Some Results on Electromagnetic Transmission Eigenvalues

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Abstract

The electromagnetic interior transmission problem is a boundary value problem which is neither elliptic nor self-adjoint. The associated transmission eigenvalue problem has important applications in the inverse electromagnetic scattering theory for inhomogeneous media. In this paper, we show that in general there do not exist purely imaginary electromagnetic transmission eigenvalues. For constant index of refraction, we prove that it is uniquely determined by the smallest (real) transmission eigenvalue. Finally, we show that complex transmission eigenvalues must lie in a certain region in the complex plane. The result is verified by examples.

1 Introduction

The interior transmission problem arises in the study of inverse scattering theory for inhomogeneous media [13, 19, 14]. Due to its importance in the study of the far field pattern and reconstruction of the index of refraction [6, 8, 26], this problem received significant attention recently in the inverse scattering community [16, 2, 3, 28]. Cakoni et al. [2] show that transmission eigenvalues could be determined from scattering data and used to give a lower bound for the index of refraction. A similar result of the Maxwell's equation is given in [8] for anisotropic media where the transmission eigenvalues are used to obtain upper and lower bounds on the norm of the index of refraction. In [4], the authors show that transmission eigenvalues can be determined from scattering data and used to determine the presence of cavities in a dielectric. In fact, the transmission eigenvalues can be determined from the near field data as well [26, 3]. In particular, an optimization technique using a numerical method for transmission eigenvalues is used by Sun in [26] to estimate the index of refraction.

The existence of a finite number of (real) transmission eigenvalues in the general case was first given by Päivärinta and Sylvester [25]. Cakoni and Haddar [11] and Kirsch [20] proved the similar result for anisotropic media and Maxwell's equations. We refer the readers to [6, 9, 12] and references therein for recent results on the existence of (real) transmission eigenvalues. In [5], Cakoni et al. show that if the domain is a disk and the constant index refraction is small enough, complex transmission eigenvalues exist. However, to the authors' knowledge, the existence of complex transmission eigenvalues for more general cases is an open problem.

Since the problem is neither elliptic nor self-adjoint, numerical computation of the interior transmission problem and the associated eigenvalue problem is challenging. Finite element methods for transmission eigenvalues were first proposed in Colton-Monk-Sun 2010 [15]. Hsiao et al. 2011 [17] employed a coupled boundary element method and finite element method for the interior transmission problem. In Sun 2011 [27], two iterative methods based on the fourth order reformulation of the transmission eigenvalues were proposed. The convergence of the methods were also given. However, the H^2 conforming Argyris elements were used for the fourth order problem. Ji et al. [18] proposed a mixed finite element method where an iterative Arnoldi method was used to search for the eigenvalues of the resulted non-Hermitian matrix problem. The idea was further extended to the case of Maxwell's equations by Monk and Sun [22].

In this paper, we consider the transmission eigenvalue problem for Maxwell's equations. Using results on Sobolev spaces related to Maxwell's equations, we first show that there does not exist purely imaginary transmission eigenvalues for general domain and index of refraction. Then we consider the inverse scattering problem of determining the index of refraction using transmission eigenvalues. In particular, we show that the constant index of refraction is uniquely determined by the lowest (real) transmission eigenvalue. In the last part of the paper, we determine regions in the complex plane where transmission eigenvalues must lie in. Numerical examples are also provided. Note that the results in this paper for Maxwell's transmission eigenvalues parallel to those in Colton-Monk-Sun 2010 [15] and Cakoni-Colton-Gintides 2010 [5] for transmission eigenvalues of the Helmholtz equation.

2 Transmission Eigenvalue Problem

In this section, we give the definition of the transmission eigenvalue problem for Maxwell's equations and show that there are no purely imaginary transmission eigenvalues.

Let $D \subset \mathbb{R}^3$ be a bounded connected region with piece-wise smooth boundary ∂D. Let ν be the unit outward normal to ∂D. We first introduce necessary functional spaces to analyze the transmission eigenvalue problem. We refer the readers to [21] for more details. The Hilbert space $H(\text{curl}; D)$ is defined as

$$H\left(\text{curl}, D\right) := \left\{ u \in L^2\left(D\right)^3 ; \nabla \times u \in \left(L^2\left(D\right)\right)^3 \right\}$$

equipped with the scalar product

$$(u, v)_{\text{curl}} = (u, v) + (\nabla \times u, \nabla \times v)$$

where $(\cdot, \cdot)$ is the L^2 inner product on D. A subspace of $H(\text{curl}; D)$ is given by

$$H_0\left(\text{curl}; D\right) := \left\{ u \in H\left(\text{curl}\, u; D\right); \nu \times u = 0 \text{ on } \partial D \right\}.$$

We also define $H(\text{div}; D)$ as

$$H(\text{div}; D) := \left\{u \in L^2(D)^3 ; \nabla \cdot u \in L^2(D)\right\}$$

equipped with the scalar product

$$(u, v)_{\text{div}} = (u, v) + (\nabla \cdot u, \nabla \cdot v) .$$

To analyze the transmission eigenvalue problem, we need a special functional space

$$H\left(\text{curl}^2; D\right) := \{u \in H(\text{curl}; D) ; \nabla \times u \in H(\text{curl}; D)\}$$

equipped with the scalar product

$$(u, v)_{\text{curl}^2} = (u, v) + (\nabla \times u, \nabla \times v) + (\nabla \times \nabla \times u, \nabla \times \nabla \times v) .$$

A useful subspace of $H\left(\text{curl}^2; D\right)$ is given by

$$H_0\left(\text{curl}^2; D\right) := \{u \in H_0(\text{curl}; D) ; \nabla \times u \in H_0(\text{curl}; D)\} .$$

Let N be a 3×3 matrix valued function defined on D with $L^\infty(D)$ real valued entries i.e. $N \in L^\infty\left(D, \mathbb{R}^{3\times 3}\right)$.

Definition 2.1. *A real matrix field $N \in L^\infty(D; \mathbb{R}^{3\times 3})$ is said to be positive definite if there exists a constant $\gamma > 0$ such that*

$$\psi \cdot N\psi \geq \gamma |\psi|^2 \quad \forall \psi \in \mathbb{C}^3 \quad \text{a.e.} \quad \text{in} \quad D$$

We assume that N, N^{-1} and either $(N - I)^{-1}$ or $(I - N)^{-1}$ are bounded positive definite real matrix fields on D. In terms of the electric field, the electromagnetic transmission eigenvalue problem can be formulated as follows.

Definition 2.2. *Find values of $k^2 \neq 0$ and $E, E_0 \in H(curl; D)$ such that*

$$\nabla \times \nabla \times E - k^2 N(x) E = 0 \quad \text{in} \quad D, \tag{2.1a}$$

$$\nabla \times \nabla \times E_0 - k^2 E_0 = 0 \quad \text{in} \quad D. \tag{2.1b}$$

$$\nu \times E = \nu \times E_0 \quad \text{on} \quad \partial D, \tag{2.1c}$$

$$\nu \times \nabla \times E = \nu \times \nabla \times E_0 \quad \text{on} \quad \partial D. \tag{2.1d}$$

An equivalent fourth order problem can be obtained by introducing $u = E - E_0$ (see [11]). Then we have that $v = NE - E_0$ and

$$E = (N - I)^{-1}(v - u), \quad E_0 = (I - N)^{-1}(Nu - v).$$

Subtracting (2.1b) from (2.1a), we obtain

$$\nabla \times \nabla \times u = k^2 v,$$

and therefore

$$E = (N - I)^{-1}\left(\frac{1}{k^2}\nabla\times\nabla\times u - u\right). \tag{2.2}$$

Substituting for E in (2.1a) and taking the boundary conditions (2.1c) and (2.1d) into account, we end up with a fourth order differential equation for u satisfying

$$\left(\nabla\times\nabla\times -k^2 N\right)(N-I)^{-1}\left(\nabla\times\nabla\times u - k^2 u\right) = 0 \tag{2.3}$$

Hence the variational formulation for the transmission eigenvalue problem can be stated as: finding $k^2 \neq 0$ and $u \in H_0(\text{curl}^2; D)$ such that

$$\left((N-I)^{-1}\left(\nabla\times\nabla\times -k^2 I\right)u, (\nabla\times\nabla\times -k^2 N)\phi\right) = 0 \tag{2.4}$$

for all ϕ in $H_0(\text{curl}^2; D)$. Note that for electric fields, we also have the condition $\nabla\cdot E = \nabla\cdot E_0 = 0$ which implies $\nabla\cdot u = 0$. Similar to Theorem 2.2 in Colton-Monk-Sun 2010 [15] for the Helmholtz equation, we show that there do not exist purely imaginary transmission eigenvalues for Maxwell's equations.

Theorem 2.3. *We assume N, N^{-1} and either $(N-I)^{-1}$ or $(I-N)^{-1}$ are bounded positive real matrices for $x \in \bar{D}$. Then there are no purely imaginary transmission eigenvalues.*

Proof. We first assume that $(N-I)^{-1}$ is a bounded positive real matrix. Following Cakoni and Haddar 2009 [11], we define

$$\begin{aligned} A_\tau(u,v) &= \left((N-I)^{-1}(\nabla\times\nabla\times u - \tau u), (\nabla\times\nabla\times v - \tau v)\right) + \tau^2(u,v),\\ B(u,v) &= (\nabla\times u, \nabla\times v), \end{aligned}$$

where we have set $\tau = k^2$. Then (2.4) can be written as

$$A_\tau(u,v) - \tau B(u,v) = 0 \quad \forall v \in V.$$

Due to the fact that $\nabla\cdot u = 0$, we define

$$V := \left\{u \in H_0(\text{curl}^2; D)\cap H(\text{div}; D); \nabla\cdot u = 0\right\}.$$

If k is purely imaginary, then $\tau = -\sigma < \delta$ for some $\delta > 0$. We have

$$0 = A_\tau(u,u) + \sigma B(u,u).$$

On the other hand

$$A_\tau(u,u) + \sigma B(u,u) \geq \sigma^2(u,u) + \sigma(\nabla\times u, \nabla\times u).$$

This implies $u = 0$.

When $(I-N)^{-1}$ is a bounded positive real matrix, we define

$$\begin{aligned} \tilde{A}_\tau(u,v) &= \left((I-N)^{-1}(\nabla\times\nabla\times u - \tau N u), (\nabla\times\nabla\times v - \tau N v)\right) + \tau^2(Nu,v)\\ &= \left(N(I-N)^{-1}(\nabla\times\nabla\times u - \tau u), (\nabla\times\nabla\times v - \tau v)\right)\\ &\qquad + (\nabla\times\nabla\times u, \nabla\times\nabla\times v). \end{aligned}$$

Again, we assume that k is purely imaginary. Letting $\tau = -\sigma < \delta$ for $\delta > 0$ and setting $v = u$, we have

$$\tilde{A}_\tau(u,u) + \sigma B(u,u) = 0.$$

On the other hand, noting that $\nabla \cdot u = 0$, we have

$$\begin{aligned}\tilde{A}_\tau(u,u) + \sigma B(u,u) &\geq (\nabla \times \nabla \times u, \nabla \times \nabla \times u) + \sigma(\nabla \times u, \nabla \times u) \\ &\geq C \left\|u\right\|_{H(\mathrm{curl}^2;D)}\end{aligned}$$

due to the Friedrichs inequality (see Corollary 3.51 of Monk 2003 [21]). Thus we have $u = 0$. Hence, for both cases, there do not exist purely imaginary transmission eigenvalues. □

3 A Uniqueness Theorem

Now we consider an inverse problem of determining the index of refraction using transmission eigenvalues. For spherically stratified media, this problem was studied in [24, 23, 5]. In this section, we will show that the lowest (real) transmission eigenvalue uniquely determines the constant index of refraction, i.e., when $N = nI$ for some constant n. The result we obtain here parallels the uniqueness theorem in Cakoni-Colton-Gintides 2010 [5] for the case of Helmholtz equation.

To this end, we define

$$\lambda(\tau, n) = \inf_{u \in V, \|\nabla \times u\| = 1} \left\{ (N - I)^{-1} \left\|\nabla \times \nabla \times u - \tau u\right\|^2 + \tau^2 \left\|u\right\|^2 \right\} \tag{3.5}$$

or

$$\lambda(\tau, n) = \inf_{u \in V, \|\nabla \times u\| = 1} \left\{ N(I - N)^{-1} \left\|\nabla \times \nabla \times u - \tau u\right\|^2 + \left\|\nabla \times \nabla \times u\right\|^2 \right\} \tag{3.6}$$

corresponding to the cases of $\gamma > 1$ and $0 < \gamma < 1$ in Definition 2.1, respectively.

From (2.3), we have that

$$\left(\nabla \times \nabla \times -k^2 nI\right) \frac{1}{n-1} \left(\nabla \times \nabla \times -k^2\right) u = 0 \tag{3.7}$$

From the proof of Theorem 2.3 we see that transmission eigenvalues satisfy

$$A_\tau(u, v) - \tau B(u, v) = 0 \tag{3.8}$$

or

$$\tilde{A}_\tau(u, v) - \tau B(u, v) = 0 \tag{3.9}$$

corresponding to the cases of $n > 1$ and $0 < n < 1$, respectively. Thus the first transmission eigenvalue k_1 is the smallest zero of (see [10])

$$\lambda(\tau, n) - \tau = 0. \tag{3.10}$$

It is easy to see that $\lambda(\tau, n)$ is a continuous function of $\tau \in (0, +\infty)$. In the following, we show that n can be uniquely determined from the smallest (real) transmission eigenvalue.

Theorem 3.1. *The constant index of refraction n is uniquely determined from a knowledge of the smallest transmission eigenvalue $k_{1,n} > 0$, provided that it is known a priori that either $n > 1$ or $0 < n < 1$.*

Proof. We first assume that $n_i > 1, i = 1, 2$ with $1 < n_1 < n_2$. It is easy to see that $\lambda(\tau, n_2) \le \lambda(\tau, n_1)$. Let k_{1,n_1} be the smallest transmission eigenvalue for (2.1) with n_1, and let $u_1 = E_1 - E_{0,1}$ where E_1, $E_{0,1}$ are the corresponding non-zero solutions. We normalize u_1 such that

$$\|\nabla \times u_1\| = 1.$$

Setting $\tau_1 = k_1^2$, we have

$$\frac{1}{n_1 - 1} \|\nabla \times \nabla \times u_1 - \tau_1 u_1\|^2 + \tau_1^2 \|u_1\|^2 - \tau_1 = 0.$$

Furthermore, we have

$$\begin{aligned} &\frac{1}{n_2 - 1} \|\nabla \times \nabla \times u - \tau u\|^2 + \tau^2 \|\nabla \times u\|^2 \\ &\qquad \le \frac{1}{n_1 - 1} \|\nabla \times \nabla \times u - \tau u\|^2 + \tau^2 \|\nabla \times u\|^2 \end{aligned}$$

for all $u \in V$ such that $\|\nabla \times u\| = 1$ and $\tau > 0$. Now for $u = u_1$ and $\tau = \tau_1$ we have that

$$\begin{aligned} &\frac{1}{n_2 - 1} \|\nabla \times \nabla \times u_1 - \tau_1 u_1\|^2 + \tau_1^2 \|u_1\|^2 \\ &\qquad < \frac{1}{n_1 - 1} \|\nabla \times \nabla \times u_1 - \tau_1 u_1\|^2 + \tau_1^2 \|u_1\|^2 \\ &\qquad = \lambda(\tau_1, n_1). \end{aligned}$$

On the other hand, we have that

$$\lambda(\tau_1, n_2) \le \frac{1}{n_2 - 1} \|\nabla \times \nabla \times u_1 - \tau_1 u_1\|^2 + \tau_1^2 \|u_1\|^2 < \lambda(\tau_1, n_1).$$

Hence we obtain

$$\lambda(\tau_1, n_2) < \lambda(\tau_1, n_1).$$

Now we look at $\lambda(\tau, n_2) - \tau = 0$. Let $\lambda(D)$ be the smallest Maxwell's eigenvalue in D. For all $\tau > 0$ small enough such that $\tau \in (0, \lambda(D)/n_2)$, we have that (see Lemma 2.9 of [9])

$$\lambda(\tau, n_2) - \tau > 0. \tag{3.11}$$

Meanwhile

$$\lambda(\tau_1, n_2) - \tau_1 < \lambda(\tau_1, n_1) - \tau_1 = 0 \tag{3.12}$$

Since λ is continuous, from (3.11) and (3.12) there exists τ_2, $0 < \tau_2 < \tau_1$, such that

$$\lambda(\tau_2, n_2) - \tau_2 = 0$$

i.e. τ_2 is a transmission eigenvalue when $N = n_2 I$. In summary, if $n_1 \neq n_2$, we have $\tau_1 \neq \tau_2$ and the unique determination is established. The case for $0 < n < 1$ can be proved similarly. □

Remark 3.2. *It was proved in [8] that the function $f(\tau) := \lambda(\tau, n) - \tau$ is differentiable. Furthermore, $f(\tau)$ is monotonically decreasing on the interval $(0, \frac{n+1}{2n}\lambda(D))$ [8, 27]. This property is used in [26] in an optimization technique to find a constant estimation for the index of refraction.*

4 Eigenvalue free zones in the Complex Plane

In the following, we will investigate the eigenvalue free zone in the complex plane for the Maxwell's transmission eigenvalues defined in (2.1). It turns out that the result for the Maxwell's equations is similar to that for the Helmholtz equation (Section 3.2 of [5]). Note that the result in this section and that in [5] are under the assumption that the index of refraction is constant while Theorem 2.3 holds for general cases.

We first assume that $N = nI$ for some $n > 1$. Let $k := x + iy$. Thus $k^2 := \tau + i\mu$ for $\tau = x^2 - y^2$, $\mu = 2xy$. Since we consider the complex eigenvalues, we will use the notation $\bar{u}$ for the conjugate of u explicitly. Recall that the transmission eigenvalue problem can be written as: find $(k, u) \in \mathbb{C} \times H_0(\mathrm{curl}^2, D)$ such that

$$\int_D \frac{1}{n-1}(\nabla \times \nabla \times u - k^2 u) \cdot (\nabla \times \nabla \times \bar{v} - k^2 n\bar{v})\,\mathrm{d}x = 0 \tag{4.13}$$

for all $v \in H_0(\mathrm{curl}^2, D)$. Letting $v = u$ in the above equation, we obtain

$$\begin{aligned}
0 &= \int_D \frac{1}{n-1}(\nabla \times \nabla \times u - k^2 u) \cdot (\nabla \times \nabla \times \bar{u} - k^2 n\bar{u})\mathrm{d}x \\
&= \int_D \frac{1}{n-1}(\nabla \times \nabla \times u - k^2 nu + k^2(n-1)u) \cdot (\nabla \times \nabla \times \bar{u} - \bar{k}^2 n\bar{u} + (\bar{k}^2 - k^2)n\bar{u})\,\mathrm{d}x \\
&= \int_D \frac{1}{n-1}\left|\nabla \times \nabla \times u - k^2 nu\right|^2 \mathrm{d}x + \int_D \frac{1}{n-1}(\nabla \times \nabla \times u - k^2 nu)(\bar{k}^2 - k^2)n\bar{u}\,\mathrm{d}x \\
&\qquad + \int_D k^2 u \cdot (\nabla \times \nabla \times \bar{u} - k^2 n\bar{u})\mathrm{d}x \\
&= \int_D \frac{1}{n-1}\left|\nabla \times \nabla \times u - k^2 nu\right|^2 \mathrm{d}x + \int_D (\bar{k}^2 - k^2)\frac{n}{n-1}\left|\nabla \times u\right|^2 \mathrm{d}x \\
&\qquad - \int_D k^2(\bar{k}^2 - k^2) \cdot \frac{n^2}{n-1}\left|u\right|^2 \mathrm{d}x + \int_D k^2 \left|\nabla \times u\right|^2 \mathrm{d}x - \int_D k^4 n \left|u\right|^2 \mathrm{d}x.
\end{aligned}$$

Setting $k^2 := \tau + i\mu$, we have

$$\begin{aligned}
0 &= \int_D \frac{1}{n-1} \left|\nabla \times \nabla \times u - (\tau + i\mu) n u\right|^2 \, \mathrm{d}x \\
&\quad + \int_D \left[(\tau + i\mu) \cdot 2i\mu \cdot \frac{n^2}{n-1} - (\tau^2 - \mu^2 + 2i\mu\tau) n\right] |u|^2 \, \mathrm{d}x \\
&\qquad + \int_D \left[-2\mu i \frac{n}{n-1} + \tau + i\mu\right] |\nabla \times u|^2 \, \mathrm{d}x \\
&= \int_D \frac{1}{n-1} \left|\nabla \times \nabla \times u - (\tau + i\mu) n u\right|^2 \, \mathrm{d}x \\
&\quad + \int_D \left[-2\mu^2 \frac{n^2}{n-1} - (\tau^2 - \mu^2) n\right] |u|^2 \, \mathrm{d}x + \int_D \tau \, |\nabla \times u|^2 \, \mathrm{d}x \\
&\qquad + i \left\{ \int_D \left[2\mu\tau \frac{n^2}{n-1} - 2\tau\mu n\right] |u|^2 \, \mathrm{d}x + \int_D \left[\mu - 2\mu \frac{n}{n-1}\right] |\nabla \times u|^2 \, \mathrm{d}x \right\}.
\end{aligned}$$

Taking the imaginary part of the equation and dividing by $\mu \neq 0$, we obtain

$$0 = -\int_D \frac{n+1}{n-1} |\nabla \times u|^2 \, \mathrm{d}x + 2\tau \int_D \frac{n}{n-1} |u|^2 \, \mathrm{d}x.$$

Since $n > 1$, the above equation implies $u = 0$ if $\tau \leq 0$. Again we have employed the Friedrichs inequality when $\tau = 0$. In terms of x and y, it implies that u is trivial if $x^2 \leq y^2$. Thus a complex eigenvalue must line in the region $x^2 > y^2$.

Taking the real part, we have

$$\begin{aligned}
0 \quad = \quad & \int_D \frac{1}{n-1} \left|\nabla \times \nabla \times u - (\tau + i\mu) n u\right|^2 \, \mathrm{d}x \\
& \qquad + \tau \int_D |\nabla \times u|^2 \, \mathrm{d}x - \int_D \left[(\tau^2 - \mu^2) n + 2\mu^2 \frac{n^2}{n-1}\right] |u|^2 \, \mathrm{d}x.
\end{aligned}$$

But we have that

$$\begin{aligned}
\tau \int_D |\nabla \times u|^2 \, \mathrm{d}x &- \int_D (\tau^2 - \mu^2) n + 2\mu^2 \frac{n^2}{n-1} |u|^2 \, \mathrm{d}x \\
&\geq \left(\tau \lambda(D) - (\tau^2 - \mu^2) n - 2\mu^2 \frac{n^2}{n-1}\right) \|u\|^2_{L^2(D)}
\end{aligned}$$

where $\lambda(D)$ is the smallest Maxwell's eigenvalue. Thus the real and imaginary part of a complex eigenvalue k^2 must satisfy

$$\tau \lambda(D) - (\tau^2 - \mu^2) n - 2\mu^2 \frac{n^2}{n-1} < 0.$$

We can write it as

$$\tau^2 - \tau \frac{\lambda(D)}{n} + \mu^2 \frac{n+1}{n-1} > 0. \tag{4.14}$$

Note that for the case of real transmission eigenvalues, i.e., $\mu = 0$, we have

$$\tau^2 - \tau\frac{\lambda(D)}{n} > 0$$

which recovers the well-known Faber-Krahn estimate for the case of Maxwell's equations (see Section 4.4 of Cakoni-Colton-Monk 2011 [7])

$$k^2 > \frac{\lambda(D)}{n}.$$

The relation (4.14) in terms of the real and imaginary part of k, x and y, can be written as

$$x^4 + y^4 + x^2y^2\frac{2n+6}{n-1} - (x^2 - y^2)\frac{\lambda(D)}{n} > 0. \tag{4.15}$$

Combining both conditions we can conclude that complex transmission eigenvalues $k = x + iy$ (if they exist) lie in the region Σ of the complex plane defined by

$$\Sigma := \begin{cases} x^4 + y^4 + x^2y^2\frac{2n+6}{n-1} - (x^2 - y^2)\frac{\lambda(D)}{n} > 0, \\ x^2 > y^2. \end{cases} \tag{4.16}$$

Next we consider the case when $n < 1$. Letting $v = u$ and dropping the constant $\frac{1}{n-1}$, we have

$$\begin{aligned} 0 &= \int_D (\nabla\times\nabla\times u - k^2u)\cdot(\nabla\times\nabla\times\bar{u} - k^2n\bar{u})\mathrm{d}x \\ &= \int_D (\nabla\times\nabla\times u - k^2u)\cdot(\nabla\times\nabla\times\bar{u} - \bar{k}^2\bar{u} + \bar{k}^2\bar{u} - k^2n\bar{u})\mathrm{d}x \\ &= \int_D |\nabla\times\nabla\times u - k^2u|^2\mathrm{d}x + \int_D (\bar{k}^2 - k^2n)|\nabla\times u|^2\mathrm{d}x + \int_D k^2(k^2n - \bar{k}^2)|u|^2\mathrm{d}x \end{aligned}$$

Setting $k^2 = \tau + i\mu$, we obtain

$$\begin{aligned} 0 &= \|\nabla\times\nabla\times u - k^2u\|^2 + \{(\tau - i\mu) - (\tau + i\mu)n\}\|\nabla\times u\|^2 \\ &\qquad + \{(\tau^2 - \mu^2 + 2\tau\mu i)n - \tau^2 - \mu^2\}\|u\|^2 \\ &= \|\nabla\times\nabla\times u - k^2u\|^2 + (1-n)\tau\|\nabla\times u\|^2 + (n\tau^2 - n\mu^2 - \tau^2 - \mu^2)\|u\|^2 \\ &\qquad + i\left\{-(n+1)\mu\|\nabla\times u\|^2 + 2n\tau\mu\|u\|^2\right\} \end{aligned}$$

We take the imaginary part to obtain

$$0 = -(n+1)\mu\|\nabla\times u\|^2 + 2n\tau\mu\|u\|^2.$$

Hence if $u \neq 0$, we have that

$$-(n+1)\|\nabla\times u\|^2 + 2n\tau\|u\|^2 = 0$$

which implies that $\tau > 0$, i.e., $x^2 > y^2$.

Taking the real part, we have

$$0 = \|\nabla \times \nabla \times u - k^2 u\|^2 + (1-n)\tau \|\nabla \times u\|^2 + (n\tau^2 - n\mu^2 - \tau^2 - \mu^2)\|u\|^2.$$

In order to have non-trivial u, we need to have that

$$(1-n)\tau \|\nabla \times u\|^2 + (n\tau^2 - n\mu^2 - \tau^2 - \mu^2)\|u\|^2 < 0.$$

Applying the Friedrichs inequality, we have

$$(1-n)\tau\lambda(D) + (n\tau^2 - n\mu^2 - \tau^2 - \mu^2) < 0.$$

Thus we obtain

$$-\tau^2 + \lambda(D)\tau - \frac{1+n}{1-n}\mu^2 < 0.$$

In terms of x and y we have that

$$-x^4 - y^4 + \lambda(D)(x^2 - y^2) - \frac{6n+2}{1-n}x^2y^2 < 0. \tag{4.17}$$

Hence for $0 < n < 1$, complex transmission eigenvalues $k = x + iy$ (if they exist) lie in the region Σ of the complex plane defined by

$$\Sigma := \begin{cases} -x^4 - y^4 + \lambda(D)(x^2 - y^2) - \frac{6n+2}{1-n}x^2y^2 < 0, \\ \\ x^2 > y^2. \end{cases} \tag{4.18}$$

The regions (4.16) and (4.18) defines the possible location of transmission eigenvalues for $n > 1$ and $0 < n < 1$, respectively. This is similar to the case for the Helmholtz equation [5]. However, we need to compute the smallest Maxwell's eigenvalue which itself has been an interesting research topic [21, 1]. In the following we show two examples to verify the above result.

Example 1. Let D be the unit ball. The smallest Maxwell's eigenvalue is $\lambda(D) \approx 7.53$ (see Bramble-Kolev-Pasciak 2005 [1]). We first let $n = 4$. Then the region Σ is given by

$$\Sigma := \begin{cases} x^4 + y^4 + \frac{14}{3}x^2y^2 - 1.88(x^2 - y^2) > 0, \\ \\ x^2 > y^2. \end{cases} \tag{4.19}$$

For constant index of refraction $N = nI$, we can write down the transmission eigenvalues using separation of variables (see Monk-Sun 2011 [22]). In fact, the transmission eigenvalues are given by the wave number k^2's satisfying

$$\begin{vmatrix} j_m(k\rho) & j_m(k\sqrt{n}\rho) \\ \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho j_m(k\rho)) & \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho j_m(k\sqrt{n}\rho)) \end{vmatrix} = 0, \quad m \geq 1 \tag{4.20}$$

corresponding to the TE mode (see also [12]) and

$$\begin{vmatrix} \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho j_m(k\rho)) & \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho j_m(k\sqrt{n}\rho)) \\ k^2 j_m(k\rho) & k^2 n j_m(k\sqrt{n}\rho) \end{vmatrix} = 0, \quad m \geq 1. \tag{4.21}$$

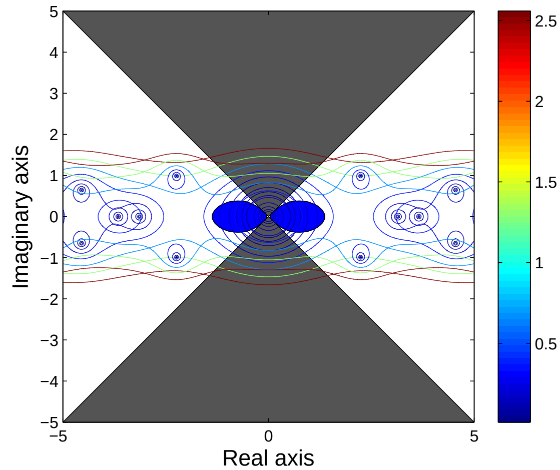

Figure 1: The distribution of transmission eigenvalues for the unit ball when $n = 4$. Σ defined in (4.19) is the unshaded region. We also plot contours of absolute values of the determinants given in (4.20) and (4.21) for $m = 1$. The transmission eigenvalues are at the positions where the values are zeros (the centers of small circular contours).

corresponding to the TM mode. Here j_m is the spherical Bessel's function of order m.

In Figure 1, we plot the region Σ given in (4.19) (unshaded part). For verification, we also give the contour plot of the absolute values of the determinants given in (4.20) and (4.21) for $m = 1$. The analytical transmission eigenvalues are corresponding to the positions where the absolute values are zeros (the centers of small circular contours). We see that these transmission eigenvalues lie in the region Σ defined in (4.19).

Example 2. The second example is the unit cube. In this case, the Maxwell's eigenvalue can be derived exactly. In fact, they are given by $k^2\pi^2$ where $k^2 = k_1^2 + k_2^2 + k_3^2$ and $k_i, i = 1, 2, 3$ are non-negative integers satisfying $k_1k_2 + k_2k_3 + k_3k_1 > 0$ (see also Bramble-Kolev-Pasciak 2005 [1]). Thus $\lambda(D) = 2\pi^2$. Letting $n = 6$, the region Σ is given by

$$\Sigma := \begin{cases} x^4 + y^4 + \frac{18}{5}x^2y^2 - \frac{\pi^2}{3}(x^2 - y^2), \\ \qquad\qquad x^2 > y^2. \end{cases} \tag{4.22}$$

For this case, there is no analytic way to derive the transmission eigenvalues and we employ the mixed finite element method to compute a few of them (see Monk-Sun 2011 [22]).

In Figure 2, we plot the region Σ given in (4.22) (unshaded part). For verification, we plot a few computed transmission eigenvalues (denoted by '*' in the figure). We see that these transmission eigenvalues, both real and complex, lie in the region Σ defined in (4.22).

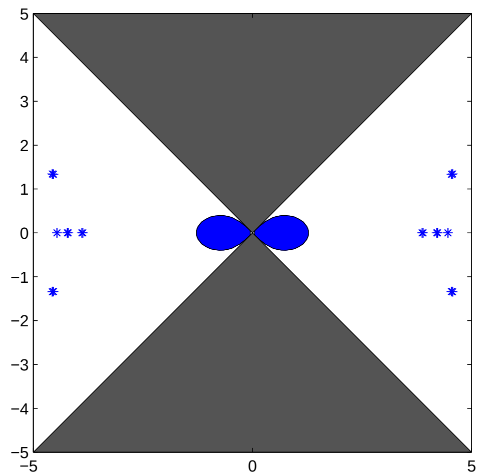

Figure 2: The distribution of transmission eigenvalues for the unit cube when $n = 6$. Σ defined in (4.22) is the unshaded region. In the figure, '*' denotes the computed transmission eigenvalues.

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References

[1] J. H. Bramble, T. V. Kolev, and J. E. Pasciak , *The approximation of the Maxwell eigenvalue problem using a least-squares method*, Math. Comp. 74 (2005), 1575-1598.

[2] F. Cakoni, D. Colton and P. Monk, *On the use of transmission eigenvalues to estimate the index of refraction from far field data*, Inverse Problems 23 (2007) 507-522.

[3] F. Cakoni, D. Colton and J. Sun, *Estimation of Dirichlet and transmission eigenvalues by near field Linear Sampling Method*, Proceedings of the 10th International Conference on the Mathematical and Numerical Aspects of Waves, 2011, 431-434.

[4] F. Cakoni, M. Cayoren and D. Colton, *Transmission eigenvalues and the nondestructive testing of dielectrics*, Inverse Problems 24 (2008) 065016.

[5] F. Cakoni, D. Colton and D. Gintides, *The interior transmission eigenvalue problem*, SIAM J. Math. Anal., 42 (2010), No. 6, 2912-2921.

[6] F. Cakoni, D. Colton, and H. Haddar, *On the determination of Dirichlet or transmission eigenvalues from far field data*, C. R. Math. Acad. Sci. Paris, Ser. I, 348 (2010) pp. 379–383.

[7] F. Cakoni, D. Colton and P. Monk, The Linear Sampling Method in Inverse Electromagnetic Scattering, SIAM, Philadelphia, 2011.

[8] F. Cakoni, D. Colton, P. Monk and J. Sun, *The inverse electromagnetic scattering problem for anisotropic media*, Inverse Problems 26 (2010), 074004.

[9] F. Cakoni, D. Gintides and H. Haddar, *The existence of an infinite discrete set of transmission eigenvalues*, SIAM J. Math. Anal., 42 (2010), No 1, 237-255.

[10] F. Cakoni and D. Gintides, *New results on transmission eigenvalues*. Inverse Probl. Imaging 4 (2010), No. 1, 3948.

[11] F. Cakoni and H. Haddar, *On the existence of transmission eigenvalues in an inhomogenous medium*, Appl. Anal., 88 (2009), No. 4, 475-493.

[12] F. Cakoni and A. Kirsch, *On the interior transmission eigenvalue problem*, Int. Jour. Comp. Sci. Math. 3 (2010), No. 1-2, 142-167.

[13] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, 2nd Edition, Springer-Verlag, 1998.

[14] D. Colton, L. Päivärinta and J. Sylvester, *The interior transmission problem*, Inverse Probl. Imaging, 1 (2007), No. 1, 13-28.

[15] D. Colton, P. Monk and J. Sun, *Analytical and computational methods for transmission eigenvalues*, Inverse Problems, 26 (2010), 045011.

[16] H. Haddar, *The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem*, Math. Methods Appl. Sci., 27 (2004), Iss. 18, 2111-2129.

[17] G. Hsiao, L. Xu, F. Liu and J. Sun, *A coupling procedure for the interior transmission problem*, J. Comput. Appl. Math., 235 (2011), Iss. 17, 5213–5221.

[18] X. Ji, T. Turner and J. Sun, *A mixed finite element method for Helmholtz transmission eigenvalues*, submitted.

[19] A. Kirsch, *The denseness of the far field patterns for the transmission problem*, IMA J. Appl. Math., 37 (1986), 213-225.

[20] A. Kirsch, *On the existence of transmission eigenvalues*, Inverse Problems and Imaging, Vol. 3 (2009), No. 2, 155–172.

[21] P. Monk, Finite Element Methods for Maxwell's Equations, Oxford University Press, 2003.

[22] P. Monk and J. Sun, *Finite element methods of Maxwell transmission eigenvalues*, submitted.

[23] J. McLaughlin and P. Polyakov, *On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues*, J. Differential Equations, 107 (1994), 351-382.

[24] J. McLaughlin, P. Polyakov, and P. Sacks, *Reconstruction of a spherically symmetric speed of sound*, SIAM J. Appl. Math., 54 (1994), 1203-1223.

[25] L. Päivärinta and J. Sylvester, *Transmission eigenvalues*, SIAM J. Math. Anal., Vol. 40 (2008), No. 2, 738-753.

[26] J. Sun, *Estimation of transmission eigenvalues and the index of refraction from Cauchy data*, Inverse Problems, 27 (2011) 015009.

[27] J. Sun, *Iterative methods for transmission eigenvalues*, SIAM J. Numer. Anal., Vol. 49 (2011), No. 5, 1860-1874.

[28] J. Sun, *An eigenvalue method using multiple frequency data for inverse scattering problems*, submitted.