



Self-colorings of graphs

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Abstract. Suppose that G and H are finite simple graphs on the same vertex set V . We will say that H is G -colorable if H is properly colorable from the list assignment N_G , i.e., the assignment of $N_G(v)$ as a list of colors to each $v \in V$. If G itself is G -colorable, we will say that G is self-colorable.

It is shown that every graph G with no isolated vertices is self-colorable. A necessary and sufficient condition on G for its complement to be G -colorable is proven. Multicolorings from the N_G list assignment are considered and questions are posed.

We owe this inquiry to one of Steve Hedetniemi's seminal questions [1].

1 Introduction

All graphs here are finite and simple. G is a graph, its vertex set will be $V(G)$ and its edge set is $E(G)$. If $v \in V(G)$ then $N_G(v) = \{u \in V(G) | uv \in E(G)\}$ is called the *open neighborhood* (or open neighborhood set) of v in G . The corresponding closed neighborhood is $N_G[v] = \{v\} \cup N_G(v)$. If $S \subseteq V(G)$, $N_G(S) = \bigcup_{v \in S} N_G(v)$.

At the Fall 2017 Clemson Mini-Conference, Steve Hedetniemi asked: for which graphs G does the indexed collection $\mathcal{N}(G) = [N_G(v) | v \in V(G)]$ of open neighborhoods have a *system of distinct representatives* (SDR). An SDR for $\mathcal{N}(G)$ is a one-to-one function $\phi : V(G) \rightarrow V(G)$ such that $\phi(v) \in N_G(v)$ for all $v \in V(G)$. Necessary and sufficient conditions were found in [1].

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Theorem 1.1 (Hedetniemi, Holliday and Johnson [1]).

Suppose that G is a graph. $\mathcal{N}(G)$ has an SDR if and only if G has a spanning subgraph the components of which are either single edges (K_2) or cycles.

Proof. Clearly, if G has a spanning subgraph X such that $\mathcal{N}(X)$ has an SDR, then $\mathcal{N}(G)$ has an SDR. It is easy to see that if every component of X is either a single edge or a cycle, then $\mathcal{N}(X)$ has an SDR. So the “if” claim is settled.

Suppose that $\mathcal{N}(G)$ has an SDR $\phi : V \rightarrow V$, $V = V(G)$. Then ϕ is a permutation of V with no fixed points. Therefore ϕ factors into “cyclic” permutations on pairwise disjoint subsets of V , with all cycle lengths ≥ 2 . Because $v\phi(v) \in E(G)$ for all $v \in V$, these cyclic permutations are associated with single edges, when the permutation is a transposition, and graph cycles, when the permutation cycle length is > 2 . $\square$

Corollary 1.2. *If G is bipartite then $\mathcal{N}(G)$ has an SDR if and only if G has a perfect matching.*

Proof. Every even cycle has a perfect matching. $\square$

A spanning subgraph X of a graph G such that every component of X is either an edge or a cycle will be called a $(1,2)$ -factor of G . This terminology is due to Tutte [4], but he means by it something a little different: in Tutte’s definition, a $(1,2)$ -factor of a graph is a spanning subgraph in which every vertex has degree either 1 or 2. Therefore, our $(1,2)$ -factors are all Tutte $(1,2)$ -factors, but the reverse inclusion does not hold. For instance, P_3 , the path on 3 vertices, is a Tutte $(1,2)$ -factor of itself, but does not have a $(1,2)$ -factor in our more restrictive sense.

Obviously Theorem 1.1 can be restated.

Theorem 1.1’. *Suppose that G is a graph. $\mathcal{N}(G)$ has an SDR if and only if G has a $(1,2)$ -factor.*

If it was not clear from the example of $[N_G(v)|v \in V(G)]$, an SDR of an indexed collection $[A_i; i \in I]$ of sets is an injective function $\phi : I \rightarrow \bigcup_{i \in I} A_i$ such that $\phi(i) \in A_i$ for all $i \in I$. There is a time-hallowed connection between SDRs and *list colorings* of graphs.

Let us explain the latter. For a set C (of *colors*) let $\mathcal{F}(C)$ denote the collection of finite subsets of C (so $\mathcal{F}(C) = 2^C$ if C itself is finite).

A *list assignment* to the vertices of a graph G is a function $L : V(G) \rightarrow \mathcal{F}(C)$, for some set C . For such an L , a *proper L -coloring* of G is a function $\phi : V(G) \rightarrow C$ such that, for all $u, v \in V(G)$,

1. $\phi(v) \in L(v)$, and
2. if $uv \in E(G)$, then $\phi(u) \neq \phi(v)$.

Lemma 1.3. *If $[A_i; i \in I]$ is an indexed collection of sets, then an SDR for this collection is the same as a proper L -coloring of $K(I)$, the complete graph on vertex set I , if L is defined by $L(i) = A_i$.*

Letting $V = V(G)$ and $N_G : V \rightarrow 2^V$ denote the list assignment that assigns $N_G(v)$ to each $v \in V$, we see that Theorem 1.1 gives a necessary and sufficient condition on a graph G for the complete graph $K(V)$ on V to be properly N_G -colorable. This perspective immediately opens a general question: Given G , for which graphs H with $V(H) = V(G)$ is there a proper N_G -coloring of H ?

2 G -colorability

Throughout, G and H will be graphs on the same vertex set V . We will say that H is *G -colorable* if there is a proper N_G -coloring of H . In other words, H is G -colorable if is a function $\phi : V \rightarrow V$ such that, for all $u, v \in V$

1. $\phi(v) \in N_G(v)$, and
2. if $uv \in E(H)$ then $\phi(u) \neq \phi(v)$.

The following proposition gives some straightforward basics about G -colorability. Proofs are omitted.

Proposition 2.1. *Suppose that G , H , and X are graphs on the same vertex set V .*

- (1) *If G has an isolated vertex, then no graph on V is G -colorable.*

- (2) If H is G -colorable and X is a spanning subgraph of H , then X is G -colorable.
- (3) If H is G -colorable and G is a spanning subgraph of X , then H is X -colorable.

Two types of extremal problems arise from Proposition 2.1 (2) and (3).

1. Given G , find the maximal graph or graphs H which are G -colorable.
2. Given H , find the minimal graph or graphs G such that H is G -colorable.

In these problems, “maximal” and “minimal” refer to the partial order on these graphs (with the same vertex set) of inclusion of edge sets.

Example 2.2. Let $G = K_{1,3}$, the “claw”, depicted with vertices labeled in Figure 1.

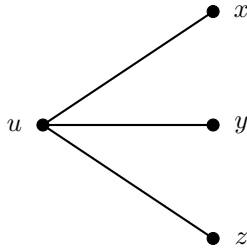


Figure 1: The claw

G is G -colorable: set $u = \phi(x) = \phi(y) = \phi(z)$ and let $\phi(u) \in \{x, y, z\}$. Further, G is the unique maximal graph on $V = \{u, x, y, z\}$ among the G -colorable graphs. (Verification left for the pleasure of the reader.) However, although G is minimal in $Q(G) = \{X | X \text{ is a graph on } V \text{ and } G \text{ is } X\text{-colorable}\}$, because deleting any edge of G creates an isolated vertex, G is not the unique minimal graph in $Q(G)$; the 3 different perfect matchings on V are also minimal in $Q(G)$.

Intriguing as we find these questions of maximality and minimality, we postpone further inspection of them for now.

If a graph G is G -colorable, we will say that G is self-colorable.

Theorem 2.3. *A graph G is self-colorable if and only if G has no isolated vertices.*

Proof. The “only if” claim follows from Proposition 2.1.

Suppose that G has no isolated vertices. We can assume that $n = |V(G)| \geq 3$ and proceed by induction on n . Therefore, we can assume that G is connected.

If $v \in V(G)$ has degree 1 then its neighbor w has degree ≥ 2 , as G is connected on at least 3 vertices. Then $G - v$ has no isolated vertices. By the induction hypothesis, $G - v$ has a self-coloring. Extend this coloring to G by coloring v with w , to obtain a self-coloring of G .

So now we can assume that all vertices of G have degree at least 2 in G . Take any $v \in V(G)$. Then $G - v$ has no isolated vertex, and is therefore self-colorable, by the induction hypothesis. Consider any such coloring of $G - v$. If any $w \in N_G(v)$ does not appear as a color on any other vertex of $N_G(v)$, then color v with w . Otherwise, every $w \in N_G(v)$ appears as a color on some other $x \in N_G(v)$ in our supposed self-coloring of $G - v$. Therefore, this coloring restricted to $N_G(v)$ is a permutation of $N_G(v)$. Take any $w \in N_G(v)$, recolor w with v and color v with the color formerly on w to obtain a G -coloring of G . $\square$

Now we know which graphs G are self-colorable, and for which G the complete graph on $V(G)$ is G -colorable. A natural next question is: for which G is its complement G -colorable? We do not know the answer to this question, but we do have a necessary condition.

Let the complement of a graph G be denoted $\overline{G}$.

Lemma 2.4. *If $\overline{G}$ is G -colorable, then for every $U \subseteq V = V(G)$ which is independent in G , $|N_G(U)| \geq |U|$.*

Proof. Let $\phi : V \rightarrow V$ be a G -coloring of $\overline{G}$. If $U \subseteq V$ is independent in G , then $\overline{G}[U]$, the subgraph of $\overline{G}$ induced by U , is a complete graph. Therefore ϕ restricted to U is a one-to-one function from U into $N_G[U]$. $\square$

A matching M in a graph G *saturates* a set $U \subseteq V(G)$ if each vertex of U is incident to an edge of M .

Lemma 2.5. *If B is a bipartite graph with bipartition U, W , then there is a matching in B which saturates U if and only if for all $S \subseteq U$, $|N_B(S)| \geq |S|$.*

This is the well known “matchings in bipartite graphs” version of Hall’s Theorem on SDRs; see [4].

For the next lemma, and therefore for the hard part of the proof of the theorem following it, we are indebted to L. Levine [2].

Lemma 2.6. *Suppose that G is a graph on vertex set V and B is the bipartite graph with bipartition $V, V' = \{v' | v \in V\}$, with $E(B) = \{uv' | uv \in E(G)\}$. Then G has a $(1,2)$ -factor if and only if B has a perfect matching.*

Proof. Suppose that B has a perfect matching M . Because $|M| = |V'|$, M saturates both V and V' . By the way adjacency is defined in B , if $uv' \in E(B)$, then $u \neq v$.

Define $f : V \rightarrow V$ by: for $u \in V$, let $f(u) = v \iff uv' \in M$. Because M is a perfect matching, f is well-defined and injective. Therefore, since V is finite, f is a permutation of V with no fixed points. Also, $vf(v) \in E(G)$ for all $v \in V$. Therefore, as in the proof of Theorem 1.1, the disjoint permutation cycles that f the permutation factors into correspond to vertex disjoint edges and cycles in G , and it follows that G has a $(1,2)$ -factor – a spanning subgraph whose components are single edges or cycles.

Now suppose that G has a $(1,2)$ -factor F . We form a perfect matching M in B as follows: for each single-edge component uv of F , put both uv' and vu' in M . For each component of F which is a cycle, C , say $C = u_1u_2 \dots u_ku_1$, put the edges $u_1u'_2, u_2u'_3, \dots, u_ku'_1$ in M . (Or, you could add edges to M by going around the cycle the other way, but don’t do both!) It is straightforward to see that if every component of F is processed per these instructions, the resulting M will be a perfect matching in B . $\square$

Theorem 2.7. *Suppose that G is a graph on finite vertex set V and $K(V)$ is the complete graph on V . The following are equivalent:*

- (a) G has a $(1,2)$ -factor;
- (b) $K(V)$ is G -colorable;
- (c) $\overline{G}$ is G -colorable.

Proof. Since $\mathcal{N}(G)$ having an SDR is, as explained earlier, equivalent to $K(V)$ being G -colorable, we have, by Theorem 1.1, that (a) and (b) are

equivalent. Also, by Proposition 2.1 part (2), (b) implies that all graphs on V are G -colorable, and that implies (c). Therefore, to finish the proof it suffices to prove that (c) $\Rightarrow$ (a).

Let B be the bipartite graph with bipartition V, V' defined with reference to G as in Lemma 2.6. By that lemma, to prove (a), assuming (c), it suffices to show that B has a perfect matching. Because $|V| = |V'|$, it suffices to show that B has a matching saturating V , and therefore, by Lemma 2.5, it suffices to show that $|N_B(S)| \geq |S|$ for all $S \subseteq V$.

Suppose that $S \subseteq V$. Let $I = \{v | v \text{ is an isolated vertex in } G[S]\}$ and let $Q = S \setminus I$. For any subset $U \subseteq V$, let $U' = \{u' | u \in U\} \subseteq V'$. Note that $|U'| = |U|$ for each such U . Assuming (c), by Lemma 2.4 we have $|N_G(I)| \geq |I|$. Since I is the set of isolated vertices in $G[S]$, $N_G(I) \cap Q = \emptyset$ and $Q \subseteq N_G(Q)$. Thus, $|N_B(S)| = |(N_G(I) \cup N_G(Q))'| = |N_G(I) \cup N_G(Q)| \geq |N_G(I) \cup Q| = |N_G(I)| + |Q| \geq |I| + |Q| = S$. $\square$

Corollary 2.8. *Let G and V be as in Theorem 2.7. Then (a)- (c) in Theorem 2.7 are also equivalent to each of: for every set $U \subseteq V$ independent in G ,*

$$(d) \quad |N_G(U)| \geq |U|;$$

(e) *there is a matching in G which saturates U .*

Proof. This is a corollary of Theorem 2.7 and of its proof. But first, note that if $U \subseteq V$ is independent in G , then no edge of G is incident to more than one $u \in U$. Therefore (e) $\Rightarrow$ (d). For the reverse implication, observe that if $U \subseteq V$ is independent in G then so is each set $S \subseteq U$. Thus (d) applied to the bipartite graph with bipartition $U, N_G(U)$, induced by the $U - N_G(U)$ edges of G , implies (e), by Lemma 2.5. Thus, (d) and (e) are equivalent.

Lemma 2.4 says that (c) implies (d), and the proof of Theorem 2.7 shows that (d) implies (a). $\square$

The surprising fact that the G -colorability of $\overline{G}$ implies the G -colorability of $K(V) = G \cup \overline{G}$ leads us to:

Conjecture 2.9. *If G and H are graphs on the same vertex set, and H is G -colorable, then $G \cup H$ is G -colorable.*

3 Graph-referential multicolorings

G , H , and V will be as before and we add to the mix functions $f : V \rightarrow \mathbb{N} = \{0, 1, 2, \dots\}$. For such a function f , a (G, f) -coloring of H is a function $\phi : V \rightarrow 2^V$ such that, for all $u, v \in V$,

1. $\phi(v) \subseteq N_G(v)$;
2. $|\phi(v)| = f(v)$; and
3. if $uv \in E(H)$ then $\phi(u) \cap \phi(v) = \emptyset$.

When f is a constant function, say $f(v) = k$ for all $v \in V$, we will write (G, k) instead of (G, f) . Clearly, a $(G, 1)$ -coloring is a G -coloring. A (G, k) -coloring of G will be called a k -self-coloring of G . For instance, every cycle C_n on $n > 3$ vertices is 2-self-colorable; further, there is exactly one $(C_n, 2)$ -coloring of C_n .

There is an obvious generalization of Proposition 2.1 to (G, k) -colorings. Let $\delta(G)$ denote the minimum degree in G .

Proposition 3.1. *Suppose that G , H , and X are graphs on the same vertex set V , and k is a non-negative integer.*

1. *If there is a (G, k) -coloring of H , then $\delta(G) \geq k$.*
2. *If there is a (G, k) -coloring of H , and X is a subgraph of H , then there is a (G, k) -coloring of X .*
3. *If there is a (G, k) -coloring of H , and G is a subgraph of X , then there is an (X, k) -coloring of H .*
4. *If $k > 0$ and there is a (G, k) -coloring of H , then there is a $(G, k-1)$ -coloring of H .*

The question of k -self-colorability is just one of the many that arise from the definition of (G, k) - (more generally, of (G, f) -) coloring, but it is the one that we will primarily focus on for the remainder of this paper. Noting that for every graph G , every simple graph on $V(G)$ is $(G, 0)$ -colorable (color each vertex with $\emptyset$), we define $hh(G) = \max\{k \in \mathbb{N}; G \text{ is } k\text{-self-colorable}\}$.

Theorem 2.3 can be restated; $hh(G) \geq 1$ if and only if G has no isolated vertices.

For a graph H , $\omega(H)$ denotes the *clique number* of H , the greatest order of a complete subgraph of H , and $\alpha(H)$ is the *vertex independence number* of H , $\alpha(H) = \omega(\overline{H})$.

Proposition 3.2. *Suppose that G and H are graphs on a vertex set V , k is a positive integer, and H is (G, k) -colorable. Then $\omega(H) \leq \lfloor \frac{|V|}{k} \rfloor$.*

Proof. This proof will resemble that of Lemma 2.4. If $\phi : V \rightarrow 2^V$ is a (G, k) -coloring of H , and $U \subseteq V$ induces a complete graph in H , then the sets $\phi(u)$, $u \in U$, are pairwise disjoint k -subsets of V . Therefore, $k|U| \leq |V|$. $\square$

Corollary 3.3. *If $k > 1$ is an integer, then for no graph G is the complete graph $K(V)$ on $V = V(G)$ (G, k) -colorable.*

Corollary 3.4. *Suppose G is a graph on a vertex set V , and k is a positive integer. If G is k -self-colorable then $\omega(G) \leq \lfloor \frac{|V|}{k} \rfloor$. If $\overline{G}$ is (G, k) -colorable, then $\alpha(G) \leq \lfloor \frac{|V|}{k} \rfloor$.*

Corollary 3.5. *For any graph G , $hh(G) \leq \min[\delta(G), \lfloor |V(G)|/\omega(G) \rfloor]$.*

Example 3.6. *The graph G in Figure 2 has $hh(G) = 1$, while $\delta(G) = |V(G)|/\omega(G) = 2$.*

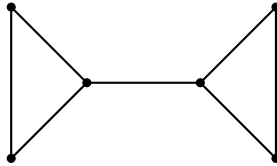


Figure 2: A graph for which the inequality in Corollary 3.5 is strict.

If we add 2 edges in Figure 2, we can get the “prism,” depicted in Figure 3.

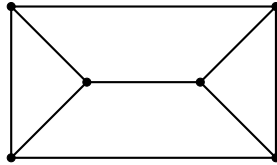


Figure 3: The prism

It is easy to see that the prism is 2-self-colorable, but it is not 3-self-colorable, by Corollary 3.5. This raises a question: is the inequality in Corollary 3.5 always equality when G is regular and connected? Turns out the answer is no.

Proposition 3.7.

- (1) *If G is triangle-free, then G is $\delta(G)$ -self-colorable.*
- (2) *If G is r -regular then G is r -self-colorable if and only if G is triangle-free.*

We leave the proofs to the reader.

By Proposition 3.7 (2), to find a counterexample to the conjecture that for every r -regular connected graph, equality in Corollary 3.5 holds, it will suffice to find a connected r -regular graph G with $|V(G)|/\omega(G) \geq r$, with $\omega(G) \geq 3$. Possibly there are many such examples, but here is an entire family of them, one for each $r \geq 4$.

Example 3.8. *For $r \geq 3$, let the vertices of C_r be replaced by K_{r-1} 's and let K_{r-1} 's corresponding to adjacent vertices on the C_r be joined by a matching consisting of $r-1$ edges.*

The resulting graph G is r -regular, with $\omega(G) = r-1$. G is triangle-free $\iff r = 3$. In all cases, $|V(G)|/\omega(G) = r(r-1)/(r-1) = r$.

Therefore, for $r \geq 4$, G is not triangle-free, whence $hh(G) < r = \delta(G) = |V(G)|/\omega(G)$.

So, what is $hh(G)$? The reader should note that what we are calling G here could be any one of a number of non-isomorphic graphs, depending on how those matchings are installed. We leave as a recreation the verification that if $r > 3$, then $hh(G) = 3$ for every such G . When $r = 3$ then there are exactly two non-isomorphic choices for G , one with triangles, the other without. When G is the triangle-free one, $hh(G) = 3$, by Proposition 3.7. Also by Proposition 3.7, when G is the one with triangles, $hh(G) < 3$. We leave it as a puzzle whether $hh(G) = 1$ or 2 in this case.

Here is another puzzle: can an r -regular, connected graph G be found such that

$$hh(G) < \left\lfloor \frac{|V(G)|}{\omega(G)} \right\rfloor < r?$$

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