all eigenvalues using Matlab's *eig* function. Attempts to compute a few eigenvalues via the sparse matrix eigenvalue solver *eigs* did not converge. This limits the maximum number of degrees of freedom that we can use. The large number of degrees of freedom per element for the Argyris method in turn restricts us to a coarse mesh in this case. We denote by N the total number of equations in the assembled matrices in all examples.

5.1 Real Transmission Eigenvalues for General Domains

We first consider the case when D is a disk of radius $R = 1/2$. The index of refraction n is chosen to be a constant 16. We use the same mesh for all three finite element methods. Table 2 shows some computed transmission eigenvalues. Compared with the values in Table 1 which are obtained by root finding software, all three methods give good approximations and the correct multiplicity.

Next we compute the transmission eigenvalues on the unit square. Again we set the index of refraction n to be 16. The results are shown in Table 3.

Table 3: Transmission eigenvalues of the unit square. The index of refraction n is 16. The mesh size $h \approx 0.1$. The total number of equations in the assembled matrices is denoted by N.

Argyris Method ($N = 2684$)	1.8651	2.4255	2.4271	2.8178
Continuous Method ($N = 330$)	1.9094	2.5032	2.5032	2.9679
Mixed Method ($N = 513$)	1.8954	2.4644	2.4658	2.8918

Table 4: Transmission eigenvalues of a triangle whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$. The index of refraction n is 16. The mesh size $h \approx 0.08$. The total number of equations in the assembled matrices is denoted by N.

Argyris Method ($N = 4366$)	1.8321	2.3034	2.3057	2.8585
Continuous Method ($N = 534$)	1.8392	2.3239	2.3264	2.9077
Mix Method ($N = 830$)	1.8097	2.2756	2.2785	2.8209

The third example is a triangle whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$. The index of refraction n is 16. Table 4 shows the computed transmission eigenvalues.

We see that in all cases the corresponding (real) transmission eigenvalues computed by the three methods agree very well. Because of the limitations discussed previously, we cannot perform a mesh convergence study except for the continuous finite element method. In Table 5 we show errors and convergence rates for the first and second real transmission eigenvalues on three uniformly refined meshes for the first case, i.e., D is a disk of radius $1/2$ and $n = 16$. We use the values obtained in Section 3 as the exact values. The method is clearly second order in this case as we might expect for a continuous element applied to a non-selfadjoint eigenvalue problem

5.2 Cylindrically Stratified Media

We now consider several examples of spherically stratified media with non-constant index of refraction $n(x)$. We use a unit disk for D (see Fig. 3). Inside

Table 5: Errors and convergence rates for the first and second transmission eigenvalues computed by the continuous finite element method. The domain D is a disk of radius $1/2$. The index of refraction n is 16. $k_i^h, i = 1, 2$ are the computed first and second eigenvalues on three meshes. We use $k_i, i = 1, 2$, the first and second eigenvalues obtained by root finding software (see Section 3), as the exact eigenvalues.

Mesh size h	$\lvert k_1^h - k_1 \rvert$	convergence order	$\lvert k_2^h - k_2 \rvert$	convergence order
$h \approx 0.1$	0.04215	-	0.08078	-
$h \approx 0.05$	0.01064	1.986	0.02052	1.977
$h \approx 0.025$	0.00266	2.002	0.00514	1.995

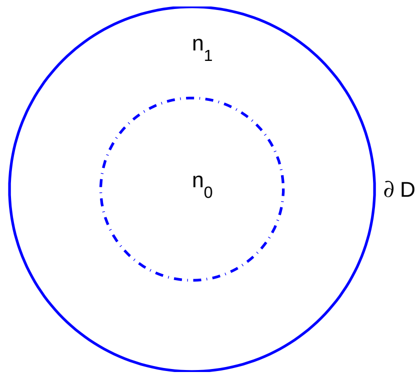

Figure 3: A schematic of the spherically stratified media.

Table 6: Real transmission eigenvalues $k_i, i = 1, \ldots, 6$ of the unit disk. Case A: $n_1 = 16$, $n_0 = 1$. Case B: $n_1 = 16$, $n_0 = 1/2$. Case C: $n_1 = 9/4$, $n_0 = 1/4$.

Cases	k_1	k_2	k_3	k_4	k_5	k_6
A	1.7135	1.7343	1.7345	1.8835	1.8837	2.0803
B	1.7462	1.7464	1.7531	1.8898	1.8900	2.0839
C	2.5895	7.5899	7.6041	8.0577	8.0799	8.6780

D, we choose a small disk D_0 of radius $1/2$. Denote the indices of refraction of D_0 and $D \setminus D_0$ by n_0 and n_1 respectively. In Case A, we choose $n_1 = 16$ and $n_0 = 1$. In Case B, we choose $n_1 = 16$ and $n_0 = 1/2$. In Case C, we choose $n_1 = 9/4$ and $n_0 = 1/4$ such that

$$\int_0^1 n(x)^{1/2}\, dx = \int_0^{1/2} \left(\frac{1}{4}\right)\, dx + \int_{1/2}^1 \left(\frac{9}{4}\right)^{1/2}\, dx = 1. \tag{5.28}$$

Note that Case C is not covered in Theorem 2 of [13] since (5.28) violates the condition in that paper. We use the continuous finite element method for this case with $h \approx 0.1$ and $N = 1074$. The computed real transmission eigenvalues for the above three cases are shown in Table 6. In Case C, n_1 is much less than those in Cases A and B and the first transmission eigenvalue is significantly larger.

5.3 Complex Transmission Eigenvalues

In Section 3, we find complex transmission eigenvalues by looking at zeros of $Z_0(k)$ for a disk of radius $R = 1/2$ and $n(x) = 16$. All three finite element methods compute the complex transmission eigenvalues as well. In Table 7, we show the first pair (in the sense of the magnitude of the norm) of complex transmission eigenvalues. The values in the first row are obtained by root finding software for $Z_0(k)$. We use different meshes such that degrees of freedoms of

Table 7: The first pair of complex transmission eigenvalues of a disk of radius $R = 1/2$. The index of refraction n is 16. Here h is the mesh size. The total number of equations in the assembled matrices is denoted by N.

Numerical Root Finding Software	$4.9009 \pm 0.5781i$
Argyris Method ($h \approx 0.1, N = 2074$)	$4.9495 \pm 0.5795i$
Continuous Method ($h \approx 0.025, N = 4066$)	$4.9130 \pm 0.5783i$
Mixed Method ($h \approx 0.05, N = 1557$)	$4.9096 \pm 0.5595i$

the three methods are of the same order. Numerical results show that the accuracy is increased by mesh refinement for the continuous method and the mixed method. We see that all three methods give good approximations.

Note that for real $n(x)$, there are no analytical results on the existence or estimation of complex transmission eigenvalues to date. Our computation clearly suggests the existence of complex transmission eigenvalues.

5.4 Verification of Estimates on the Transmission Eigenvalues

We will now verify and compare some estimates for transmission eigenvalues, in particular, Theorem 1.1 introduced in Section 1 and the following Faber-Krahn type inequality obtained by Colton et al.[13]

$$k_{1,D}^2 > \frac{\lambda_0(D)}{\sup_D n(x)}. \tag{5.29}$$

We first choose D to be the unit square. Then the ball B_1 with radius $1/2$ is the largest ball such that $B_1 \subset D$ and the ball B_2 with radius 0.8 is a ball such that $D \subset B_2$. Let $n(x) = 16$ for all cases and let $k_{1,D}$, k_{1,B_1} and k_{1,B_2} be the first real transmission eigenvalues of the above domains respectively. The finite element computation shows that

$$k_{1,B_2} \approx 1.2443, k_{1,D} \approx 1.8651, k_{1,B_1} \approx 1.9912,$$

which satisfies the estimate of Theorem 1.1. By comparison, the estimate (5.29) gives $k_{1,D} > 1.2337$ which is a slightly worse estimate than that given by Theorem 1.1.

Next let D be the triangle whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$. Then the ball B_1 with radius $1/2$ is the largest ball such that $B_1 \subset D$ and the ball B_2 with radius 1 is a ball such that $D \subset B_2$. Let $n(x) = 16$ for all cases. Our computation shows that

$$k_{1,B_2} \approx 0.9956, k_{1,D} \approx 1.7885, k_{1,B_1} \approx 1.9912,$$

which again verifies the estimate of Theorem 1.1. By comparison, the estimate (5.29) gives $k_{1,D} > 1.0507$. In this case, the Faber-Krahn type inequality (5.29) gives a better estimate for $k_{1,D}$ than Theorem 1.1.

6 Conclusions

In this paper, we presented some analytical and computational results for transmission eigenvalues. We proved an existence theorem for spherically stratified media. For the general case, we showed that there are no purely imaginary transmission eigenvalues under mild restriction on the index of refraction.

To compute transmission eigenvalues numerically, we proposed three finite element methods. The Argyris finite element is based on a 4th order reformulation of the original problem and does not compute spurious eigenvalues. However, the method is difficult to implement since the use of normal vectors to the edges is not respected by affine transformations. The resulting discrete system is large due to the many local degrees of freedom (21 for each triangle). The mixed finite element method is relatively easy to implement. Although it suffers from spurious eigenvalues, we are able to pick the correct eigenvalues due to the fact that there are no purely imaginary eigenvalues. The continuous finite element method is the easiest to implement compared to the Argyris triangles and the Raviart-Thomas element needed in the previous two methods. In addition, the trivial eigenvalues $k = 0$ are also computed accurately (the norms are less than 10^{-6} on a rather coarse triangular mesh). Numerical results show that the three finite element methods give consistent transmission eigenvalues. Some estimates of the first (real) transmission eigenvalue are also investigated and these results suggest that the use of simple disks to approximate D does not necessarily provide better estimates than the Faber-Krahn type inequality.

To date, there is no analytical result on the existence of complex transmission eigenvalues. However, the numerical results and the discussion in Section 3 clearly indicate the existence of complex transmission eigenvalues. When $n(x) > 1$ is sufficiently close to 1, the computation shows that the transmission eigenvalues of least norm are complex. An error analysis of the numerical methods presented here is a worthwhile future project.

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