A mixed finite element method for Helmholtz transmission eigenvalues

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Transmission eigenvalue problem has important applications in inverse scattering. Since the problem is not self-adjoint, the computation of transmission eigenvalues needs special treatment. Based on a fourth order reformulation of the transmission eigenvalue problem, we choose a mixed finite element method. The method has two major advantages: 1) the formulation leads to a generalized eigenvalue problems naturally without the need to invert a related linear system, and 2) the non-physical zero transmission eigenvalue, which has an infinitely dimensional eigenspace, is eliminated. To solve the resulting non-Hermitian eigenvalue problem, we propose an iterative algorithm using restarted Arnoldi method. To make the computation efficient, the search interval is decided using a Fabra-Khan type inequality for transmission eignevalues and the interval is updated at each iteration. The algorithm is implemented using Matlab. The code can be easily used in the qualitative methods in inverse scattering and be modified to compute transmission eigenvalues for other models such as elasticity problem.

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1. INTRODUCTION

Transmission eigenvalue problem has important applications in inverse scattering theory and attracted attention of many researchers recently [Colton et al. 2007] [Päivärinta and Sylvester 2008] [Cakoni and Haddar 2009] [Colton et al. 2010]. For the case of scattering of time-harmonic acoustic waves by a bounded simply connected inhomogeneous medium $D \subset \mathbb{R}^2$, the transmission eigenvalue problem is to find $k \in \mathbb{C}$,

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$w, v \in L^2(D)$, $w - v \in H^2(D)$ such that

$$\Delta u + k^2(1 + q(x))u = 0, \quad \text{in } D, \tag{1a}$$

$$\Delta v + k^2 v = 0, \quad \text{in } D, \tag{1b}$$

$$u - v = 0, \quad \text{on } \partial D, \tag{1c}$$

$$\frac{\partial u}{\partial \nu} - \frac{\partial v}{\partial \nu} = 0, \quad \text{on } \partial D. \tag{1d}$$

where ν is the unit outward normal to the boundary ∂D. The index of refraction $n(x) := 1+q(x)$ is assumed to be positive. Values of k such that there exists a nontrivial solution to (1) are called transmission eigenvalues.

Although stated in a simple form, the transmission eigenvalue problem is difficult. It is non-standard and the classical theory can not be applied directly. Recent investigation focuses on the existence of real transmission eigenvalues [Cakoni and Haddar 2009] [Kirsch 2009] [Cakoni and Gintides 2010]. It has been shown numerically that there exist complex transmission eigenvalues [Colton et al. 2010]. However the existence of complex transmission eigenvalues is an open problem except for some very special cases.

In this paper we will focus on the computation of a few lowest real transmission eigenvalues which are of practical importance [Cakoni et al. 2010]. Numerical treatment for transmission eigenvalues is quite limited so far. Computation of transmission eigenvalues faces two major difficulties. The first one is the infinite dimensional eigenspace corresponding to the non-physical transmission eigenvalue $k = 0$. It is readily seen that any harmonic function on D is an eigenfunction by setting $k = 0$ in (1) such that (1a) and (1b) become the same. The second difficulty is due to the non-selfadjointness. The resulting matrix eigenvalue problem from standard finite element method is non-Hermitian. Moreover, the problem is in general large and sparse. Thus there is a need for efficient eigensolvers. Note that the lowest real transmission eigenvalue is of interest and it may not be the transmission eigenvalue of the smallest norm due to the existence of complex transmission eigenvalues. This also generates difficulty in computation. In [Colton et al. 2010], three finite element method are proposed. However, two of them compute the zero transmission eigenvalues and one needs to implement an H^2 conforming finite element. All of them use the direct solver for the resulting generalized eigenvalue problem which puts a strict limit on the size of the problem. In [Sun 2010], Sun propose an iterative method based on the formulation of transmission eigenvalues as the roots of a non-linear equation involving positive definite forth order eigenvalue problems. The convergence of the scheme is proved as well. However, at each step, a forth order eigenvalue problem needs to be solved.

In this paper, we propose an efficient method to compute the lowest a few transmission eigenvalues based on a mixed finite element method. We first rewrite the problem into a forth order problem which naturally eliminates the non-physical zero transmission eigenvalue. Then we use a mixed finite element method with additional advantage as can be seen in Section 2. To compute eigenvalues of the non-Hermitian matrix problem resulted from the mixed finite element method, we resort to the Arnoldi method [Golub and Von Loan 1989] [Saad 1980]. For efficient search, we would like to specify an accurate interval which is relative small and contains the desired eigenvalues. This is done by a combination of estimation of a lower bound for transmission eigenvalues using the Fabra-Khran type inequality and adaptive update of the search interval. The proposed method has been used in qualitative method in inverse scattering theory and proved to be robust [Sun 2011b]. The code can be adapted to compute other transmission eigenvalue problems such as elasticity.

The rest of the paper is organized as follows. In Section 2, we describe the mixed finite element method for the transmission eigenvalue problem and end up with a large, sparse, non-Hermitian generalized matrix eigenvalue problem. In Section 3, we propose an adaptive algorithm based on the Arnoldi method to efficiently compute the lowest a few real transmission eigenvalues and briefly describe the implementation of the method. We refer the reader to Appendix for mored details. Finally we provide some numerical examples in Section 4.

2. A MIXED FINITE ELEMENT METHODS

We first rewrite the transmission eigenvalue problem into a fourth order equation. We assume that D is a convex Lipschitz domain and $q(x) \geq \delta > 0$ on D. Let $z = v - w \in H_0^2(D)$. Then we have

$$(\triangle + k^2(1+q))z = -k^2 qv.$$

Thus $q^{-1}(\triangle + k^2(1+q))z = -k^2 v$. We apply $(\triangle + k^2)$ to both sides of the above equation to obtain

$$(\triangle + k^2)\frac{1}{q}(\triangle + k^2(1+q))z = 0.$$

The transmission eigenproblem can be stated as: Find $(k^2, z) \in \mathbb{C} \times H_0^2(D)$ such that

$$\left(\frac{1}{q}(\triangle + k^2(1+q))z, (\triangle + k^2)\phi\right) = 0, \quad \forall \phi \in H_0^2(D). \tag{2}$$

It is obvious that $k = 0$ is not a non-trivial eigenvalue any longer since $\left(\frac{1}{q}\triangle z, \triangle z\right) = 0$ implies that $z = 0$ [Ciarlet 2002].

Before we move on to discuss the numerical method, we quote the existence results of (real) transmission eigenvalues (Theorem 3.1 and 3.2 in [Cakoni and Gintides 2010]). Again we denote the index of refraction $n(x) := 1 + q(x)$. Let $n_* = \inf_D(n(x))$ and $n^* = \sup_D(n(x))$ and assume that the origin of the coordinative system is inside D. Then the following results hold.

THEOREM 2.1. *Let $n \in L^\infty(D)$ satisfying either one of the following assumptions*

$$\begin{aligned} &1) \quad 1 + \alpha \leq n_* \leq n(x) \leq n^* < \infty, \\ &2) \quad 0 < n_* \leq n(x) \leq n^* < 1 - \beta, \end{aligned}$$

for some constant $\alpha > 0$ or $\beta > 0$. Then there exists at least one transmission eigenvalue.

THEOREM 2.2. *Assume that the index of refraction $n > 0$ is a positive constant such that $n \neq 1$. Then there exists an infinite discrete set of transmission eigenvalues with $+\infty$ as accumulation point.*

It is possible to use H^2-conforming finite element such as Argyris element [Argyris et al. 1968] or high regularity partition of unity element [Sun 2011a] to discretize the above problem directly. In general, it will end up with additional programming effort. Moreover, the problem becomes a quadratic eigenvalue problem which is difficult to handle as well. In this paper, we follow the mixed finite element approach as in [Ciarlet and Raviart 1974] [Monk 1987]. It will be seen an additional important advantage as a byproduct of the mixed formulation.

We shall use letters u and v for different unknowns. Let $u = w$ and $v = \frac{1}{q}(\Delta + k^2(1+q))w = 0$. Thus we have

$$\begin{aligned}(\Delta + k^2)v &= 0,\\ \frac{1}{q}(\Delta + k^2(1+q))u &= v.\end{aligned}$$

Following the mixed method approach [Ciarlet and Raviart 1974], [Monk 1987], we obtain the following weak problem. Find $(k^2, u, v) \in \mathbb{C} \times H_0^1(D) \times H^1(D)$ such that

$$\begin{aligned}(\nabla v, \nabla \phi) &= k^2(v, \phi), \quad \forall \phi \in H_0^1(D),\\ (\nabla u, \nabla \varphi) + (qv, \varphi) &= k^2((1+q)u, \varphi), \quad \forall \varphi \in H^1(D).\end{aligned}$$

Given finite dimensional spaces $S_h \subset H^1(D)$ and $S_h^0 \subset H_0^(D)$ such that $S_h^0 \subset S_h$, the discrete problem is to find $(k_h^2, u_h, v_h) \in \mathbb{C} \times S_h^0 \times S_h$ such that

$$\begin{aligned}(\nabla v_h, \nabla \phi_h) &= k_h^2(v_h, \phi_h), \quad \forall \phi_h \in S_h^0,\\ (\nabla u_h, \nabla \varphi_h) + (qv_h, \varphi_h) &= k_h^2((1+q)u_h, \varphi_h), \quad \forall \varphi \in S_h.\end{aligned}$$

In our , we can use standard piecewise linear finite elements to discretize the problem

$$\begin{aligned}S_h &= \text{the space of continuous piecewise linear finite elements on } D,\\ S_h^0 &= S_h \cap H_0^1(D)\\ &= \text{the subspace of functions in } S_h \text{ that have vanishing DoF on } \partial D\end{aligned}$$

where DoF stands for degree of freedom. Let $\psi_1, \ldots, \psi_K$ be a basis for S_h^0 and $\psi_1, \ldots, \psi_K, \psi_{K+1}, \ldots, \psi_T$ be a basis for S_h. Let $u_h = \sum_{i=1}^K u_i\psi_i$ and $v_h = \sum_{i=1}^T u_i\psi_i$. Furthermore, let $\mathbf{u} = (u_1, \ldots, u_K)^T$ and $\mathbf{v} = (v_1, \ldots, v_T)^T$. Then matrix problem corresponding to the above problem is

$$\begin{aligned}S_{K\times T}\mathbf{v} &= k_h^2 M_{K\times T}\mathbf{v},\\ S_{T\times K}\mathbf{u} + M_{T\times T}^q\mathbf{v} &= k_h^2 M_{T\times K}^{1+q}\mathbf{u},\end{aligned}$$

where the matrices are defined as the following

Matrix	Dimension	Definition
$S_{K\times T}$	$K \times T$	$S_{K\times T}^{i,j} = (\nabla\psi_i, \nabla\psi_j), 1 \le i \le K, 1 \le j \le T$
$S_{T\times K}$	$T \times K$	$S_{K\times T}^{i,j} = (\nabla\psi_i, \nabla\psi_j), 1 \le i \le T, 1 \le j \le K$
$M_{K\times T}$	$K \times T$	$M_{K\times T}^{i,j} = (\psi_i, \psi_j), 1 \le i \le K, 1 \le j \le T$
$M_{T\times K}^{1+q}$	$T \times K$	$(M_{T\times K}^{1+q})^{i,j} = ((1+q)\psi_i, \psi_j), 1 \le i \le T, 1 \le j \le K$
$M_{T\times T}^q$	$T \times T$	$(M_{T\times T}^q)^{i,j} = (q\psi_i, \psi_j), 1 \le i \le T, 1 \le j \le T$

The generalized eigenvalue problem we need to solve is

$$\begin{pmatrix} S_{K\times T} & 0_{K\times K} \\ M_{T\times T}^q & S_{T\times K} \end{pmatrix}\begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix} = k_h^2 \begin{pmatrix} M_{K\times T} & 0_{K\times K} \\ 0_{T\times T} & M_{T\times K}^{1+q} \end{pmatrix}\begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix}.$$

In contrast to mixed methods for bi-harmonic or Dirichlet eigenvalue problem which need the inversion of certain matrix, here we have the general eigenvalue problem directly. This is certainly an advantage thanks to the property of the original problem.

For simplicity, we shall write the above problem as

$$A\mathbf{x} = \lambda B\mathbf{x} \tag{3}$$

where

$$A = \begin{pmatrix} S_{K\times T} & 0_{K\times K} \\ M^q_{T\times T} & S_{T\times K} \end{pmatrix}, \quad B = \begin{pmatrix} M_{K\times T} & 0_{K\times K} \\ 0_{T\times T} & M^{1+q}_{T\times K} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \mathbf{v} \\ \mathbf{u} \end{pmatrix}.$$

3. AN ADAPTIVE ARNOLDI METHOD AND IMPLEMENTATION

The generalized eigenvalue problem obtained in last section is large, sparse and non-Hermitian. Direct method is prohibitive even on a rather coarse mesh [Colton et al. 2010]. Since we only need a few lowest real transmission eigenvalues in inverse scattering theory, iterative methods become the apparent choice. For this purpose we will devise an adaptive algorithm using Arnoldi method [Golub and Von Loan 1989] [Saad 1980] to compute the transmission eigenvalues. This is mainly due to the fact that Matlab has an implemented Arnoldi solver named 'sptarn' which can be integrated into our finite element code easily.

Sptarn uses Arnoldi with spectral transformation. To guarantee efficiency, we need to specify a small interval, i.e., to accurately estimate an interval contains desired lowest a few transmission eigenvalues. In [Colton et al. 2007] Colton et. al have proved the following Faber-Krahn type inequality.

THEOREM 3.1. *Let $k_{1,n(x)}$ be the lowest transmission eigenvalue and let λ_0 be the first Dirichlet eigenvalue for $-\Delta$ in D. If $n(x) > \alpha > 1$ for $x \in \overline{D}$. Then*

$$k^2_{1,n(x)} \geq \frac{\lambda_0}{\sup_D n}.$$

The above theorem provides a lower bound for transmission eigenvalues as long as we have the first Dirichlet eigenvalue. In fact, this can be done easily since we have the necessary matrices for the mixed finite element already. The discrete Dirichlet eigenvalue problem is simply the following generalized eigenvalue problem

$$S_{K\times K}\mathbf{x} = \lambda M_{K\times K}\mathbf{x} \tag{4}$$

where $S_{K\times K}$ and $M_{K\times K}$ are the stiffness matrix and the mass matrix defined in Section 2.

Since 'sptarn' might compute complex transmission eigenvalues, we need to exclude them as well. Assuming a triangular mesh $\mathcal{T}$ is already generated for D, the following adaptive algorithm computes the desired lowest a few transmission eigenvalues efficiently.

Algorithm: Mixed FEM for TEs

input a regular triangular mesh for D
input the index of refraction $n(x)$ and $n^* \sup_D(n(x))$
input the number of transmission eigenvalues noe to be computed
construct matrices S, M, M_n
construct matrices A, B from S, M, M_n
compute λ_0 from S and M
set $TE = \emptyset, lb = \dfrac{\lambda_0}{\sup_D n}, rb = lb + 1$
while $length(TE) < noe$
    $it = it + 1,$
    $[V, D] = sptarn(A, B, lb, rb)$
    delete complex values in D
    $TE = TE \cup D$
    $lb = rb, rb = lb + it + 1$
end

The algorithm is implemented using Matlab. The input are a triangular mesh $\mathcal{T}$ for domain D, the supreme of the index of refraction $n(x)$, and the number of transmission eigenvalues to be computed. The function for index of refraction $n(x)$ needs to be predefined in the file (see Appendix). The mesh are assumed to obey the Matlab PDEtool format, i.e., including the vertex matrix mesh.p, the triangle matrix mesh.t and the boundary edge matrix mesh.e. The construction of the stiffness and mass matrix is standard. Then the matrices are used to compute the lowest Dirichlet eigenvalue and set up the matrices A and B in (3).

These matrices are sent to the adaptive Arnoldi method to search for a few lowest transmission eigenvalue. This is done by first compute the left bound lb of an interval using Theorem 3.1 and set the right bound of the search interval $rb = lb + 1$. We would to keep this interval small since a larger interval might contains many transmission eigenvalues and keeps 'sptarn' searching forever. In fact, the distribution of real transmission eigenvalue are quite complicate [Colton et al. 2010]. Then we use 'sptarn' to search for real transmission eigenvalues. The search interval is moved to right by one unit until all desired transmission eigenvalues are found. The detail documentation can be found in the Appendix.

4. NUMERICAL EXAMPLES

Now we provide some numerical examples to show the effectiveness of our algorithm. We first consider the case when the index of refraction is constant. Here we choose $n(x) = 16$. We choose two geometries for D: a disk centered at $(0, 0)$ with radius $1/2$ and a unit square given by $[-1/2, 1/2] \times [-1/2, 1/2]$.

Note that for the case of disk, using Bessel's functions, the transmission eigenvalues can be found analytically. In fact they are the roots k of d_m defined by (see [Colton et al. 2010])

$$d_m(k) = J_1(k/2)J_0(2k) - 4J_0(k/2)J_1(2k), \quad m = 0, \tag{5}$$

$$d_m(k) = J_{m-1}(k/2)J_m(2k) - 4J_m(k/2)J_{m-1}(2k), \quad m = 1, 2, 3, \ldots \tag{6}$$

The computed transmission eigenvalues are shown in Table I which are consistent with the values in [Colton et al. 2010].

Table I.

domain	index of refraction n	1st	2nd	3rd	4th
disk ($r = 1/2$)	16	1.9986	2.6334	2.6343	3.2641
unit square	16	1.8873	2.4596	2.4599	2.8928

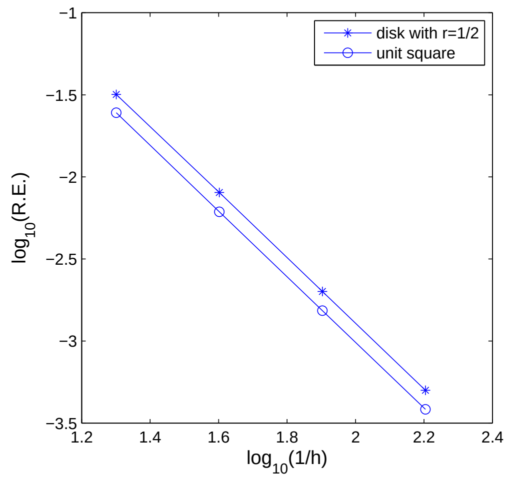

Fig. 1. The plot of $log_{10}(R.E.)$ against $log_{10}(1/h)$ for the lowest transmission eigenvalue.

Table II.

domain	index of refraction n	1st	2nd	3rd	4th
disk ($r = 1/2$)	$8 + 4\|x\|$	2.7770	3.5571	3.5584	4.3605
unit square	$8 + x_1 - x_2$	2.8373	3.5632	3.5642	4.1582

Next we check the convergence numerically. We start with a quasi-uniform triangular mesh $\mathcal{T}$ for D with $h \approx 0.1$. Then we uniformly refine the mesh a couple of times. In Figure 1, we plot the convergence of the relative error of the computed lowest transmission eigenvalues on a series of uniformly refined meshes. It is clearly we obtain a second order convergence rate numerically.

Finally, we compute the transmission eigenvalues when the index of refraction is a function. We set $n(x) = 8 + 4|x|$ for the disk and $8 + x_1 - x_2$ for the unit square. The lowest a few transmission eigenvalues are shown in Table II. The computed values are consistent with the results in [Sun 2010] and [Colton et al. 2010]. In particular, the lowest transmission eigenvalues are consistent with the values in [Sun 2011b] which are computed from the near field data using inverse scattering algorithm.

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Online Appendix to:
A mixed finite element method for Helmholtz transmission eigenvalues

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mixFemTE.m

The main m-file is 'mixFemTE.m'. To compute transmission eigenvalues, first type 'mixFemTE' in Matlab Command Window. The program first asks the user to input the mesh file by display

```
Input name of the mesh file -->
```

The user types the mesh file name, for example,

```
'halfcircle'
```

Then the program asks the user to input the supreme of index of refraction by display

```
Input the supreme of index of refraction -->
```

The user could type, for example,

```
16
```

Note that the actual definition for the index of refraction $n(x)$ is defined in 'Rindex.m'. Finally the program asks how many transmission eigenvalues to be computed by displaying

```
Input the number of transmission eigenvalues -->
```

The user could type, for example,

```
16
```

Taking all above input, the program does the following

1. Construct the stiffness matrix S, mass matrix M and weighted mass matrix Mn (by calling subroutine 'assemble').
2. Identify the interior and boundary nodes and store them in 'Inode' and 'Bnode' (by calling subroutine 'intnode').
3. Compute the first Dirichlet eigenvalue (by calling subroutine 'DirichletEig').
4. Construct the matrices A and B for the generalized eigenvalue problem (by calling subroutine 'MixMethod').
5. Compute transmission eigenvalues (by calling subroutine 'sptranite').

function [Smat, Mmat, Mnmat] = assemble(mesh)

The following illustrates the data structure of triangular mesh from Matlab PDE tool:

— In the Point matrix p, the first and second rows contain x- and y-coordinates of the points in the mesh.

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— In the Edge matrix e, the first and second rows contain indices of the starting and ending point, the third and fourth rows contain the starting and ending parameter values, the fifth row contains the edge segment number, and the sixth and seventh row contain the left- and right-hand side subdomain numbers.
— In the Triangle matrix t, the first three rows contain indices to the corner points, given in counter clockwise order, and the fourth row contains the subdomain number.

The construction of stiffness matrix S, mass matrix M and weighted mass matrix Mn is fairly standard. It uses 3 points quadrature rule for linear finite element.

function [Inode,Bnode]=intnode(mesh)

This subroutine takes the mesh and find the interior and boundary nodes and store them in 'Inode' and 'Bnode' respectively.

function lambda=DirichletEig(A, M)

This subroutine takes the stiff and mass matrices and compute the first Dirichlet eigenvalue by calling Matlab command 'eigs'. Note that this problem need zero boundary values.

function [A,B]=MixMethod(S, Ma, Mn, Inode, Bnode)

This subroutine takes the matrices S, Ma, Mn and constructs the matrices A and B described in Section 2.

function k=sptranite(A,B,lb,noe)

This subroutine takes the matrices A and B and compute transmission eigenvalues using Matlab command 'sptarn'. At beginning, the left bound is given by $lb = \lambda_0/\sup_D(n)$ and the right bound is given by $rb = lb + 1$. It calls 'sptarn' to compute generalized eigenvalues. Complex eigenvalues are excluded and real eigenvalues are stored. Then the interval is shifted to right by one unit and 'sptarn' is called again until 'noe' eigenvalues are found.

function n=Rindex(x,index)

The actual index of refraction needs to be defined here. The coordinate is x and 'index' is the index of the medium containing x. This index is related to the mesh construction phase. For example, the whole domain can be divided into three regions with index 1, 2 and 3 respectively.