ITERATIVE METHODS FOR TRANSMISSION EIGENVALUES

JIGUANG SUN*

Abstract. Transmission eigenvalues have important applications in inverse scattering theory. They can be used to obtain useful information of the physical properties, such as the index of refraction, of the scattering target. Despite considerable effort devoted to the existence and estimation for the transmission eigenvalues, the numerical treatment is limited. Since the problem is non-standard, classical finite element methods result in non-Hermitian matrix eigenvalue problems. In this paper, we focus on the computation of a few lowest transmission eigenvalues which are of practical importance. Instead of a non-Hermitian problem, we work on a series of generalized Hermitian problems. We first use a forth order reformulation of the transmission eigenproblem to construct functions involving an associated generalized eigenvalue problem. The roots of these functions are the transmission eigenvalues. Then we apply iterative methods to compute the transmission eigenvalues. We show the convergence of the numerical schemes. The effectiveness of the methods is demonstrated using various numerical examples.

Key words. Transmission Eigenvalues, Inverse Scattering, Iterative Method, Finite Element Method.

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1. Introduction. Recently the transmission eigenvalue problem has attracted many researchers in inverse scattering community [5, 6, 10, 12, 11, 15, 16, 8]. Since the transmission eigenvalues can be determined from the far field pattern, they can be used to obtain estimates for the material properties of the scattering object [5, 7]. Furthermore, transmission eigenvalues have theoretical importance in the uniqueness and reconstruction in inverse scattering theory [11].

It can be proved that the transmission eigenvalues form at most a discrete set with infinity as the only possible accumulation point by applying the analytic Fredholm theory [12]. However, little was known about the existence of the transmission eigenvalues except the spherically stratified medium until very recent. In [16], Päivärinta and Sylvester show the existence of a finite number of transmission eigenvalues provided that the index of refraction is large enough. Cakoni and Haddar [6] extend the idea of [16] and prove the existence of finitely many transmission eigenvalues for a larger class of problems. The idea is further extended to show the existence of an infinite discrete set of transmission eigenvalues that accumulate at infinity [5].

Despite considerable papers devoted to the existence and estimation, the numerical treatment for the transmission eigenvalues is quite limited. To the author's knowledge, the recent paper by Colton et.al. [10] contains the first numerical study where three finite element methods are proposed. However, the finite element mesh has to be kept rather coarse in [10] since the resulting sparse matrix problem is non-Hermitian.

Inspired by the existence proofs in [6] and the fact that only a few lowest real transmission eigenvalues are of practical importance in the inverse scattering theory to estimate the index of refraction [7], we propose two iterative methods. The methods depend on a forth order reformulation of the problem. We first construct functions involving an associated generalized eigenvalue problem. Then the roots of these functions are shown to be the transmission eigenvalues. Finally, iterative methods are applied to search the roots of these functions. The associated generalized eigenvalue

*Department of Mathematical Sciences, Delaware State University, Dover, DE 19901 (jsun@desu.edu).

problems are computed by the finite element method. We prove the convergence of the iterative methods using the derivative of the generalized eigenvalues [14, 13, 2]. The proposed methods are effective and fast which is demonstrated by various examples.

The paper is organized as follows. In Section 2, we introduce the transmission eigenvalue problem and derive a forth order reformulation. Using the associated generalized eigenvalue problems, we form functions whose roots are the transmission eigenvalues. In Section 3, we propose two iterative methods for the computation of the transmission eigenvalues and prove the convergence of the methods. Various numerical examples are shown in Section 4. Finally in Section 5, we make conclusions and discuss some future works.

2. The transmission eigenvalue problem. In this paper, we study the transmission eigenvalues corresponding to the scattering of acoustic waves by a bounded simply connected inhomogeneous medium $D \subset \mathbb{R}^2$. The transmission eigenvalue problem is to find $k \in \mathbb{C}$, $w, v \in L^2(D)$, $w - v \in H^2(D)$ such that

$$\Delta w + k^2 n(x) w = 0, \qquad \text{in } D, \tag{2.1a}$$

$$\Delta v + k^2 v = 0, \qquad \text{in } D, \tag{2.1b}$$

$$w - v = 0, \qquad \text{on } \partial D, \tag{2.1c}$$

$$\frac{\partial w}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \partial D, \tag{2.1d}$$

where ν is the unit outward normal to ∂D and the index of refraction $n(x)$ is positive. Values of $k \neq 0$ such that there exists a nontrivial solution to (2.1) are called the transmission eigenvalues (see [10]).

We first rewrite (2.1) as a forth order problem. Define

$$H_0^2(D) = \left\{ u \in H^2(D) : u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

Let $u = w - v \in H_0^2(D)$. Subtracting (2.1b) from (2.1a), we obtain

$$(\triangle + k^2) u = -k^2 (n(x) - 1) w.$$

Dividing $n(x) - 1$ and applying $(\triangle + k^2 n(x))$ to both sides of the above equation, we obtain

$$(\triangle + k^2 n(x)) \frac{1}{n(x) - 1} (\triangle + k^2) u = 0.$$

Denote by (u, v) the $L^2(D)$ inner product. The weak formulation for the transmission eigenvalue problem can be stated as: find $(k^2 \neq 0, u) \in \mathbb{C} \times H_0^2(D)$ such that

$$\left(\frac{1}{n(x) - 1} (\triangle u + k^2 u), \triangle v + k^2 n(x) v \right) = 0 \quad \text{for all } v \in H_0^2(D). \tag{2.2}$$

In the rest of this section, we will introduce the associated generalized eigenvalue problems and the existence of transmission eigenvalues. More details can be found in [6] and we present the results in a way such that we can derive numerical methods directly.

2.1. The associated generalized eigenvalue problems. We first introduce the associated generalized eigenvalue problems. These generalized eigenvalue problems correspond to a family of positive definite and self-adjoint operators with respect to a non-negative compact operator. Thus standard finite element methods can be used to compute the generalized eigenvalues [9, 3].

First we define

$$\mathcal{A}_\tau(u,v) = \left(\frac{1}{n(x)-1}(\triangle u + \tau u), (\triangle v + \tau v)\right) + \tau^2(u,v), \tag{2.3a}$$

$$\begin{aligned}\tilde{\mathcal{A}}_\tau(u,v) &= \left(\frac{1}{1-n(x)}(\triangle u + \tau n(x)u), (\triangle v + \tau n(x)v)\right) + \tau^2(n(x)u,v) \\ &= \left(\frac{n(x)}{1-n(x)}(\triangle u + \tau u), (\triangle v + \tau v)\right) + (\Delta u, \Delta v),\end{aligned} \tag{2.3b}$$

$$\mathcal{B}(u,v) = (\nabla u, \nabla v), \tag{2.3c}$$

where $\tau := k^2$. For simplicity, we also call τ a transmission eigenvalue if k is. From (2.2), it is straightforward to show that the transmission eigenvalues are τ's such that

$$\mathcal{A}_\tau(u,v) - \tau\mathcal{B}(u,v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.4}$$

when $n(x) > 1$ and

$$\tilde{\mathcal{A}}_\tau(u,v) - \tau\mathcal{B}(u,v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.5}$$

when $n(x) < 1$.

The following lemma provides useful properties of the generalized eigenvalue problems. Because of its importance for the proposed iterative methods, we sketch its proof and refer the readers to [6] for more details.

Lemma 2.1. *Let the index of refraction $n(x)$ satisfy*

$$\frac{1}{n(x)-1} > \gamma > 0, \quad a.e. \text{ in } D, \tag{2.6}$$

or

$$\frac{n(x)}{1-n(x)} > \gamma > 0, \quad a.e. \text{ in } D. \tag{2.7}$$

Then $\mathcal{A}_\tau$ or $\tilde{\mathcal{A}}_\tau$ is a coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$. Moreover, $\mathcal{B}$ is symmetric and non-negative on $H_0^2(D)$.

Proof. Assuming that $n(x)$ satisfies (2.6), we have

$$\begin{aligned}\mathcal{A}_\tau(u,u) &\geq \gamma\|\Delta u + \tau u\|_{L^2}^2 + \tau^2\|u\|_{L^2}^2 \\ &\geq \gamma\|\Delta u\|_{L^2}^2 - 2\gamma\tau\|\Delta u\|_{L^2}\|u\|_{L^2} + (\gamma+1)\tau^2\|u\|_{L^2}^2 \\ &= \epsilon\left(\tau\|u\|_{L^2} - \frac{\gamma}{\epsilon}\|\Delta u\|_{L^2}\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|_{L^2}^2 + (1+\gamma-\epsilon)\tau^2\|u\|_{L^2}^2 \\ &\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|_{L^2}^2 + (1+\gamma-\epsilon)\tau^2\|u\|_{L^2}^2\end{aligned}$$

for $\gamma < \epsilon < \gamma + 1$. Moreover, letting $\lambda_0(D)$ be the first Dirichlet eigenvalue of $-\Delta$ in D and using the Poincaré inequality, we have

$$\|\nabla u\|_{L^2(D)}^2 \leq \frac{1}{\lambda_0(D)}\|\Delta u\|_{L^2(D)}^2$$

since $\nabla u \in H_0^1(D)^2$. Thus $\mathcal{A}_\tau$ is a coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$, i.e.,

$$\mathcal{A}_\tau(u,u) \geq C_\tau \|u\|^2_{H^2(D)} \tag{2.8}$$

for some positive constant C_τ.

Similarly, it can be shown that $\tilde{\mathcal{A}}_\tau$ is a coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$ provided (2.7) is satisfied. The conclusion on $\mathcal{B}$ is obvious. □

Now we define the following bounded self-adjoint linear operators

$$A_\tau : H_0^2(D) \to H_0^2(D), \quad (A_\tau u, v) = \mathcal{A}_\tau(u,v), \tag{2.9}$$
$$\tilde{A}_\tau : H_0^2(D) \to H_0^2(D), \quad (\tilde{A}_\tau u, v) = \tilde{\mathcal{A}}_\tau(u,v), \tag{2.10}$$
$$B : H_0^2(D) \to H_0^2(D), \quad (Bu, v) = \mathcal{B}_\tau(u,v). \tag{2.11}$$

Lemma 2.1 shows that B is a non-negative operator, A_τ is a positive definite operator if $\frac{1}{n(x)-1} > \gamma > 0$, and $\tilde{A}_\tau$ is a positive definite operator if $\frac{n(x)}{1-n(x)} > \gamma > 0$. Since $H_0^1(D)^2$ is compactly embedded in $L^2(D)^2$, B is a compact operator. In addition, A_τ and $\tilde{A}_\tau$ depend continuously on $\tau \in (0, \infty)$.

Now we consider the following generalized eigenvalue problem

$$\mathcal{A}_\tau(u,v) - \lambda(\tau)\mathcal{B}(u,v) = 0 \text{ for all } v \in H_0^2(D), \tag{2.12}$$

or

$$\tilde{\mathcal{A}}_\tau(u,v) - \lambda(\tau)\mathcal{B}(u,v) = 0 \text{ for all } v \in H_0^2(D). \tag{2.13}$$

Then $\lambda(\tau)$ is a continuous function of τ. From (2.4) and (2.5), a transmission eigenvalue is the root of

$$f(\tau) := \lambda(\tau) - \tau. \tag{2.14}$$

2.2. Existence of the transmission eigenvalues. We first introduce a theorem in [6] which provides the conditions for the existence of solutions of (2.14).

THEOREM 2.2. *Let $\tau \to A_\tau$ be a continuous mapping from $(0,\infty)$ to the set of self-adjoint and positive definite bounded linear operators on a Hilbert space U, and let B be a self-adjoint and non-negative compact bounded linear operator on U. We assume that there exists two positive constant $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$A_{\tau_0} - \tau_0 B$ is positive on U,*
2. *$A_{\tau_1} - \tau_1 B$ is non-positive on a k-dimensional subspace W_k of U.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, k$ has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the jth eigenvalue (counting multiplicity) of A_τ with respect to B, i.e. $ker(A_\tau - \lambda_j(\tau)B) \neq \{0\}$.

Under the following assumptions on $n(x)$, the operators A_τ or $\tilde{A}_\tau$ with B satisfy the conditions of the above theorem with $U = H_0^2(D)$. Let $n_* = \inf_D(n)$, $n^* = \sup_D(n)$, and $\mu_p(D) > 0$ be the $(p+1)$th clamped plate eigenvalue (counting the multiplicity) on D. Set

$$\theta_p(D) := 4\frac{\mu_p(D)^{1/2}}{\lambda_0(D)} + 4\frac{\mu_p(D)}{\lambda_0(D)^2}. \tag{2.15}$$

The following theorem is a modification of Theorem 3.1 in [6].

THEOREM 2.3. *Let $n(x) \in L^\infty(D)$ satisfying either one of the following assumptions*

$$1) \quad 1+\theta_p(D) \le n_* \le n(x) \le n^* < \infty \tag{2.16}$$

and

$$2) \quad 0 < n_* \le n(x) \le n^* < \frac{1}{1+\theta_p(D)}. \tag{2.17}$$

Then, there exists $p+1$ transmission eigenvalues (counting multiplicity) in the interval $[\tau_0, \tau_1]$ where

$$\tau_0 = \frac{\lambda_0(D)}{\sup_D(n)} - \epsilon, \quad \tau_1 = \frac{\lambda_0(D) - 2M\mu_p(D)^{1/2}}{2+2M}, \quad M = \frac{1}{n_*-1} \tag{2.18}$$

for case 1) and

$$\tau_0 = \lambda_0(D) - \epsilon, \quad \tau_1 = \frac{\lambda_0(D) - 2M\mu_p(D)^{1/2}}{2M}, \quad M = \frac{n^*}{1-n^*} \tag{2.19}$$

for case 2) with any $\epsilon > 0$.

The assumptions of Theorem 2.3 are restrictive (see Section 4). The following result in [5] can also be used to determine an interval containing transmission eigenvalues under mild conditions on $n(x)$. Let $\epsilon > 0$ such that D contains $m = m(\epsilon)$ disjoint disks B_ϵ of radius ϵ. Also let B_{r_1} be the largest ball of radius r_1 such that $B_{r_1} \subset D$ and B_{r_2} be the smallest ball of radius r_2 such that $D \subset B_{r_2}$.

THEOREM 2.4. *Assume that $n(x) \in L^\infty(D)$ and α, β are positive constants. Let k_{1,n_*} and k_{1,n^*} be the first transmission eigenvalues corresponding to the ball B_1 of radius one with the index of refraction n_* and n^* respectively. Let $k_{1,D}$ be the first transmission eigenvalue of D with index of refraction $n(x)$.*

1) If $1+\alpha \le n_* \le n(x) \le n^* < \infty$, then

$$0 < \frac{k_{1,n^*}}{r_2} \le k_{1,D} \le \frac{k_{1,n_*}}{r_1}. \tag{2.20}$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n^*}}{r_2}, \frac{k_{1,n_*}}{\epsilon}]$.

2) If $0 \le n_* \le n(x) \le n^* < 1-\beta$, then

$$0 < \frac{k_{1,n_*}}{r_2} \le k_{1,D} \le \frac{k_{1,n^*}}{r_1}. \tag{2.21}$$

There are at least $m(\epsilon)$ transmission eigenvalues in the interval $[\frac{k_{1,n_*}}{r_2}, \frac{k_{1,n^*}}{\epsilon}]$.

In general, the values in (2.20) and (2.21) provide alternative bounds for the transmission eigenvalues under milder conditions than Theorem 2.3. However, we do need to know the transmission eigenvalues of disks with constant index of refraction.

3. Iterative methods for the transmission eigenvalues. Our numerical methods are based on finding the root of a discrete version of (2.14). Since $\lambda(\tau)$ is the generalized eigenvalue of operator A_τ or $\tilde{A}_\tau$ with respect to B, we need to compute an approximation $\lambda_h(\tau)$ for $\lambda(\tau)$. This is done by using finite element methods for the generalized eigenvalue problem (2.12) and (2.13). In particular, we use the

H^2 conforming finite element space of Argyris element [9], denoted by S_h. For a mesh $\mathcal{T}$ for D, assume that $\lambda_{j,h}(\tau)$ is the jth eigenvalues of the discrete eigenvalue problem

$$A_{\tau,h}\mathbf{x} = \lambda_{j,h}(\tau) B_h \mathbf{x} \tag{3.1}$$

where $A_{\tau,h}$ (or $\tilde{A}_{\tau,h}$) and B_h are the finite element matrices for (2.12) (or (2.13)). Note that $\lambda_{j,h}(\tau)$ depends on τ continuously. To compute the jth transmission eigenvalue, we fix a index j and compute the jth eigenvalues $\lambda_{j,h}(\tau)$ of (3.1). These values are then used to compute the root of (2.14). For simplicity, we drop the index j in the following except j needs to be specified. The following lemma is obvious [3, 9].

Lemma 3.1. *Assume that we apply the Argyris finite element method for* (2.12) *or* (2.13) *on a Lipschitz domain D and the index of refraction $n(x)$ satisfies the condition of Lemma 2.1. Let $\lambda_h(\tau)$ be the finite element approximation of a generalized eigenvalue $\lambda(\tau)$ on a triangular mesh $\mathcal{T}$ with mesh size h. Then for any $\epsilon > 0$, there exists an h_0 such that if $h \le h_0$ then*

$$|\lambda_h(\tau) - \lambda(\tau)| \le \epsilon.$$

In the following, we propose two iterative methods to compute the root of

$$f_h(\tau) := \lambda_h(\tau) - \tau. \tag{3.2}$$

3.1. A bisection method. We start with finding τ_0 and τ_1 such that the desired transmission eigenvalues are in $[\tau_0, \tau_1]$. According to the discussion in the previous section, we can either compute τ_0 and τ_1 using the lowest Dirichlet eigenvalue and the clamp plate eigenvalues ((2.18) or (2.19) of Theorem 2.3) for D or the transmission eigenvalues for disks containing or contained D ((2.20) or (2.21) of Theorem 2.4).

Now we propose a bisection algorithm to compute N lowest transmission eigenvalues. The tolerance is denoted by *tol*.

Algorithm 1 (Bisection Method): $bisectionTE(n(x), tol, N)$
  generate a regular triangular mesh for D
  compute τ_0 and τ_1 and construct matrix B_h
  for each $i, 1 \le i \le N$ do the following
  while abs$(\tau_0 - \tau_1) > tol$
    $\tau = (\tau_0 + \tau_1)/2$
    construct matrix $A_{\tau,h}$ depending on τ
    compute ith smallest eigenvalue $\lambda_{i,h}$ of $A_{\tau,h}\mathbf{x} = \lambda B_h \mathbf{x}$
    if $\lambda_{i,h} - \tau > 0$
      $\tau_0 = \tau$
    elseif $\lambda_{i,h} - \tau < 0$
      $\tau_1 = \tau$
    else
      break
    end
  end

In the following, we will establish the convergence of the above method using the derivatives of eigenvalues [14, 13, 2]. Let λ_h be a generalized eigenvalue of (3.1) and X be a matrix of eigenvectors associated with λ_h such that $X^T B X = I$. Thus we have

$$A_{\tau,h} X = B_h X \Lambda_h$$

where $\Lambda_h = \lambda_h I$. In general, the repeated eigenvalue λ_h will separate as τ changes and the derivative of the eigenvalue λ_h with multiplicity m is not a scalar. We will denote it by $\Lambda'_h = diag(\lambda'_{1,h}, \dots, \lambda'_{m,h})$. It is well-known that the choice of X is not unique [13] and there exists a suitable matrix $\Gamma \in \mathbb{R}^{m\times m}$ such that $\Gamma^T\Gamma = I$ and the columns of orthogonal transformation $Z = X\Gamma$ are the eigenvectors for which a derivative can be defined.

Differentiating $A_{\tau,h}Z = B_h Z \Lambda_h$, we obtain

$$A'_{\tau,h}Z + A_{\tau,h}Z' = B'_h Z\Lambda_h + B_h Z'\Lambda_h + B_h Z \Lambda'_h.$$

Collecting similar terms we obtain

$$(A_{\tau,h} - \lambda_h B_h)Z' = (\lambda_h B'_h - A'_{\tau,h})Z + B_h Z\Lambda'_h.$$

Multiplying X^T, substituting $Z = X\Gamma$ and using the fact that $X^T(A_{\tau,h} - \lambda_h B_h) = 0$, we have

$$X^T(A'_{\tau,h} - \lambda_h B'_h)X\Gamma = \Gamma\Lambda'_h.$$

Note that B_h does not depend on τ, we have $B'_h = 0$ and thus

$$\Lambda'_h = (X\Gamma)^T(A'_{\tau,h})(X\Gamma). \tag{3.3}$$

If λ_h is a distinct eigenvalue, we have

$$\lambda'_h = \mathbf{x}^T A'_{\tau,h}\mathbf{x}$$

where $\mathbf{x}$ is the associated eigenvector such that $\mathbf{x}^T B_h \mathbf{x} = 1$.

Now we show that $f'_h(\tau)$ is negative on an interval right to τ_0. Let $\lambda_h(\tau)$ be a generalized eigenvalue and X be the associated matrix of eigenvectors such that $X^T B_h X = I$. In addition, let $Z = X\Gamma$ be the transformation whose columns are the eigenvectors for which a derivative can be defined. This is true since we have generalized Hermitian eigenvalue problems.

Lemma 3.2. *Let $\mathcal{A}'_{\tau,h}$ and $\tilde{\mathcal{A}}'_{\tau,h}$ represent the derivatives of $\mathcal{A}_{\tau,h}$ and $\tilde{\mathcal{A}}_{\tau,h}$ respectively. For simplicity we denote $\|\cdot\|_{L^\infty}$ by $\|\cdot\|$. If $|\nabla\frac{1}{n(x)-1}| < c_g$ for some constant c_g for $n(x) > 1$ or $|\nabla\frac{1}{n(x)-1}| < c_g$ for some constant c_g for $n(x) < 1$, we have $f'_h(\tau) < 0$ when*

$$\tau < \frac{\left(1 + \frac{2}{n^*-1} - c_g - \frac{c_g}{\lambda_0(D)}\right)\lambda_0(D)}{2\left(\frac{1}{n_*-1} + 1\right)} \tag{3.4}$$

for $n(x) > 1$ and

$$\tau < \frac{\left(1 + \frac{2}{1-n_*} - c_g - \frac{c_g}{\lambda_0(D)}\right)\lambda_0(D)}{\frac{2n^*}{1-n^*}} \tag{3.5}$$

for $n(x) < 1$.

Proof. Assume that the index of refraction $n(x) > 1$ and $|\nabla\frac{1}{n(x)-1}| < c_g$ for some constant c_g. By simple calculations, we have

$$\mathcal{A}'_{\tau,h}(u,v) = -\left(\nabla\frac{u}{n(x)-1}, \nabla v\right) - \left(\nabla u, \nabla\frac{v}{n(x)-1}\right) + 2\tau\left(\frac{1}{n(x)-1}u, v\right) + 2\tau(u,v).$$

Letting $v = u$, we have

$$\begin{aligned}\mathcal{A}'_{\tau,h}(u,u) &= -2\left(\left(\nabla\frac{1}{n(x)-1}\right)u,\nabla u\right) - 2\left(\frac{1}{n(x)-1}\nabla u,\nabla u\right)\\ &\qquad\qquad\qquad\qquad +2\tau\left(\frac{1}{n(x)-1}u,u\right) + 2\tau(u,u)\\ &\le c_g(\|u\|^2+\|\nabla u\|^2) - \frac{2}{n^*-1}\|\nabla u\|^2 + \frac{2\tau}{n_*-1}\|u\|^2 + 2\tau\|u\|^2\\ &\le c_g(\|u\|^2+\|\nabla u\|^2) - \frac{2}{n^*-1}\|\nabla u\|^2 + 2\tau\left(\frac{1}{n_*-1}+1\right)\|u\|^2.\end{aligned}$$

Let $\mathbf{x}$ be a column of Z and u be the corresponding function of $\mathbf{x}$ in S_h. Note that $(\nabla u, \nabla u) = 1$. Let $A'_{\tau,h}$ be the matrix corresponding to $\mathcal{A}'_{\tau,h}$. Then we have

$$\begin{aligned}\lambda'_h(\tau) &= Z^T A'_{\tau,h} Z\\ &\le c_g\|u\|^2 + c_g - \frac{2}{n^*-1} + 2\tau\left(\frac{1}{n_*-1}+1\right)\|u\|^2\\ &\le c_g\frac{1}{\lambda_0(D)} + c_g - \frac{2}{n^*-1} + 2\tau\left(\frac{1}{n_*-1}+1\right)\frac{1}{\lambda_0(D)}\end{aligned}$$

where we have applied the Poincaré inequality. Thus if

$$c_g\frac{1}{\lambda_0(D)} + c_g - \frac{2}{n^*-1} + 2\tau\left(\frac{1}{n_*-1}+1\right)\frac{1}{\lambda_0(D)} < 1,$$

i.e.

$$\tau < \frac{\left(1+\frac{2}{n^*-1} - c_g - \frac{c_g}{\lambda_0(D)}\right)\lambda_0(D)}{2\left(\frac{1}{n_*-1}+1\right)}, \tag{3.6}$$

we have $f'_h(\tau) = \lambda'_h(\tau) - 1 < 0$ which implies $f(\tau)$ is monotonically decreasing.

Similarly, let $\tilde{\mathcal{A}}'_{\tau,h}$ represent the derivative of $\tilde{\mathcal{A}}_{\tau,h}$ with respect to τ. Assume that the index of refraction $n(x) < 1$ and $|\nabla\frac{n(x)}{1-n(x)}| < c_g$ for some constant c_g. We have

$$\tilde{\mathcal{A}}'_{\tau,h}(u,v) = -\left(\nabla u, \nabla\frac{n(x)v}{1-n(x)}\right) - \left(\nabla\frac{n(x)u}{1-n(x)},\nabla v\right) + 2\tau\left(\frac{n(x)}{1-n(x)}u,v\right).$$

Letting $v = u$, we have

$$\begin{aligned}\tilde{\mathcal{A}}'_{\tau,h}(u,u) &= -2\left(\left(\nabla\frac{n(x)}{1-n(x)}\right)u,\nabla u\right) - 2\left(\frac{n(x)}{1-n(x)}\nabla u,\nabla u\right) + 2\tau\left(\frac{n(x)}{1-n(x)}u,u\right)\\ &\le c_g(\|u\|^2+\|\nabla u\|^2) - \frac{2}{1-n_*}\|\nabla u\|^2 + \frac{2\tau n^*}{1-n^*}\|u\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\lambda'_h(\tau) &= Z^T\tilde{A}'_{\tau,h}Z\\ &= c_g\|u\|^2 + c_g - \frac{2}{1-n_*} + \frac{2\tau n^*}{1-n^*}\|u\|^2\\ &\le c_g\frac{1}{\lambda_0(D)} + c_g - \frac{2}{1-n_*} + \frac{2\tau n^*}{1-n^*}\frac{1}{\lambda_0(D)}\end{aligned}$$

where again we applied the Poincaré inequality. Thus if

$$c_g \frac{1}{\lambda_0(D)} + c_g - \frac{2}{1-n_*} + \frac{2\tau n^*}{1-n^*}\frac{1}{\lambda_0(D)} < 1,$$

i.e.,

$$\tau < \frac{\left(1 + \frac{2}{1-n_*} - c_g - \frac{c_g}{\lambda_0(D)}\right)\lambda_0(D)}{\frac{2n^*}{1-n^*}}, \tag{3.7}$$

we have $f_h'(\tau) = \lambda_h'(\tau) - 1 < 0$. Note that the above derivation does not depend on the mesh size h. □

In the case of constant index of refraction for a simple eigenvalue, the results can be simplified. Assuming $n > 1$ is constant and using integration by part, we obtain

$$A_{\tau,h} = H - \frac{2\tau}{n-1}G + \tau^2\frac{n}{n-1}M \tag{3.8}$$

where H, G, M are matrices corresponding to $\frac{1}{n-1}(\triangle u, \triangle v)$, $(\nabla u, \nabla v)$ and (u, v) respectively. Then we have

$$A_{\tau,h}' = -\frac{2}{n-1}G + 2\tau\frac{n}{n-1}M.$$

Assume that $\mathbf{x}$ is an eigenvector associated with λ_h. Let u be the corresponding function of $\mathbf{x}$ in S_h, we have $(\nabla u, \nabla u) = 1$. Hence

$$\begin{aligned}
\lambda_h'(\tau) &= \mathbf{x}^T A_{\tau,h}'\mathbf{x} \\
&= -\frac{2}{n-1}(\nabla u, \nabla u) + 2\tau\frac{n}{n-1}(u, u) \\
&\le -\frac{2}{n-1} + 2\tau\frac{n}{n-1}\frac{1}{\lambda_0(D)}.
\end{aligned}$$

Thus we have $f_h'(\tau) < 0$ if

$$-\frac{2}{n-1} + 2\tau\frac{n}{n-1}\frac{1}{\lambda_0(D)} < 1,$$

i.e.,

$$\tau < \frac{n+1}{2n}\lambda_0(D). \tag{3.9}$$

For the case of $\tilde{A}_{\tau,h}$, assuming the index of refraction $0 < n < 1$ is a constant, we have

$$\tilde{A}_{\tau,h}' = -\frac{2n}{1-n}G + 2\tau\left(\frac{n^2}{1-n} + n\right)\tau M.$$

Hence

$$\begin{aligned}
\lambda_h'(\tau) &= \mathbf{x}^T\tilde{A}_{\tau,h}'\mathbf{x} \\
&= -\frac{2n}{1-n}(\nabla u, \nabla u) + 2\tau\left(\frac{n^2}{1-n} + n\right)(u, u) \\
&\le -\frac{2n}{1-n} + 2\tau\left(\frac{n^2}{1-n} + n\right)\frac{1}{\lambda_0(D)}.
\end{aligned}$$

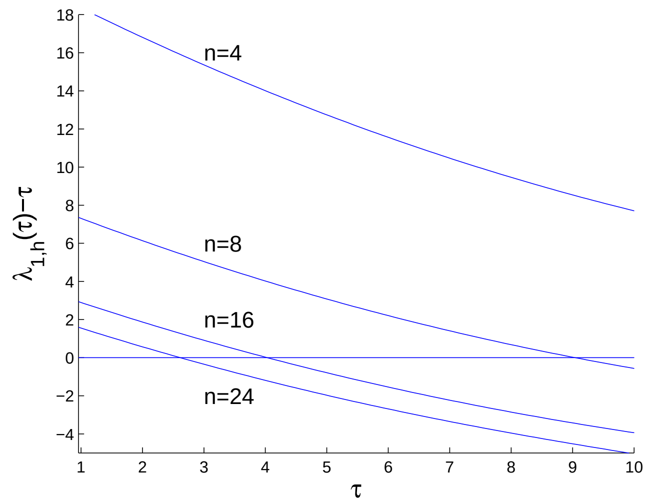

FIG. 3.1. $\lambda_{1,h}(\tau)-\tau$ *with* $n=24,16,8,4$ *when* D *is a disk of radius* $1/2$.

Thus we have $f_h'(\tau)<0$ if

$$-\frac{2n}{1-n}+2\tau\left(\frac{n^2}{1-n}+n\right)\frac{1}{\lambda_0(D)}<1,$$

i.e.,

$$\tau<\frac{n+1}{2n}\lambda_0(D).$$

Now we show some numerical study of function $f_h(\tau)$ as a verification of the above results. We consider the case when D is a disk with radius $1/2$. The computation is done on a mesh $\mathcal{T}$ for D whose size $h\approx 0.05$. In Fig. 3.1, we plot $f_{1,h}=\lambda_{1,h}(\tau)-\tau$ with $n=24,16,8,4$. We see that $f_{1,h}$ is positive for small positive τ and monotonically decreasing in an interval right to zero. From (3.9), we have

$$\tau_2:=\frac{n+1}{2n}\lambda_0(D)\approx 12.2311.$$

According to Lemma 3.2, for each j, $f_{j,h}(\tau)=\lambda_{j,h}(\tau)-\tau$ is monotonically decreasing on (τ_0,τ_2). This conclusion is verified in Fig. 3.2

The following lemma states that the root of (3.2) approximates the root of (2.14) well if the mesh size is small enough.

LEMMA 3.3. *Let* $f(\tau)$ *and* $f_h(\tau)$ *be two continuous functions. For a small enough* $\epsilon>0$*, we assume that* $f_h'(\tau)\le-\delta<0$ *for some* $\delta>0$ *and* $|f(\tau)-f_h(\tau)|<\epsilon$ *on an interval* $[a-\epsilon/\delta,b+\epsilon/\delta]$ *for some* $0<a<b$*. If* $f_h(\tau_0)=0$ *for some* $\tau_0\in[a,b]$*, then there exists a* τ_* *such that* $f(\tau_*)=0$ *and*

$$|\tau_*-\tau_0|<\epsilon/\delta.$$

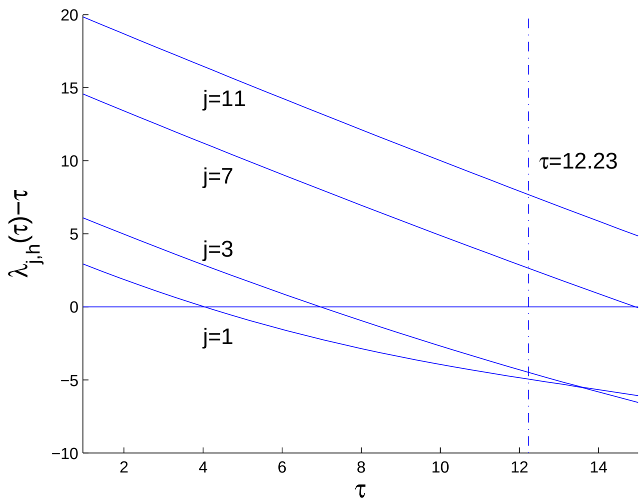

FIG. 3.2. $\lambda_{j,h}(\tau)-\tau$ *for* $j=1,3,7,11$ *when* $n=16$ *when* D *is a disk of radius* $1/2$.

Proof. Since $f_h'(\tau)\le -\delta<0$, if ϵ is small enough, there must exist τ_1 and τ_2 such that $f_h(\tau_1)>\epsilon$ and $f_h(\tau_2)<-\epsilon$. Furthermore, $|f(\tau)-f_h(\tau)|<\epsilon$ for all τ implies that $f(\tau_1)>0$ and $f(\tau_2)<0$. The existence of τ_* such that $f(\tau_*)=0$ follows immediately since $f(\tau)$ is continuous.

Assume that $|\tau_*-\tau_0|\ge\epsilon/\delta$. Since $f_h(\tau_0)=0$, we have $f_h(\tau_*)=f_h'(\xi)(\tau_*-\tau_0)$ for ξ between τ_0 and τ_*. Thus we have either $f_h(\tau_*)>\epsilon$ or $f_h(\tau_*)<-\epsilon$. Both contradict the fact that $|f_h(\tau_*)-f(\tau_*)|<\epsilon$. This completes the proof. □

Combining Lemmas 3.1, 3.2 and 3.3 and assuming we carry out the bisection method using the tolerance *tol*, we have the following convergence result.

THEOREM 3.4. *Assume that we apply the conforming finite element method for* (2.12) *or* (2.13) *using Argyris element on a regular mesh* $\mathcal{T}$ *with mesh size* h *and the conditions in Lemmas 3.1, 3.2 and 3.3 are satisfied. Let* τ_* *be the root of* (2.14) *and* τ_h *be the approximation of* τ_* *computed by the bisection method. Assume that* τ *satisfies* (3.4) *and* (3.5)*, then for any* $\epsilon>0$*, there exists* h_0 *such that for* $h<h_0$ *we have*

$$|\tau_h-\tau_*|\le\epsilon/\delta+tol$$

for some fixed $\delta>0$ *not depending on* ϵ.

Proof. Let $\lambda_h(\tau)$ be the finite element approximation of $\lambda(\tau)$ for the generalized eigenproblems (2.4) or (2.5). Then by Lemma 3.1, for any $\epsilon>0$, there exist h_0 such that for a regular mesh with $h<h_0$, we have

$$|\lambda_h(\tau)-\lambda(\tau)|<\epsilon.$$

Assume τ_0 is the root of $f_h(\tau)$, i.e., $\lambda_h(\tau_0)-\tau_0=0$. It is obvious that $|\tau_h-\tau_0|<tol$. If τ satisfies (3.4) and (3.5), from the derivation of Lemma 3.2, there exist $\delta>0$ such that $f_h'(\tau)<-\delta$ in a neighborhood of τ_0. Using Lemma 3.3, we have

$$|\tau_*-\tau_0|<\epsilon/\delta.$$

Then an application of the triangle inequality completes the proof. □

3.2. A secant method. To use the above bisection method, we need to decide an interval $[\tau_0, \tau_1]$ which contains the desired transmission eigenvalues. However, computation of τ_0 and τ_1 using Theorem 2.3 would require the Dirichlet and the clamped plate eigenvalues. Conditions (2.16) and (2.17) of Theorem 2.3 also put a strict condition on the index of refraction $n(x)$ (see Section 4). Theorem 2.4 provides an alternative way to decide τ_0 and τ_1 under mild restriction on $n(x)$. However, it requires the computation of the transmission eigenvalues of disks with constant index of refraction. To overcome these difficulties, we propose the following secant method to search the roots of $f_h(\tau)$. The method turns out to be very efficient in general.

Algorithm 2 (Secant Method): $secantTE(x_0, x_1, n(x), tol, N, maxit)$
generate a regular triangular mesh for D
for each $i, 1 \le i \le N$ do the following
set $it = 0$ and $\delta =$abs$(x_1 - x_0)$
$it = it + 1$
$t = x_0$
construct matrix corresponding to $A_{t,h}$
compute the ith smallest generalized eigenvalue λ_A of $A_{t,h}\mathbf{x} = \lambda B_h \mathbf{x}$
$t = x_1$
construct matrices corresponding to $A_{t,h}$
compute the ith generalized eigenvalue λ_B of $A_{t,h}\mathbf{x} = \lambda B_h \mathbf{x}$
while $\delta > tol$ and $it < maxit$
  $t = x_1 - \lambda_B \frac{x_1 - x_0}{\lambda_B - \lambda_A}$
  construct the matrix corresponding to $A_{t,h}$
  compute the ith smallest eigenvalue λ_t of $A_{t,h}\mathbf{x} = \lambda B_h \mathbf{x}$
  $\delta = \text{abs}(\lambda_t - t)$
  $x_0 = x_1, x_1 = t, \lambda_A = \lambda_B, \lambda_B = \lambda_t, it = it + 1.$
end

Here x_0 and x_1 are initial values which are chosen close to zero and $x_0 < x_1$. This is due to the fact that $f_h(\tau)$ is monotonically decreasing in an interval I right to zero. The $maxit$ is the maximum number of iterations. Similar to the bisection method, we have the following convergence theorem whose proof is straightforward (see [1]).

THEOREM 3.5. *Assume we apply the conforming finite element method for* (2.4) *or* (2.5) *using Argyris element on a regular mesh* $\mathcal{T}$ *with mesh size* h. *Let* $f_h'(\tau) < -\delta < 0$ *for* $\delta > 0$ *on some interval* $[a, b]$ *where* $a = \tau_0$ *is given by* (2.18) *and* b *is given by* (3.6) *for* $n(x) > 1$ *(*(2.19) *and* (3.7) *for* $n(x) < 1$*). Let* ϵ *be an arbitrary positive number. Assume that* τ_* *is the root of* $f(\tau)$ *such that* $\tau_* \in [a + \epsilon/\delta, b - \epsilon/\delta]$. *Let* τ_0 *be the root of* $f_h(\tau)$ *computed by the secant method. Then there exist an* h_0 *such that for* $h < h_0$ *we have*

$$|\tau_0 - \tau_*| \le \epsilon/\delta + tol.$$

4. Numerical Examples. In this section, we present some numerical examples. All computations are done using Matlab on a Macbook Pro with 4G memory. We use linear finite element to compute the lowest Dirichlet eigenvalue and Argyris element to compute the clamped plate eigenvalues and the generalized eigenvalue problems. In all examples, we use a regular mesh with mesh size $h \approx 0.05$ and $tol = 1.0E-6$. The transmission eigenvalues computed here are consistent with the values computed by the methods in [10].

TABLE 4.1
The 1st transmission eigenvalue computed by the bisection method using Theorem 2.3 for three domains: a disk D_1 of radius $R = 1/2$, the unite square D_2 and a triangle D_3 whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$.

Domain	Size of $A_{\tau,h}$	n	τ_0	τ_1	$k_1^2(D)$
D_1	17846×17846	24	0.9640	9.3825	2.5872
D_2	5630×5630	24	0.8225	7.9462	2.3275
D_3	2183×2183	24	0.7360	7.0675	2.1712
D_1	17846×17846	$\frac{1}{24}$	23.1373	225.1791	55.8562
D_2	5630×5630	$\frac{1}{24}$	19.7392	190.7097	62.0928
D_3	2183×2183	$\frac{1}{24}$	17.6641	169.6204	52.1111

TABLE 4.2
The 1st transmission eigenvalue when index of refraction is not constant for two domains: a disk D_1 of radius $R = 1/2$ and the unite square D_2 centered at the origin. The third column is the values from [17] computed by the inverse scattering scheme. The fourth column is computed by the bisection method.

Domain	$n(x)$	$k_1(D)$ (inverse scattering)	$k_1(D)$ (bisection method)
D_1	$8 + 4\|x\|$	2.78	2.8292
D_2	$8 + x_1 - x_2$	2.90	2.8834

The major advantages of the proposed iterative methods over the finite element methods in [10] are the accuracy and speed. For example, it is impossible to use the Argyris method in [10] on a mesh with mesh size $h < 0.05$ for a disk with radius 1/2 since solving the non-Hermitian eigenvalue problem using Matlab's *eig* leads to *Out of memory*. Instead of *eig*, one might use *sptarn* which is much more efficient and needs less memory. However, a search interval need to be specified precisely otherwise *sptarn* might not converge for our problem. In addition, there are no convergence results for the non-Hermitian iterative solvers up to date [4].

4.1. The bisection method using Theorem 2.3. We compute the 1st transmission eigenvalue for three different domains: a disk D_1 of radius $R = 1/2$ centered at the origin, the unite square D_2 centered at the origin and a triangle D_3 whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$. The mesh size $h \approx 0.05$ for all three domains. Table 4.1 shows the results of the bisection method when $n = 24$ and $n = \frac{1}{24}$. The sizes of the matrices are also shown in the table.

Next we consider the case when the index of refraction is not constant. We choose two domains: a disk D_1 of radius $R = 1/2$ and the unite square D_2 centered at the origin. The indices of refraction are given by $8 + 4|x|$ and $8 + x_1 - x_2$, respectively. We make these choices because the 1st transmission eigenvalues of both cases are obtained in [17] via the inverse scattering scheme and thus we can make comparison. The meshes for both domains keep the same. The result is shown in Table 4.2. We can see that the values computed by the bisection method (direct way) and by the inverse scattering scheme agree very well. Since [17] uses k_1 instead of k_1^2, we also show k_1 in Table 4.2.

4.2. The bisection method using Theorem 2.4. A major draw back of using Theorem 2.3 is the restriction on the index of refraction. It becomes severe if we want to compute several transmission eigenvalues. For example, suppose we want to compute five lowest transmission eigenvalues. Since $\mu_5(D) \approx 25,337.6304$,

Table 4.3
Secant method: lowest 6 transmission eigenvalues for a disk with radius 1/2 and $n = 24$.

j	k_j^2	number of iterations
1	2.5872	4
2	4.5364	4
3	4.5389	4
4	6.9483	4
5	6.9525	4
6	8.7960	5

we obtain $\theta_5(D) \approx 216.8401$. This would require

$$n(x) > 1 + \theta_5(D) \approx 217.8401$$

for the condition in Theorem 2.3 to be satisfied. As an alternative we can use the bounds given in Theorem 2.4 which requires the transmission eigenvalues of disks with constant index of refraction. We refer the readers to [10] for some discussion on how to obtain these transmission eigenvalues.

Let $n(x) = 16$ and D be the unit square. Then $B_1 = \{x; |x| < 1/2\}$ is the largest disk such that $B_1 \subset D$ and $B_2 = \{x; |x| < 0.8\}$ is a disk such that $D \subset B_2$. Note that the condition in Theorem 2.3 is not satisfied since $16 < 1 + \theta_0(D) \approx 21.8749$. Let $k_{1,D}$, k_{1,B_1} and k_{1,B_2} be the first transmission eigenvalues of the above domains respectively. From [10] we have

$$k_{1,B_1} \approx 1.9912, k_{1,B_2} \approx 1.2443$$

Using these bounds in the bisection method, we obtain $k_{1,D} \approx 1.8651$.

Next let D be the triangle whose vertices are given by $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(0, 1)$. Then $B_1 = \{x; |x| < 1/2\}$ satisfies $B_1 \subset D$ and $B_2 = \{x; |x| < 1\}$ satisfies $D \subset B_2$. Again $n(x) = 16$ violates the condition of Theorem 2.3. We have

$$k_{1,B_1} \approx 1.9912, k_{1,B_2} \approx 0.9956.$$

Using these bounds in the bisection method, we obtain $k_{1,D} \approx 1.7885$.

4.3. The secant method. The secant method only needs the value of the function and converge quickly for the lowest a few transmission eigenvalues. In Table 4.3, we show six lowest transmission eigenvalues computed by the secant method for a disk with radius 1/2 and $n = 24$. The secant method converges much faster than the bisection method. For example, for the lowest transmission eigenvalue, the bisection method uses 27 iterations comparing to 4 iterations by the secant method.

5. Conclusions and future work. In this paper, we propose two iterative methods to compute transmission eigenvalues. The major advantage of these methods is the accuracy and effectiveness since we only need to compute eigenvalues of Hermitian problems instead of non-Hermitian problems. This fits the practical need in the sense that in general only the lowest transmission eigenvalue is needed to estimate the index of refraction in inverse scattering theory [7]. We prove the convergence of the proposed methods and the effectiveness is verified by numerical examples.

In future, we plan to extend the methods to compute transmission eigenvalues for anisotropic Maxwell's equations.

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