# Reconstruction of obstacles embedded in periodic waveguides

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**Abstract**

Reconstruction of obstacles embedded in a periodic waveguide with arbitrary geometry is considered in this paper. The measurement is on a line segment of the scattered field due to point sources inside the waveguide. A linear sampling type method is proposed to characterize the obstacles by using the solutions of the near field equations. Due to the fact that the Green function cannot be written down analytically for periodic waveguide with arbitrary geometry, we employ a method based on the limiting absorption principle and the recursive doubling technique. Furthermore, we devise an algorithm to speed up the sampling procedure. Numerical examples are presented to demonstrate the performance of the proposed method.

**Key words**: Inverse scattering problems, periodic structures, linear sampling method, limiting absorption principle

## 1 Introduction

Inverse problems for periodic structures have a long history and have been investigated by many researchers, for example [15, 13, 16, 14, 17, 19, 20, **?**, 18, 1]. Most of these works address the reconstruction of structure profiles. In this paper, we consider the reconstruction of obstacles embedded in waveguides. Recently this issue has drawn much attention. For example, in [10] Xu et al. applied a method using generalized dual space indicator for imaging an obstacle in a shallow water waveguide. They used the scattered field on a straight line and solved some ill-posed integral equations. Dediu and McLaughlin [11] proposed an eigensystem decomposition to recover weak inhomogeneities in a waveguide from far-field data. In [12], Bourgeois and Lunèville also use far field data to reconstruct the inhomogeneity. Their method is based on the idea of the linear sampling method and a factorization of the far field operator.

A major difficulty of the reconstruction of obstacles embedded in the waveguide is the computation of Greens functions. This is analogous to the case when applying linear sampling method to characterize objects embedded in an inhomogeneous complex background in open space. Methods of avoiding, at least partially, the computation of background Green function have been developed in [5, 4]. Except some rare cases, the background Green function cannot be written as a closed form in contrast to the free space case. This accounts for why most works on the reconstruction of inhomogeneities in waveguide only consider the homogeneous waveguide, which motivates us to develop a method to compute Greens function of a waveguide with arbitrary geometry. In this paper, we employ the method develop in [2] to compute the background Green's function.

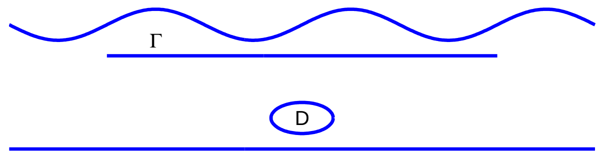


Figure 1: Schematic picture: scattering by an obstacle in a waveguide. The waveguide has a complicate geometry. The obstacle is $D$. The point sources and measurement of the scattered field are on $\Gamma$.

We assume that the incident field is due to a point source on a line $\Gamma$ and the scattered field is measured on the same line (see Fig. 1). For the inverse problem, we will use the near field version of the linear sampling method, i.e., we solve linear integral equations and plot the solution norms in a sampling domain. The major ingredient of the linear sampling method is the computation of the Green function. Since we need the Greens functions for all the sampling points, we propose a Fourier expansion method to significantly reduce the computational cost in the discrete case.

The rest of the paper is organized as follows. Section 2 discusses the scattering of inhomogeneities due to point source incident field in the waveguide. In Section 3, we present the inverse method to characterize the inhomogeneities using the near field equations. In Section 4, we present a fast Fourier expansion method to compute the Green functions. We present some numerical examples in Section 5.

## 2 The direct problem

The inverse problem we consider is to imaging an object $D$ embedded in a two-dimensional periodic waveguide $\Omega$ (see Fig. 1). Let us indicate the top boundary of $\Omega$ as $\partial\Omega^+$ and the bottom boundary of $\Omega$ as $\partial\Omega^-$. For simplicity, we assume that $\partial\Omega^1$ is a straight line and $\partial\Omega^+$ is a periodic function of $x$. The top boundary $\partial\Omega^+$ is supposed sound-soft and the bottom boundary $\partial\Omega^-$ soft-hard. The point sources are located on a curve $\Gamma$ above $D$. This implies that the incident field is simply the background Green function at $\mathbf{r}_s \in \Gamma$ which solves

$$\Delta_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}_s) + k^2 G(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \qquad \text{in } \Omega, \tag{1a}$$

$$G(\mathbf{r}, \mathbf{r}_s) = 0, \qquad \text{on } \partial\Omega^+, \tag{1b}$$

$$\partial_\nu G(\mathbf{r}, \mathbf{r}_s) = 0, \qquad \text{on } \partial\Omega^-. \tag{1c}$$

Let $\Phi = \frac{1}{4i} H_0^{(1)}(k|\mathbf{r} - \mathbf{r}_s|)$ be the Green function in $\mathbb{R}^2$. Then it is known [12] that

$$G(\mathbf{r}, \mathbf{r}_s) \sim \Phi(\mathbf{r}, \mathbf{r}_s) \sim \frac{1}{2\pi} \log(k|\mathbf{r} - \mathbf{r}_s|), \quad \mathbf{r} \to \mathbf{r}_s.$$

Note that the above asymptotic relation also holds for any complex $k$ with $\Re k > 0$, $\Im k > 0$ and $\Im k < \Re k$. For a homogeneous waveguide, it is possible to derive an analytical series expansion for $G$, see [12]. For a periodic waveguide with arbitrary geometric, this is not possible in general.

We suppose the object $D$ is sound-soft. This means that the scattered field $u^s(\cdot, \mathbf{r}_s)$ due to the appearance of $D$ solves the following equation

$$\Delta_{\mathbf{r}} u^s(\mathbf{r}, \mathbf{r}_s) + k^2 u^s(\mathbf{r}, \mathbf{r}_s) = 0, \qquad \mathbf{r} \in \Omega \setminus \bar{D}, \tag{2a}$$
$$u^s(\mathbf{r}, \mathbf{r}_s) = 0, \qquad \mathbf{r} \in \partial\Omega^+, \tag{2b}$$
$$\partial_\nu u^s(\mathbf{r}, \mathbf{r}_s) = 0, \qquad \mathbf{r} \in \partial\Omega^-, \tag{2c}$$
$$u^s(\mathbf{r}, \mathbf{r}_s) = -G(\mathbf{r}, \mathbf{r}_s), \qquad \mathbf{r} \in \partial D. \tag{2d}$$

To ensure the well-posedness, both $G$ and $u^s$ should also satisfy appropriate condition along the longitudinal axis of the waveguide as $|x| \to \infty$. For this purpose, we assume that the *limiting absorption principle* (LAP) holds for the specific geometry $\Omega$ and the specific wave number $k$. The precise meaning of this principle is as follows. For any $\delta > 0$, replacing $k^2$ with $k^2 + i\delta$ in (2a) we obtain a well-posed problem with damping. Let us denote by $u^s_\delta \in H^1(\Omega \setminus D)$ the solution of the damped equation. The LAP implies that $u^s_\delta$ converges in $H^1_{loc}(\Omega \setminus D)$ as $\delta \to 0^+$. Though the physical meaning of LAP is rather natural, the rigorous mathematical justification of this principle for the considered problem, to the authors' knowledge, is still an open problem.

We have the following important reciprocity property of $G(\mathbf{r}, \mathbf{r}_s)$.

**Lemma 2.1.** *The Green function has the following reciprocity relation:*

$$G(\mathbf{r}_f, \mathbf{r}_s) = G(\mathbf{r}_s, \mathbf{r}_f), \ \forall \mathbf{r}_f, \mathbf{r}_s \in \Omega.$$

*Proof.* Let $k_\delta = \sqrt{k^2 + i\delta}$ with $\delta > 0$ and $G_\delta(\mathbf{r}, \mathbf{r}_s)$ be the Green function of (2a) by replacing $k^2$ with $k^2 + i\delta$, i.e.,

$$\Delta_{\mathbf{r}} G_\delta(\mathbf{r}, \mathbf{r}_s) + (k^2 + i\delta) G_\delta(\mathbf{r}, \mathbf{r}_s) = \delta(\mathbf{r} - \mathbf{r}_s), \ \forall \mathbf{r} \in \Omega.$$

For any fixed $\mathbf{r}_f, \mathbf{r}_s \in \Omega$, let $\epsilon > 0$ be small enough and we define

$$D_{\mathbf{r}_s,\epsilon} = \{\mathbf{r} \in \Omega : |\mathbf{r} - \mathbf{r}_s| < \epsilon\},$$
$$D_{\mathbf{r}_f,\epsilon} = \{\mathbf{r} \in \Omega : |\mathbf{r} - \mathbf{r}_f| < \epsilon\},$$

such that $\overline{D}_{\mathbf{r}_s,\epsilon} \cap \overline{D}_{\mathbf{r}_f,\epsilon} = \emptyset$. Set

$$\Omega_\epsilon = \Omega \setminus (\overline{D}_{\mathbf{r}_s,\epsilon} \cap \overline{D}_{\mathbf{r}_f,\epsilon}).$$

Let us denote

$$(*) \overset{def}{=} G_\delta(\mathbf{r}, \mathbf{r}_f) \frac{\partial G_\delta(\mathbf{r}, \mathbf{r}_s)}{\partial n} - G_\delta(\mathbf{r}, \mathbf{r}_s) \frac{\partial G_\delta(\mathbf{r}, \mathbf{r}_f)}{\partial n}.$$

Applying Green formula on $\Omega_\epsilon$, we obtain

$$\int_{\partial\Omega_\epsilon} (*) \ ds = \int_{\Omega_\epsilon} [G_\delta(\mathbf{r}, \mathbf{r}_f) \triangle G_\delta(\mathbf{r}, \mathbf{r}_s) - G_\delta(\mathbf{r}, \mathbf{r}_s) \triangle G_\delta(\mathbf{r}, \mathbf{r}_f)] \ d\mathbf{r} = 0.$$

Thus we have

$$\int_{\partial D_{\mathbf{r}_s,\epsilon}} (*) \ ds + \int_{\partial D_{\mathbf{r}_f,\epsilon}} (*) \ ds = \int_{\partial\Omega} (*) \ ds.$$

The right hand side vanishes due to the boundary conditions on $\partial\Omega^\pm$, which implies

$$\int_{\partial D_{\mathbf{r}_s,\epsilon}} (*) \ ds + \int_{\partial D_{\mathbf{r}_f,\epsilon}} (*) \ ds = 0. \tag{3}$$

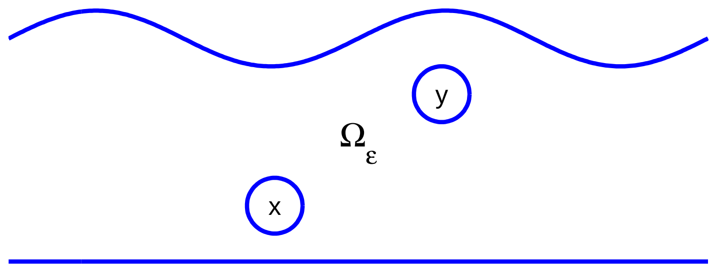


Figure 2: Schematic picture: inverse scattering by an inhomogeneity in waveguide.

On $\partial D_{\mathbf{r}_s,\epsilon}$, there exists a constant $M_1$ depending on $\mathbf{r}_{s,f}$ and $\delta$ such that

$$\left|\frac{\partial G_\delta(\mathbf{r},\mathbf{r}_f)}{\partial n}\right| = |\nabla_{\mathbf{r}} G_\delta(\mathbf{r},\mathbf{r}_f)\cdot n| \le M_1.$$

Besides, by setting

$$h = G_\delta(\mathbf{r},\mathbf{r}_s) - \frac{1}{2\pi}\log(k_\delta|\mathbf{r}-\mathbf{r}_s|),$$

there exists a constant $M_2$ depending on $\mathbf{r}_{s,f}$ and $\delta$ such that

$$\|h\|_{\mathcal{C}^1(\partial D_{\mathbf{r}_s,\epsilon})} \le M_2.$$

Hence

$$\left|\int_{\partial D_{\mathbf{r}_s,\epsilon}} \left(G_\delta(\mathbf{r},\mathbf{r}_s)\frac{\partial G_\delta(\mathbf{r},\mathbf{r}_f)}{\partial n}\right) ds\right| \le \left(\frac{1}{2\pi}|\log k_\delta \epsilon| + M_2\right) M_1 2\pi\epsilon \to 0,$$

and

$$\int_{\partial D_{\mathbf{r}_s,\epsilon}} G_\delta(\mathbf{r},\mathbf{r}_f)\frac{\partial G_\delta(\mathbf{r},\mathbf{r}_s)}{\partial n}\, ds = \epsilon\int_0^{2\pi} G_\delta(\mathbf{r}_s+\epsilon e^{i\theta},\mathbf{r}_f)\left(\frac{1}{2\pi\epsilon} + \nabla_{\mathbf{r}} h\cdot n\right) d\theta.$$

Since $|\nabla h\cdot n| \le 2M_2$, we have

$$\lim_{\epsilon\to 0}\epsilon\int_0^{2\pi} G_\delta(\mathbf{r}_s+\epsilon e^{i\theta},\mathbf{r}_f)\left(\nabla_{\mathbf{r}} h\cdot n\right) d\theta = 0.$$

Finally

$$\lim_{\epsilon\to 0}\int_{\partial D_{\mathbf{r}_s,\epsilon}} G_\delta(\mathbf{r},\mathbf{r}_f)\frac{\partial G_\delta(\mathbf{r},\mathbf{r}_s)}{\partial n}\, ds = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_0^{2\pi} G_\delta(\mathbf{r}_s+\epsilon e^{i\theta},\mathbf{r}_f)\, d\theta = G_\delta(\mathbf{r}_s,\mathbf{r}_f).$$

Similarly, we can show that the second term of the left hand side of (3) is $-G_\delta(\mathbf{r}_f,\mathbf{r}_s)$. Thus we have

$$G_\delta(\mathbf{r}_f,\mathbf{r}_s) = G_\delta(\mathbf{r}_s,\mathbf{r}_f).$$

Applying the LAP, we have $G(\mathbf{r}_f,\mathbf{r}_s) = G(\mathbf{r}_s,\mathbf{r}_f)$ by taking $\delta\to 0^+$. $\square$

Similarly, the following lemma hold.

**Lemma 2.2.** *Let $u$ be the solution forward problem defined above. For all $x \in \Omega \setminus \overline{D}$,*

$$u(x) = \int_{\partial D} \left( u(y) \frac{\partial G(x,y)}{\partial \nu_y} - \frac{\partial u(y)}{\partial \nu_y} G(x,y) \right) ds(y),$$

*where $\nu_y$ is the outward unit normal.*

**Lemma 2.3.** *Let $u^s(\cdot, y)$ be the scattered field of $D$ due to a point source at $y$. Then*

$$u^s(x,y) = u^s(y,x), \quad \forall x, y \in \Omega \setminus \bar{D}.$$

To compute the Greens function $G$ and the scattered field numerically, we employ the fast algorithm based on the recursive doubling procedure in [2].

# 3 The inverse problem

The inverse problem is to reconstruct the support of the obstacle $D$, if we have the total scattered field information on the same line $\Gamma$. With the reciprocity property of $G$, we can prove the following uniqueness theorem similarly to the one in [12] and thus we omit its proof here.

**Theorem 3.1.** *Denote by $D_1$ and $D_2$ two sound soft obstacles with Lipschitz continuous boundaries. If for all incident waves $G(\cdot, \mathbf{r}_s)$ with $\mathbf{r}_s \in \Gamma$, the corresponding scattered fields $u_1^s(\cdot, \mathbf{r}_s)$ and $u_2^s(\cdot, \mathbf{r}_s)$ coincide on $\Gamma$, then $D_1 = D_2$.*

Since we measure the near field data, to apply the linear sampling method, we will need the following near field operator $\mathcal{N} : L^2(\Gamma) \to L^2(\Gamma)$ such that for any $g \in L^2(\Gamma)$,

$$(\mathcal{N}g)(\mathbf{r}) = \int_\Gamma u^s(\mathbf{r}, \mathbf{r}_s) g(\mathbf{r}_s)\, ds(\mathbf{r}_s), \ \forall \mathbf{r} \in \Gamma. \tag{4}$$

Let $S$ be a region containing the object $D$. The linear sampling type methods is the following. For any $\mathbf{r}_s \in S$, we consider the following integral equation

$$\mathcal{N}g = G(\cdot, \mathbf{r}_s). \tag{5}$$

It is well-known that the above equation does not have a solution in general. However, it is possible to find an approximate solution to (5) except a discrete set of wavenumber $k$'s. The following type of theorem justifies the linear sampling method.

**Theorem 3.2.** *Suppose $k^2$ is not a Dirichlet eigenvalue for $D$. Let $\mathcal{N}$ be the near-field operator defined by (4) and $u^s(\cdot, \mathbf{r}_s)$ be the scattered field due to a point source at $\mathbf{r}_s \in \Gamma$.*

1. *If $\mathbf{r}_s \in D$, then for any $\epsilon > 0$ there exists an approximate solution $h^\epsilon(\cdot, \mathbf{r}_s)$ of (5) such that*

$$\|(\mathcal{N}h^\epsilon)(\cdot, \mathbf{r}_s) - G(\cdot, \mathbf{r}_s)\|_{L^2(\Gamma)} \leq \epsilon.$$

   *In addition, $h^\epsilon(\cdot, \mathbf{r}_s)$ converges in an appropriate function space as $\epsilon \to 0$.*

2. *If $\mathbf{r}_s \to \partial D$ and $h^\epsilon(\cdot, \mathbf{r}_s)$ satisfies*

$$\|(\mathcal{N}h^\epsilon)(\cdot, \mathbf{r}_s) - G(\cdot, \mathbf{r}_s)\|_{L^2(\Gamma)} \leq \epsilon,$$

   *then the norm of $h^\epsilon(\cdot, \mathbf{r}_s)$ in an appropriate function space tends to infinity as $\epsilon \to 0$.*

*Proof.* We first assume $D \subset C$. Similar to [8], consider the linear equation

$$(\mathcal{N}\phi)(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_s), \quad \forall \mathbf{r} \in \partial C. \tag{6}$$

Note that $(\mathcal{N}\phi)(\mathbf{r})$ is the scattered field of $D$ due to the incident field

$$(S\phi)(\mathbf{r}) = \int_{\partial C} G(\mathbf{r}, \mathbf{r}_s)\phi(\mathbf{r}_s)\, ds(\mathbf{r}_s). \tag{7}$$

If $\mathbf{r}_s \in D$, $\phi$ is a solution to the near field equation (6) if $S\phi$ solves

$$\Delta u + k^2 u = 0 \qquad \text{in } D, \tag{8a}$$

$$u = -G(\cdot, \mathbf{r}_s), \qquad \text{on } \partial D. \tag{8b}$$

It is known that $S\phi$ with $\phi \in L^2(\partial C)$ is dense in

$$\mathbb{H}(D) = \{u \in H^1(D); \triangle u + k^2 u = 0\}.$$

Hence there exists a $\phi$ such that $S\phi$ approximate the solution of (8) for $z \in D$. As $z$ approaches $\partial D$, the norm of $G(\cdot, z)$ on $\partial D$ blows up, hence the norm of the approxiamae solution of (8).

If $\Gamma$ is a line segment, then $\Gamma$ is a subset of $\partial C$ for some domain $C$ with $D$ inside. Following [7], it is suffices to show that the set of functions

$$(S\phi)(x) = \int_{\partial C} \Phi(x, y)\phi(y)\, ds(y), \quad \phi \in L^2(\partial C) \quad \text{with support in} \quad \Gamma \subset \partial C$$

is complete in $L^2(\partial)$. Then the approximation property follows Theorem 5.4 of [6]. Let $\varphi \in L^(\partial C)$ and suppose that for a fixed $\Gamma \subset \partial C$

$$\int_{\partial C} \varphi(x) \left[ \int_\Gamma \phi(y) G(x, y)\, ds(y) \right] ds(x) = 0$$

for every $\phi \in L^2(\Gamma)$. Interchanging the order of integration, we obain

$$\int_\Gamma \phi(y) \left[ \int_{\partial C} \varphi(x) \Phi(x, y)\, ds(x) \right] ds(y) = 0$$

for every $\phi \in L^2(\partial C)$. This implies the single-layer potential

$$(S\varphi)(y) = \int_{\partial C} \varphi(x) G(x, y)\, ds(x)$$

is zero on $\partial C$. By analyticity we have $(S\varphi)(y) = 0$ on $\partial C$. Since the single-layer potential is a solution to the Helmholtz equation and $k^2$ is not a Dirichlet eigenvalue for $\Omega$, we obtain $\varphi = 0$. □

The implementation of the above method requires us to choose a sampling domain $S$ containing $D$. Then for each point $\mathbf{r}_s \in S$, we need to find an approximate solution $g_{\mathbf{r}_s}$ of (5). According to Theorem 3.2, the norm of $g_{\mathbf{r}_s}$ is relative small if $\mathbf{r}_s \in D$ and becomes larger as $\mathbf{r}_s$ approaching $\partial D$. The procedure is exactly the same as the linear sampling method or the reciprocity gap method (see [4]).

A major numerical difficulty we are facing is the evaluation of many Green functions. Since we need $G(x,z)$ for each $x \in \Gamma$ and $z \in S$, the computation cost is prohibitive. This is similar to the case of using LSM to characterize an object embedded in inhomogeneous complex background in open space. Methods of (partially) speed up or avoiding the computation are discussed in [5, 4].

For our the problem in waveguide, we propose a Fourier expansion method to significantly reduce the computation of $G(\mathbf{r}, \mathbf{r}_s)$ in the discrete case. Let $\{x_i\}, i = 1, 2, \ldots, N_\Gamma$ be a discrete set of points on $\Gamma$ and $\{z_j\}, j = 1, 2, \ldots, N_S$ be a discrete set of points on $S$. We need $G(x_i, z_j)$, $i = 1, 2, \ldots, N_\Gamma$, $j = 1, 2, \ldots, N_S$ to implement the inverse scheme.

Suppose $\Gamma_\delta \subset \Omega$ is a neighborhood of $\Gamma$ satisfying $\Gamma_\delta \cap S = \emptyset$. Since

$$\Delta_{\mathbf{r}} G(\mathbf{r}, \mathbf{r}_s) + k^2 G(\mathbf{r}, \mathbf{r}_s) = 0, \ \forall \mathbf{r}, \mathbf{r}_s \in \Gamma_\delta \cup S,$$

by reciprocity we have

$$\Delta_{\mathbf{r}_s} G(\mathbf{r}, \mathbf{r}_s) = \Delta_{\mathbf{r}_s} G(\mathbf{r}_s, \mathbf{r}) = -k^2 G(\mathbf{r}_s, \mathbf{r}) = -k^2 G(\mathbf{r}, \mathbf{r}_s),$$

which implies that

$$\Delta_{\mathbf{r}_s} G(\mathbf{r}, \mathbf{r}_s) + k^2 G(\mathbf{r}, \mathbf{r}_s) = 0.$$

Note that we need $G(x_i, z)$ for all $z \in S$ for a fixed point $x_i \in \Gamma$. By reciprocity, we need $G(z, x_i)$ for all $z \in S$. It is easy to see that $G(z, x_i)$ satisfies the following Helmholtz equation:

$$\Delta_z G(z, x_i) + k^2 G(z, x_i) = 0, \quad \text{in } S. \tag{9}$$

Thus if we know the value of $G(z, x_i)$ for $z \in \partial S$, we will have $G(z, x_i)$ for all $z \in S$ by solving a Dirichlet boundary problem of the above Helmholtz equation. To further simplify the computation, we first choose $S$ as a disk containing $D$. Let $z_k, k = 1, 2, \ldots, N_S^0$ be a discrete set of points $S$. For a fixed point $x_i$ on $\Gamma$, we first compute $G(x_i, z_k), k = 1, 2, \ldots, N_S^0$. Since the solution of the Helmholtz equation on a disk can be represented by a series of products of Hankel functions and trigonometric functions [3], $G(x_i, z)$ can be obtained after a Fourier expansion of the Dirichlet data on $\partial S$.

It is obvious the method is not restrict to the case of wave guide and can be applied to open space problem. It is especially useful for three dimensional case since it reduce the computation of Green functions in a three dimension sampling domain to a two dimension surface.

## 4 Numerical Examples

We consider a waveguide with period 1. The lower boundary of the waveguide is the x-axis. The upper boundary is given by $f(x) = 1 + 0.05\sin(x)$. In Fig. 4, we plot the bank structure of the above waveguide. In Fig. 5, we show the computed background Greens functions for $k = 1$, $k = 3$, and $k = 9$.

A circular obstacle is located at $(x, y) = (0.4, 0.4)$ with radius of 0.1. We compute the scattering field and recorded on the same segment. We then employ the near field linear sampling method. For the ill-posed integral equations, we use Tikhonov regularization with parameter $\alpha = 10^{-6}$. Note that the choice of the regularization parameter is ad-hoc, i.e., trial and error. We show the reciprocal of the norm of the linear integral equations on the left of Fig. 6. It can be seen that the location and the size of the obstacle can be obtained. However, it seems difficult to recover the exact shape of the obstacle.

Would you please add more details about the numerical implementation?

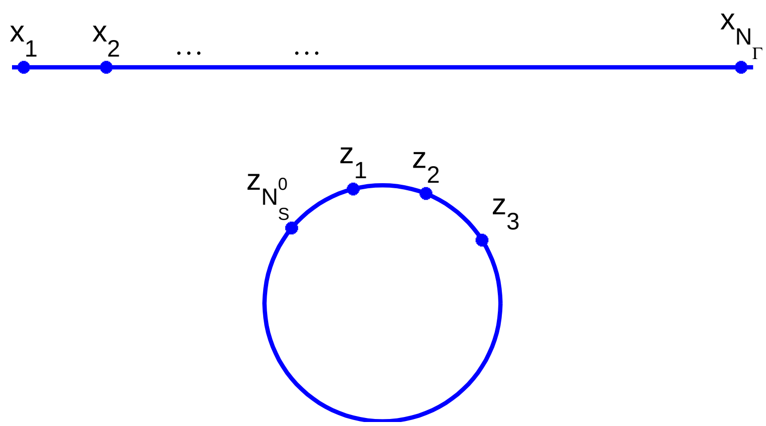


Figure 3: Schematic picture: computation of $G(x, z)$ for all points in the sampling domain $S$ (the disk) due point sources on $\Gamma$.

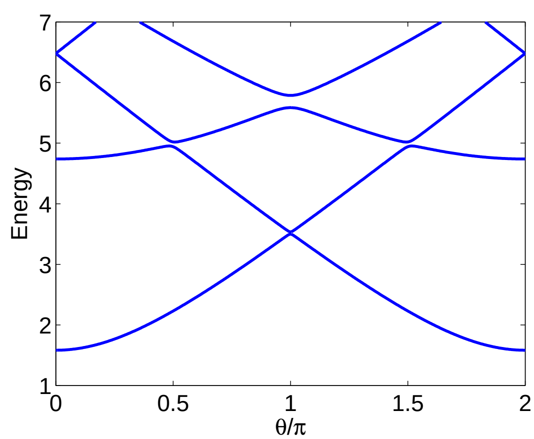


Figure 4: Frequency band structure.

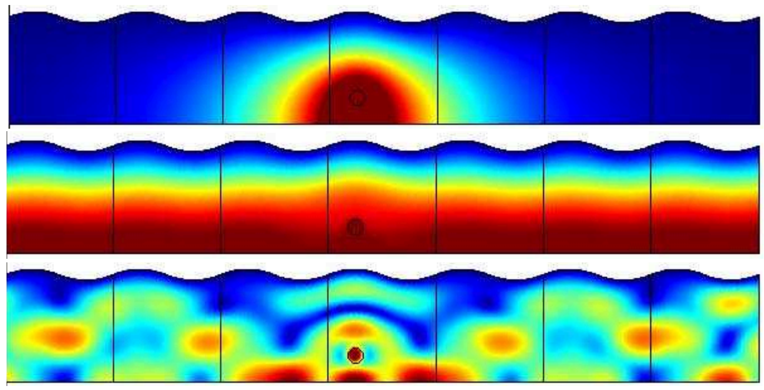

Figure 5: The background Greens functions for $k = 1$, $k = 3$, and $k = 9$.

Then we choose two circular obstacles are located at $(x, y) = (0.4, 0.4), (0.6, 0.4)$ with radius of 0.05. Tikhonov regularization parameter is still $\alpha = 10^{-6}$. We show the reciprocal of the norm of the linear integral equations on the right of Fig. 6. The construction is similar to the single obstacle case.

What is the wavenumber $k$ you are using for the two examples?

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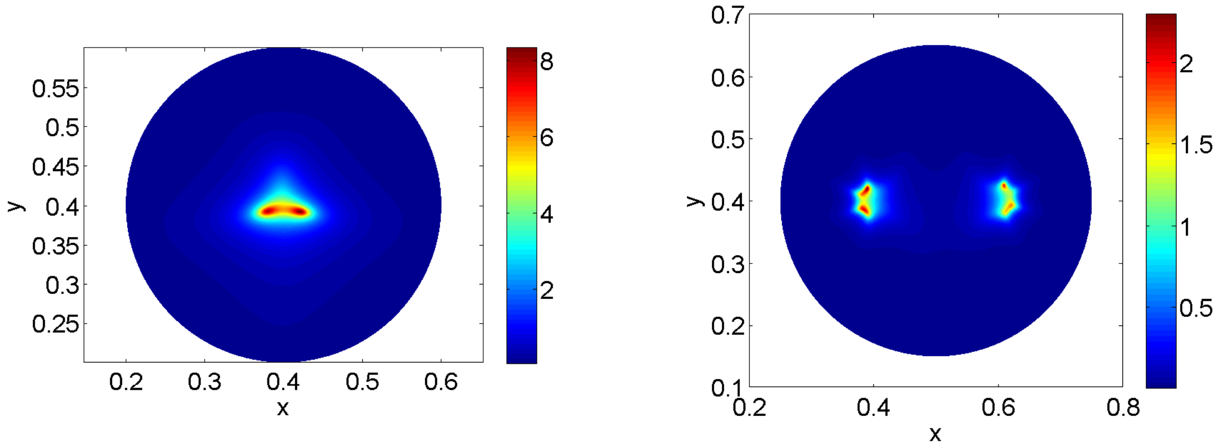


Figure 6: The plot of the norm of the solution. Left: One circular target located at $(x, y) = (0.4, 0.4)$ with radius of 0.1. Right: Two circular obstacles are located at $(x, y) = (0.4, 0.4), (0.6, 0.4)$ with radius of 0.05.


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